Math 231b
Lecture 09

G. Quick

9. Lecture 9: Existence and uniqueness of Stiefel-Whitney classes II

We continue the proof of the following theorem that shows that there exist unique Stiefel-Whitney classes.

Theorem 9.1. *There is a unique sequence of functions $w_1, w_2, \ldots$ assigning to each real vector bundle $E \to B$ over a a space B a class $w_i(E) \in H^i(B; \mathbb{Z}/2)$, depending only on the isomorphism type of E, such that*
a) $w_i(f^*E) = f^*(w_i(E))$ for a pullback along a map $f \colon B' \to B$ which is covered by a bundle map.
b) $w(E_1 \oplus E_2) = w(E_1)w(E_2)$ where $w = 1 + w_1 + w_2 + \ldots \in H^*(B; \mathbb{Z}/2)$.
c) $w_i(E) = 0$ if $i > \dim E$.
d) For the canonical line bundle γ_1^1 on $\mathbb{P}^1$, $w_1(\gamma_1^1)$ is non-zero.

9.1. Existence of Stiefel-Whitney classes.

In the previous lecture we defined the Stiefel-Whitney classes $w_i(E)$ for any vector bundle $\pi \colon E \to B$. Recall that for simplicity we assume that the base space B is paracompact. The idea was the following.

Our bundle induces a map $g \colon E \to \mathbb{R}^\infty$ which is linear and injective on each fiber. We can projectivize it by deleting zero vectors and then take the quotient by scalar multiplication. This gives us a map $\mathbb{P}(g) \colon \mathbb{P}(E) \to \mathbb{P}^\infty$. Let y be a generator of $H^1(\mathbb{P}^\infty; \mathbb{Z}/2)$ and let $x = \mathbb{P}(g)^*(y) \in H^1(\mathbb{P}(E); \mathbb{Z}/2)$. Then the powers $x_i := x^i \in H^i(\mathbb{P}(E); \mathbb{Z}/2)$ for $i = 0, \ldots, n-1$ are the desired classes since a linear injection $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty$ induces an embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^\infty$ for which y pulls back to a generator of $H^1(\mathbb{P}^{n-1}; \mathbb{Z}/2)$ (because the classes are nonzero).

Note that the classes x^i do not depend on the choice of g. For any two linear injections $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty$ are homotopic through linear injections, so the induced embeddings $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^\infty$ of different fibers of $\mathbb{P}(E)$ are all homotopic. The second assertion of the lemma then implies the claim.

Hence, by the Leray-Hirsch theorem, $H^*(\mathbb{P}(E); \mathbb{Z}/2)$ is a free $H^*(B; \mathbb{Z}/2)$-module with basis $1, x, \ldots, x^{n-1}$. Consequently, x^n can be expressed uniquely as a linear combination of these basis elements with coefficients in $H^*(B;\mathbb{Z}/2)$. Thus there is a unique relation of the form

$$x^n + w_1(E)x^{n-1} + \ldots + w_n(E) = 0$$

for certain classes $w_i(E) \in H^i(B;\mathbb{Z}/2)$. Together with the convention $w_i(E) = 0$ for $i > n$ and $w_0(E) = 1$ this is our definition of the Stiefel-Whitney classes of E. It remains to show that these classes satisfy the desired properties.

a) Consider a pullback bundle $f^*E = E'$:

$$\begin{array}{ccc} E' & \xrightarrow{f'} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

If $g\colon E \to \mathbb{R}^\infty$ is a map that is a linear injection on fibers then so is gf'. It follows that $\mathbb{P}(f')^*$ takes the canonical class $x = x(E)$ in $H^1(\mathbb{P}(E);\mathbb{Z}/2)$ to the canonical class $x(E')$ in $H^1(\mathbb{P}(E');\mathbb{Z}/2)$. Then

$$\begin{aligned} \mathbb{P}(f')^*(\textstyle\sum_i \mathbb{P}(\pi)^*(w_i(E)) \cdot x(E)^{n-i}) &= \textstyle\sum_i [\mathbb{P}(f')^* \circ \mathbb{P}(\pi)^*(w_i(E))] \cdot [\mathbb{P}(f')^*(x(E)^{n-i})] \\ &= \textstyle\sum_i \mathbb{P}(\pi')^* \circ f^*(w_i(E) \cdot x(E'))^{n-i} \end{aligned}$$

in $H^*(E';\mathbb{Z}/2)$. This shows that the relation

$$x(E)^n + w_1(E)x(E)^{n-1} + \ldots + w_n(E) = 0 \text{ defining } w_i(E)$$

pulls back to the relation

$$x(E')^n + f^*w_1(E)x(E')^{n-1} + \ldots + f^*w_n(E) = 0 \text{ defining } w_i(E').$$

By the uniqueness of this relation in the free $H^*(B;\mathbb{Z}/2)$-module $H^*(E;\mathbb{Z}/2)$, we get $w_i(E') = f^*(w_i(E))$.

b) The inclusions of E_1 and E_2 into $E_1 \oplus E_2$ give inclusions of $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ into $\mathbb{P}(E_1 \oplus E_2)$ with $\mathbb{P}(E_1) \cap \mathbb{P}(E_2) = \emptyset$. Let $U_1 = \mathbb{P}(E_1 \oplus E_2) - \mathbb{P}(E_1)$ and $U_2 = \mathbb{P}(E_1 \oplus E_2) - \mathbb{P}(E_2)$. These are open sets in $\mathbb{P}(E_1 \oplus E_2)$ which cover $\mathbb{P}(E_1 \oplus E_2)$ and that deformation retract onto $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ respectively. This means that the inclusions $\mathbb{P}(E_1) \hookrightarrow U_2$ and $\mathbb{P}(E_2) \hookrightarrow U_1$ are homotopy equivalences.

A map $g\colon E_1 \oplus E_2 \to \mathbb{R}^\infty$ which is a linear injection on fibers restricts to such a map on E_1 and E_2. By the way we constructed the canonical classes, this implies that the canonical class $x \in H^1(\mathbb{P}(E_1 \oplus E_2;\mathbb{Z}/2)$ for $E_1 \oplus E_2$ restricts to the canonical classes for E_1 and E_2.

If E_1 and E_2 have dimensions m and n, we consider the classes

$$\omega_1 = \sum_j w_j(E_1)x^{m-j} \text{ and } \omega_2 = \sum_j w_j(E_2)x^{n-j} \text{ in } H^*(\mathbb{P}(E_1 \oplus E_2);\mathbb{Z}/2).$$

Their cup product is

$$\omega_1 \cdot \omega_2 = \sum_j [\sum_{r+s=j} w_r(E_1)w_r(E_2)]x^{m+n-j}.$$

By the definition of the classes $w_j(E_1)$, the class ω_1 restricts to zero in $H^m(\mathbb{P}(E_1);\mathbb{Z}/2)$. Hence ω_1 pulls back to a class in the relative group

$$H^m(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_1);\mathbb{Z}/2) \cong H^m(\mathbb{P}(E_1 \oplus E_2), U_2;\mathbb{Z}/2).$$

and ω_2 pulls back to a class in the relative group

$$H^n(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_2);\mathbb{Z}/2) \cong H^n(\mathbb{P}(E_1 \oplus E_2), U_1;\mathbb{Z}/2).$$

The following commutative diagram then shows that $\omega_1 \cdot \omega_2 = 0$:

$$\begin{array}{ccc}
H^m(\mathbb{P}(E_1 \oplus E_2), U_2;\mathbb{Z}/2) \times H^n(\mathbb{P}(E_1 \oplus E_2), U_1;\mathbb{Z}/2) & \longrightarrow & H^{m+n}(\mathbb{P}(E_1 \oplus E_2), U_1 \cup U_2;\mathbb{Z}/2) = 0 \\
\downarrow & & \downarrow \\
H^m(\mathbb{P}(E_1 \oplus E_2);\mathbb{Z}/2) \times H^n(\mathbb{P}(E_1 \oplus E_2);\mathbb{Z}/2) & \longrightarrow & H^{m+n}(\mathbb{P}(E_1 \oplus E_2);\mathbb{Z}/2).
\end{array}$$

This shows that

$$\omega_1 \cdot \omega_2 = \sum_j [\sum_{r+s=j} w_r(E_1)w_r(E_2)]x^{m+n-j} = 0$$

is the defining relation for the Stiefel-Whitney classes of $E_1 \oplus E_2$. Thus

$$w_j(E_1 \oplus E_2) = \sum_{r+s=j} w_r(E_1)w_r(E_2).$$

c) holds by definition.

d) Recall that the canonical line bundle γ^1 on $\mathbb{P}^\infty$ is given by

$$E(\gamma^1) = \{(\ell,v) \in \mathbb{P}^\infty \times \mathbb{R}^\infty | v \in \ell\}.$$

The map $\mathbb{P}(\pi)$ is the identity in this case, i.e. γ^1 is equal to its own projective bundle. The map $g\colon E \to \mathbb{R}^\infty$ which is a linear injection on fibers can be taken to be

$$g(\ell,v) = v.$$

So $\mathbb{P}(g)$ is also the identity and $x(E)$ is a generator of $H^1(\mathbb{P}^\infty;\mathbb{Z}/2)$ and restricts to the generator in $H^1(\mathbb{P}^1;\mathbb{Z}/2)$. This proves the existence of Stiefel-Whitney classes.

9.2. **Uniqueness.** To show the uniqueness we will use an important property of vector bundles, the *splitting principle*:

Proposition 9.2. *For each vector bundle* $\pi\colon E \to B$ *there is a space* $F(E)$ *and a map* $P\colon F(E) \to B$ *such that the pullback* $p^*(E) \to F(E)$ *splits as a direct sum of line bundles, and* $p^*\colon H^*(B;\mathbb{Z}/2) \to H^*(F(E);\mathbb{Z}/2)$ *is injective.*

Now we can finish the proof of Theorem 9.1 and show the uniqueness of Stiefel-Whitney classes. Property d) determines $w_1(\gamma^1)$ for the canonical line bundle $\gamma^1 \to \mathbb{P}^\infty$. Property c) then determines all the $w_i(\gamma^1)$'s. We will now use the following property of the line bundle γ^1.

Remark 9.3. The canonical line bundle γ^1 on $\mathbb{P}^\infty$ is the universal line bundle in the following sense. Given a line bundle ξ, then there is a bundle map $f\colon \xi \to \gamma^1$ which is unique up to homotopy. For let ξ be given by a map $\pi\colon E \to B$. We have seen in the previous lecture that we can find a map $g\colon E \to \mathbb{R}^\infty$ that is linear and injective on fibers. Then we can define f by

$$f(e) = (g(\text{fiber through } e), g(e)) \in \gamma^1.$$

Using the universality of γ^1, we see that property a) therefore determines the classes w_i for all line bundles. Property b) extends this to sums of line bundles. Finally, the splitting principle implies that the $w_i's$ are determined for all bundles.