Math 231b
Lecture 05

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5. Lecture 5: Euclidean bundles, orthogonal complements and orientations

Recall that we defined the Whitney sum of two bundles:

Let ξ_1, ξ_2 be two vector bundles over the same space B. Let

$$d\colon B \to B \times B$$

denote the diagonal embedding. The bundle $d^*(\xi_1 \times \xi_2)$ over B is called the *Whitney sum* of ξ_1 and ξ_2, and will be denoted $\xi_1 \oplus \xi_2$. Each fiber $E_b(\xi_1 \oplus \xi_2)$ is canonically isomorphic to the direct sum of the fibers $E_b(\xi_1) \oplus E_b(\xi_2)$.

Definition 5.1. Consider two vector bundles ξ and η over the same base space B with $E(\xi) \subset E(\eta)$. Then ξ is a *sub-bundle* of η, written $\xi \subset \eta$, if each fiber $E_b(\xi)$ is a sub-vector space of the corresponding fiber $E_b(\eta)$.

Lemma 5.2. *Let ξ_1 and ξ_2 be sub-bundles of η such that each vector space $E_b(\eta)$ is equal to the direct sum of the sub-spaces $E_b(\xi_1)$ and $E_b(\xi_2)$. Then η is isomorphic to the Whitney sum $\xi_1 \oplus \xi_2$.*

Proof. Define a map

$$f\colon E(\xi_1 \oplus \xi_2) \to E(\xi) \text{ by } f(b, e_1, e_2) = e_1 + e_2.$$

The lemma of the previous lecture shows that f is an isomorphism of bundles since it maps the fibers isomorphically onto each other. □

5.1. Euclidean vector bundles.

Let V be a finite dimensional real vector space. Recall that a real valued function $q\colon V \to \mathbb{R}$ is called *quadratic* if q satisfies $q(av) = a^2 q(v)$ for every $v \in V$ and $a \in \mathbb{R}$ and the map $b\colon V \times V \to \mathbb{R}$ defined by

$$b(v,w) := \frac{1}{2}(q(v+w) - q(v) - q(w))$$

is a symmetric bilinear pairing. We also write $v \cdot w$ for $b(v,w)$. We have in particular: $v \cdot v = q(v)$. The quadratic function q is called *positive definite* if $q(v) > 0$ for every $v \neq 0$.

Definition 5.3. A *Euclidean vector space* is a real vector space V together with a positive definite quadratic function

$$q\colon V \to \mathbb{R}.$$

The real number $v \cdot w$ is called *inner product* of the vectors v and w. The number $q(v) = v \cdot v$ is also denoted by $|v|^2$.

Definition 5.4. A *Euclidean vector bundle* is a real vector bundle ξ together with a continuous map

$$q\colon E(\xi) \to \mathbb{R}$$

such that the restriction of q to each fiber of ξ is positive definite and quadratic. The map q is called a *Euclidian metric* on ξ.

In the case of the tangent bundle τ_M of a smooth manifold, a Euclidian metric $q\colon DM \to \mathbb{R}$ is called a *Riemannian metric*, and M together with q is called a *Riemannian manifold*.

Example 5.5. a) The trivial bundle ϵ_B^n on a space B can be given the Euclidean metric

$$q(b,x) = x_1^2 + \ldots + x_n^2.$$

b) Since the tangent bundle of $\mathbb{R}^n$ is trivial it follows that the smooth manifold $\mathbb{R}^n$ possesses a standard Riemannian metric. Moreover, any smooth manifold $M \subset \mathbb{R}^n$, the composition

$$DM \subset D\mathbb{R}^n \xrightarrow{q} \mathbb{R}$$

makes M into a Riemannian manifold.

Lemma 5.6. *Let ξ be a trivial bundle of dimension n over a space B and let q be any Euclidean metric on ξ. Then there exist n sections $s_1, \ldots, s_n$ of ξ which are normal and orthogonal in the sense that*

$$s_i(b) \cdot s_j(b) = \delta_{ij}$$

for each $b \in B$ where δ_{ij} is the Kronecker symbol.

Proof. The lemma of the previous lecture shows that ξ admits n nowhere dependent sections. Pointwise application of the Gram-Schmidt orthonormalization process yields orthonormal sections. □

5.2. **Orthogonal complements.** Given a sub-bundle $\xi \subset \eta$, is there a complementary sub-bundle so that η splits as a Whitney sum? If η is a Euclidean bundle, we can always find such a complement. We can construct it as follows.

Let $E_b(\xi^\perp)$ denote the subspace of $E_b(\eta)$ consisting of all vectors v such that $v \cdot w = 0$ for all $E_b(\xi)$. Let $E(\xi^\perp)$ denote the union of all $E_b(\xi^\perp)$.

Theorem 5.7. *The space $E(\xi^\perp)$ is the total space of a sub-bundle $\xi^\perp \subset \eta$, and η is isomorphic to the Whitney sum $\xi \oplus \xi^\perp$. The bundle $\xi^\perp$ is called the* orthogonal complement *of ξ in η.*

Proof. It is clear that each fiber $E_b(\eta)$ is the direct sum of the subspaces $E_b(\xi)$ and $E_b(\xi^\perp)$. Thus it remains to show the local triviality of $\xi^\perp$. The lemma of the previous lecture then implies that the map $(v,w) \mapsto v + w$ is an isomorphism of vector bundles.

Given any point $b_0 \in B$, let U be a neighborhood of b_0 which is sufficiently small that both $\xi|U$ and $\eta|U$ are trivial. Since $\xi|U$ is trivial, we can choose orthonormal sections $s_1, \dots, s_m$ of $\xi|U$. We may enlarge this set of sections to a set of n independent local sections of $\eta|U$ by first choosing $s'_{m+1}, \dots, s'_n$ in the fiber $E_{b_0}(\eta)$. By the continuity of the determinant function, there is a neighborhood $V \subset U$ of b_0 such that $s_1(b), \dots, s_m(b), s'_{m+1}(b), \dots, s'_n(b)$ are linearly independent for all $b \in V$ and such that the $s_i(b)$ vary continuously with b in V. Applying the Gram-Schmidt orthonormalization process to $s_1, \dots, s_m, s'_{m+1}, \dots, s'_n$ in each fiber to obtain new sections $s_1, \dots, s_n$. The formulae for this process show that the s_i vary continuously with $b \in V$. We can now define a trivialization

$$h\colon V \times \mathbb{R}^{n-m} \to E(\xi^\perp)$$

by the formula

$$h(b,x) = x_1 s_{m+1}(b) + \ldots + x_{n-m} s_n(b).$$

□

5.3. **Stably trivial bundles.** The direct sum of two trivial bundles is of course again trivial. But the direct sum of two nontrivial bundles can also be trivial. If one bundle is trivial, this phenomenon has been given a name.

Definition 5.8. A vector bundle ξ over B is called *stably trivial* if the direct sum $\xi \oplus \epsilon^n$ is a trivial bundle for some n.

Example 5.9. The direct sum of the tangent bundle τ and the normal bundle ν to S^{n-1} in $\mathbb{R}^n$ is the trivial bundle $S^{n-1} \times \mathbb{R}^n$. For the elements of the direct sum $\tau \oplus \nu$ are triples $(x,v,tx) \in S^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n$ with $x \perp v$, and the map

$$(x, v, tx) \mapsto (x, v + tx)$$

gives an isomorphism of $\tau \oplus \nu$ with $S^{n-1} \times \mathbb{R}^n$. Since the normal bundle ν is trivial, this shows that τ is stably trivial.

But there are also examples where both bundle are nontrivial whereas their Whitney sum is trivial.

Example 5.10. Let γ^1_n be the canonical line bundle on $\mathbb{R}\mathrm{P}^n$. Then the map $(\ell, v, w) \mapsto (\ell, v + w)$ for $v \in \ell$ and $w \perp \ell$ defines an isomorphism $\gamma^1_n \oplus (\gamma^1_n)^\perp \cong \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}$.

Example 5.11. Specializing the previous example to the case $n = 1$, we see that

$$\gamma_1^1 \oplus (\gamma_1^1)^\perp \cong \mathbb{R}\mathrm{P}^1 \times \mathbb{R}^2 \cong S^1 \times \mathbb{R}^2.$$

The map that rotates a vector by 90 degrees defines an isomorphism between $(\gamma_1^1)^\perp$ and γ_1^1. Since γ_1^1 is isomorphic to the Möbius bundle over S^1, this shows that the direct sum of the Möbius bundle with itself is the trivial bundle.

5.4. **Oriented bundles.** We start with a first working definition of orientation of a vector bundle. Later we will discuss orientations in a more general context and relate it elements in the cohomology groups of the total space.

Recall that an *orientation* of a real vector space V of dimension $n > 0$ is an equivalence class of bases, where two ordered bases $v_1, \dots, v_n$ and $v'_1, \dots, v'_n$ are said to be equivalent if and only if the matrix (a_{ij}) defined by the equation

$$v'_i = \sum a_{ij} v_j$$

has positive determinant. Evidently every such vector space V has precisely two distinct orientations.

Example 5.12. The vector space $\mathbb{R}^n$ has a canonical orientation corresponding to its canonical ordered basis.

Definition 5.13. Let ξ be a real vector bundle given by the map $\pi\colon E \to B$. An *orientation* of ξ is a function assigning an orientation to each fiber in such a way that near each point of B there is a local trivialization $h\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$ carrying the canonical orientation of $\mathbb{R}^n$ in the fibers of $U \times \mathbb{R}^n$ to the orientations of the fibers in $\pi^{-1}(U)$.

An oriented vector bundle ξ is a real vector bundle together with a choice of orientation.

Note: Not all bundles can be oriented.

Example 5.14. a) Every trivial bundle is orientable. Hence the existence of an orientation is a necessary condition for triviality.

b) The Möbius bundle is not orientable.

We will see in the next lecture that the Stiefel-Whitney class measures exactly if a bundle is orientable or not.