Math 231b
Lecture 08

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8. Lecture 8: Existence and uniqueness of Stiefel-Whitney classes I

Before we show that Stiefel-Whitney classes with the described properties actually exist we are going to see another interesting application of Stiefel-Whitney classes.

8.1. Immersions of projective spaces into Euclidean space.

Stiefel-Whitney classes also help us decide whether a manifold can be immersed into a Euclidean space. For if an n-dimensional manifold M can be immersed into $\mathbb{R}^{n+k}$ then

$$1 = w(\tau_{\mathbb{R}^{n+k}}) = w(\nu \oplus \tau_M)$$

where ν denotes the normal bundle of the embedding $M \subset \mathbb{R}^{n+k}$. Hence by the Whitney product formula

$$w_i(\nu) = \bar{w}_i(M)$$

where $\bar{w}_i(M)$ denotes the ith component of the multiplicative inverse of the total Stiefel-Whitney class $w(M)$. Since ν is a k-dimensional bundle, this shows

$$\bar{w}_i(M) = 0 \text{ for } i > k.$$

Example 8.1. A typical example is the real projective space $\mathbb{P}^9$. By our calculations in the previous lecture we know

$$w(\mathbb{P}^9) = (1+a)^{10} = 1 + \sum_{i=1}^{9} \binom{10}{i} a^i = 1 + a^2 + a^8$$

since all other terms have an even coefficient. As a multiplicative inverse we get

$$\bar{w}(\mathbb{P}^9) = 1 + a^2 + a^4 + a^6,$$

for

$$\begin{aligned} & (1+a^2+a^8)(1+a^2+a^4+a^6) \\ = \; & 1 + a^2 + a^4 + a^6 + a^2 + a^4 + a^6 + a^8 + a^8 + a^{10} + a^{12} + a^{14} \\ = \; & 1 + 2a^2 + 2a^4 + 2a^6 + 2a^8 \\ = \; & 1. \end{aligned}$$

Since $\bar{w}_6(\mathbb{P}^9) \neq 0$, k must be at least 6 if $\mathbb{P}^9$ can be immersed into $\mathbb{R}^{9+k}$.

If $n = 2^r$ is a power of 2, then

$$w(\mathbb{P}^n) = (1+a)^{2^r+1} = (1+a^n)(1+a) = 1 + a + a^n$$

and

$$\bar{w}(\mathbb{P}^n) = 1 + a + a^2 + \ldots + a^{n-1}$$

since

$$\begin{aligned} & (1+a+a^{2^r})(1+a+\ldots+a^{n-1}) \\ = \;& 1 + a + \ldots + a^{n-1} + a + a^2 + \ldots + a^n + a^n \\ = \;& 1 + 2(a + a^2 + \ldots + a^n) \\ = \;& 1. \end{aligned}$$

Together with the previous argument we get the following classical result.

Theorem 8.2. *If $\mathbb{P}^{2^r}$ can be immersed in $\mathbb{R}^{2^r+k}$, then k must be at least $2^r - 1$.*

Example 8.3. Since the theorem tells us that $\mathbb{P}^8$ cannot be immersed in $\mathbb{R}^{14}$, it follows that $\mathbb{P}^9$ cannot be immersed in $\mathbb{R}^{14}$ either. This gives another proof that the minimal dimension of a Euclidean space in which $\mathbb{P}^9$ can be immersed is 15.

8.2. **Existence of Stiefel-Whitney classes.** We still need to show that there cohomology classes that satisfy the axioms of Stiefel-Whitney classes.

Theorem 8.4. *There is a unique sequence of functions $w_1, w_2, \ldots$ assigning to each real vector bundle $E \to B$ over a a space B a class $w_i(E) \in H^i(B; \mathbb{Z}/2)$, depending only on the isomorphism type of E, such that*
a) $w_i(f^*E) = f^*(w_i(E))$ for a pullback along a map $f\colon B' \to B$ which is covered by a bundle map.
b) $w(E_1 \oplus E_2) = w(E_1)w(E_2)$ where $w = 1 + w_1 + w_2 + \ldots \in H^*(B; \mathbb{Z}/2)$.
c) $w_i(E) = 0$ if $i > \dim E$.
d) For the canonical line bundle γ_1^1 on $\mathbb{P}^1$, $w_1(\gamma_1^1)$ is non-zero.

There are different methods to prove this theorem. We will prove it using the following fundamental result of Leray and Hirsch on the cohomology of a fiber bundle. Roughly speaking, a fiber bundle is the same thing as a vector bundle except that we replace $\mathbb{R}^n$ by any topological space F.

Let $p\colon E \to B$ be a fiber bundle with fiber F. Then we can make $H^*(E; \mathbb{Z}/2)$ into a module over the ring $H^*(B; \mathbb{Z}/2)$ by setting $\alpha\beta = p^*(\alpha)\beta$ for $\alpha \in H^*(B; \mathbb{Z}/2)$ and $\beta \in H^*(E; \mathbb{Z}/2)$. The Leray-Hirsch theorem then tells us that $H^*(E; \mathbb{Z}/2)$ is a free $H^*(B; \mathbb{Z}/2)$-module provided that for each fiber F the inclusion $\iota\colon F \hookrightarrow E$ induces a surjection on $H^*(F; \mathbb{Z}/2)$ and $H^n(F; \mathbb{Z}/2)$ is a finite dimensional $\mathbb{Z}/2$-vector space for each n. A basis for $H^*(E; \mathbb{Z}/2)$ as a $H^*(B; \mathbb{Z}/2)$-module can be chosen as any set of elements in $H^*(E; \mathbb{Z}/2)$ that map to a basis in $H^*(F; \mathbb{Z}/2)$

under ι^*.

The precise statement of the Leray-Hirsch theorem is:

Theorem 8.5. *Let $F \xrightarrow{\iota} E \xrightarrow{p} B$ be a fiber bundle such that, for some commutative ring R:*
a) $H^n(F;R)$ is a finitely generated free R-module for each n;
b) there exist classes $c_j \in H^{k_j}(E;R)$ whose restrictions $\iota^*(c_j)$ form a basis for $H^*(F;R)$ in each fiber F.
Then the map $\varphi\colon H^*(B;R)\otimes_R H^*(F;R) \to H^*(E;R)$, $\sum_{ij} b_i \otimes \iota^*(x_j) \mapsto \sum_{ij} p^*(b_i)x_j$, is an isomorphism.

Now let us prove Theorem 8.4. For simplicity, we will assume that the base base is paracompact.

Let ξ be a vector bundle of dimension n given by the map $\pi\colon E \to B$. It comes along with a projective bundle $\mathbb{P}(\xi)$ given by the induced map $\mathbb{P}(\pi)\colon \mathbb{P}(E) \to B$. It is a fiber bundle whose fiber at b in B are the spaces of all lines through the origin in the fiber $E_b(\xi)$. The map $\mathbb{P}(\pi)$ is the natural projection sending each line in $\pi^{-1}(b)$ to b. We topologize $\mathbb{P}(E)$ as a quotient of the complement of the zero section of E modulo scalar multiplication in each fiber. Over a neighborhood U in B where E is a product $U \times \mathbb{R}^n$, this quotient is $U \times \mathbb{P}^{n-1}$. Hence $\mathbb{P}(\xi)$ is a fiber bundle over B with fiber $\mathbb{P}^{n-1}$.

Now we would like to apply the Leray-Hirsch theorem to the fiber bundle $\mathbb{P}(\xi)$. Therefore we need classes $x_i \in H^i(\mathbb{P}(E);\mathbb{Z}/2)$ restricting to generators of $H^i(\mathbb{P}^{n-1};\mathbb{Z}/2)$ in each fiber $\mathbb{P}^{n-1}$ for $i = 0, \ldots, n-1$.

We will use the following lemma.

Lemma 8.6. *There is a map $g\colon E \to \mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$ that is a linear injection on each fiber. Any two such maps are homotopic through maps that are linear injections on fibers.*

Proof. Since B is paracompact there is a countable open cover U_j of B such that E is trivial over each U_j and there is a partition of unity $\{\varphi_j\}$ with φ_j supported on U_j. Let $g_j\colon \pi^{-1}(U_j) \to \mathbb{R}^n$ be the composition of a trivialization $\pi^{-1}(U_j) \to U_j \times \mathbb{R}^n$ with the projection onto $\mathbb{R}^n$. The map

$$(\varphi_j\pi)g_j\colon \pi^{-1}(U_j) \to \mathbb{R}^n,\ v \mapsto \varphi_j(\pi(v))g_j(v)$$

extends to a map $E \to \mathbb{R}^n$ that is zero outside $\pi_{-1}(U_j)$. Near each point of B only finitely many φ_j's are nonzero, and at least one φ_j is nonzero. Hence these extended maps $(\varphi_j\pi)g_j$ are the coordinates of a map $g\colon E \to (\mathbb{R}^n)^\infty = \mathbb{R}^\infty$ that is a linear injection on each fiber.

Now let g_0 and g_1 be two such maps that are linear injections on fibers. Then let L_t be the homotopy

$$L_t\colon \mathbb{R}^\infty \to \mathbb{R}^\infty,\ L_t(x_1, x_2, \ldots) = (1-t)(x_1, x_2, \ldots) + t(x_1,0,x_2,0,\ldots).$$

For each t, this is a linear map whose kernel is easily computed to be 0. Hence L_t is injective. Composing L_t with g_0 moves the image of g_0 into the odd-numbered coordinates. Similarly, we can move the image of g_1 into the even-numbered coordinates. By abuse of notation we denote the resulting shifted maps still by g_0 and g_1 respectively. Then we set

$$g_t = (1-t)g_0 + tg_1.$$

This is a linear map which is injective on fibers for each t since g_0 and g_1 are linear and injective on fibers. □

Given the linear injection g of the lemma, we can projectivize it by deleting zero vectors and then take the quotient by scalar multiplication. This gives us a map $\mathbb{P}(g)\colon \mathbb{P}(E) \to \mathbb{P}^\infty$. Let y be a generator of $H^1(\mathbb{P}^\infty;\mathbb{Z}/2)$ and let $x = \mathbb{P}(g)^*(y) \in H^1(\mathbb{P}(E);\mathbb{Z}/2)$. Then the powers $x_i := x^i \in H^i(\mathbb{P}(E);\mathbb{Z}/2)$ for $i = 0,\ldots,n-1$ are the desired classes since a linear injection $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty$ induces an embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^\infty$ for which y pulls back to a generator of $H^1(\mathbb{P}^{n-1};\mathbb{Z}/2)$ (because the classes are nonzero).

Note that the classes x^i do not depend on the choice of g. For any two linear injections $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty$ are homotopic through linear injections, so the induced embeddings $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^\infty$ of different fibers of $\mathbb{P}(E)$ are all homotopic. The second assertion of the lemma then implies the claim.

Hence, by the Leray-Hirsch theorem, $H^*(\mathbb{P}(E);\mathbb{Z}/2)$ is a free $H^*(B;\mathbb{Z}/2)$-module with basis $1, x, \ldots, x^{n-1}$. Consequently, x^n can be expressed uniquely as a linear combination of these basis elements with coefficients in $H^*(B;\mathbb{Z}/2)$. Thus there is a unique relation of the form

$$x^n + w_1(E)x^{n-1} + \ldots + w_n(E) = 0$$

for certain classes $w_i(E) \in H^i(B;\mathbb{Z}/2)$. Together with the convention $w_i(E) = 0$ for $i > n$ and $w_0(E) = 1$ this is our definition of the Stiefel-Whitney classes of E. It remains to show that these classes satisfy the desired properties.