# Math 231b
# Lecture 10

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## 10. Lecture 10: Splitting principle and the projective bundle formula

There are two leftovers from the proof of the existence and uniqueness of Stiefel-Whitney classes. One is the splitting principle, the other one is the Leray-Hirsch theorem.

### 10.1. The splitting principle.

**Proposition 10.1.** *For each vector bundle $\pi\colon E \to B$ there is a space $F(E)$ and a map $p\colon F(E) \to B$ such that the pullback $p^*(E) \to F(E)$ splits as a direct sum of line bundles, and $p^*\colon H^*(B;\mathbb{Z}/2) \to H^*(F(E);\mathbb{Z}/2)$ is injective.*

*Proof.* Consider the pullback $\mathbb{P}(\pi)^*(E)$ of $E$ via the map $\mathbb{P}(\pi)\colon \mathbb{P}(E) \to B$. This pullback contains a natural one-dimensional sub-bundle

$$L = \{(\ell,v) \in \mathbb{P}(E) \times E | v \in \ell\}.$$

Assuming $B$ is paracompact (although this holds for any $B$) we can equip $E$ with an inner product. This inner product pulls back to an inner product on $\mathbb{P}(\pi)^*(E)$. Hence we get a splitting of the pullback as a sum $L \oplus L^\perp$. The orthogonal bundle $L^\perp$ now has dimension less than $E$. By the Leray-Hirsch theorem we know $H^*(\mathbb{P}(E);\mathbb{Z}/2)$ is the free $H^*(B;\mathbb{Z}/2)$-module with basis $1, x, \ldots, x^{n-1}$. In particular, the induced map

$$H^*(B;\mathbb{Z}/2) \to H^*(\mathbb{P}(E);\mathbb{Z}/2)$$

is injective since one of the basis elements is 1.
Now we can repeat this construction for the bundle $L^\perp \to \mathbb{P}(E)$ instead of $E \to B$. After finitely many steps we obtain the desired result. □

**Remark 10.2.** We can describe $F(E)$ as follows. The complement $L^\perp$ consist of pairs $(\ell,v) \in \mathbb{P}(E) \times E$ with $v \perp \ell$. At the next stage we construct $\mathbb{P}(L^\perp)$, whose points are pairs $(\ell,\ell')$ where $\ell$ and $\ell'$ are orthogonal lines in $E$. Continuing this way, we see that the final space $F(E)$ is the space of all orthogonal splittings $\ell_1 \oplus \ldots \oplus \ell_n$ of fibers of $E$ as sums of lines, and the vector bundle over $F(E)$ consists of all $n$-tuples of vectors in these lines.

In the previous proof we used the following result.

**Proposition 10.3.** *Let $B$ be a paracompact space and $\xi$ a real vector bundle given by the map $\pi\colon E \to B$. Then $\xi$ can be given the structure of a Euclidean vector bundle.*

*Proof.* See problem set 1. □

10.2. **The Leray-Hirsch theorem.** The precise statement of the Leray-Hirsch theorem is:

**Theorem 10.4.** *Let $F \xrightarrow{\iota} E \xrightarrow{p} B$ be a fiber bundle such that, for a principal ideal ring $R$:*
*a) $H^n(F;R)$ is a finitely generated free $R$-module for each $n$;*
*b) there exist classes $c_j \in H^{k_j}(E;R)$ for $j = 1,\ldots,r$ whose restrictions $\iota^*(c_j)$ form a basis for the $R$-module $\oplus_n H^n(F;R)$ in each fiber $F$.*
*Then the map $\varphi\colon H^*(B;R)\otimes_R H^*(F;R) \to H^*(E;R)$, $\sum_{ij} b_i \otimes \iota^*(c_j) \mapsto \sum_{ij} p^*(b_i)c_j$, is an isomorphism.*

**Remark 10.5.** 1. Note that the theorem makes only an assertion on the structure of $H^*(E;R)$ as an $H^*(B;R)$-module. It does not specify the ring structure of $H^*(E;R)$. In fact, there are examples where the map

$$\varphi\colon H^*(B;R) \otimes_R H^*(F;R) \to H^*(E;R)$$

is not a ring isomorphism.

2. An example of a fiber bundle where the assertion of the theorem does not hold is the Hopf bundle

$$S^1 \to S^3 \xrightarrow{f} S^2.$$

(Recall that $f$ can be defined as $f\colon S^3 \to \mathbb{CP}^1 = S^2$, viewing $S^3$ as the unit sphere in the complex plane $\mathbb{C}^2$. Such an $f$ is the attaching map in the complex projective plane $\mathbb{CP}^2 = S^2 \cup_f e^4$ where $e^4$ is a disk of dimension 4.)

We know that $H^*(S^3;R)$ is not isomorphic to $H^*(S^2;R) \otimes_R H^*(S^1;R)$. For we have

$$H^1(S^3;R) = 0 \text{ but } H^0(S^2;R) \otimes_R H^1(S^1;R) \cong R.$$

The assumptions of the theorem require that the map $\iota^*\colon H^*(E;R) \to H^*(F;R)$ is surjective in each degree. This is obviously not the case for the Hopf bundle.

**Sketch of a proof of Theorem 10.4 for compact base spaces**:

Throughout the proof we write $H^*(X)$ for $H^*(X;R)$. We only sketch a proof for the case that $B$ is compact, though the theorem holds for arbitrary base spaces.

Let $U$be an open subset of $B$ such that there is a homeomorphism

$$h\colon E_U := \pi^{-1}(U) \to U \times F.$$

Let $j_U\colon E_U \hookrightarrow E$ be the natural inclusion and $\pi_U$ be the restriction of $\pi$ to $U$. Then the Künneth Theorem says that the map $\pi_{U*}\colon H^*(U) \to H^*(E_U)$ is injective and the elements $j_U^*(c_1), \ldots, j_U^*(c_r)$ form a basis of the $H^*(U)$-module $H^*(E_U)$.

Now assume that the theorem is true over the open subsets $U$, $V$ and $U \cap V$. We want to show that it is also true over $U \cup V$. Therefore we introduce two functors $K^n(W)$ and $L^n(W)$ on the open subsets $W$ of $B$ as follows. Let $t_j$ be an indeterminant of degree $k_j$. (The $t_j$ have no real meaning, they are just useful to define something else.) We set

$$K^n(W) := \sum_{j=1}^{r} H^{n-k_j}(W)t_j, \text{ and } L^n(W) := H^n(E_W).$$

For every $W$ we have the homomorphism

$$\theta_W\colon K^n(W) \to L^n(W),\ \sum_j x_j t_j \mapsto \sum_j \pi^*(x_j)c_j.$$

Then we convince ourselves that the theorem is true over $W$ if and only if $\theta_W$ is an isomorphism.

The functor $W \mapsto L^n(W)$ is just the restriction of a functor which satisfies the Mayer-Vietoris property. The functor $W \mapsto K^n(W)$ is a direct sum of functors which satisfy the Mayer-Vietoris property. Hence we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
K^{n-1}(U) \oplus K^{n-1}(V) & \longrightarrow & K^{n-1}(U \cap V) & \longrightarrow & K^n(U \cup V) & \longrightarrow & K^n(U) \oplus K^n(V) & \longrightarrow & K^n(U \cap V) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \theta_{U\cup V}} & & \downarrow & & \downarrow \\
L^{n-1}(U) \oplus L^{n-1}(V) & \longrightarrow & L^{n-1}(U \cap V) & \longrightarrow & L^n(U \cup V) & \longrightarrow & L^n(U) \oplus L^n(V) & \longrightarrow & L^n(U \cap V)
\end{array}$$

By our assumption the theorem is true for $U$, $V$ and $U \cap V$ and hence the four unlabeled vertical maps are isomorphisms. By the 5-Lemma, the map $\theta_{U\cup V}$ is thus an isomorphism too. Hence the theorem also holds over $U \cup V$.

Now it remains to cover $B$ by finitely many open sets $B = U_1 \cup \ldots \cup U_n$ such that our bundle becomes trivial over each $U_i$. This completes the proof for a compact base space.

More sophisticated arguments using the Serre spectral sequence associated to the fibration sequence $F \xrightarrow{\iota} E \xrightarrow{p} B$ also prove the general case. A more elementary proof of the general statement can be found in Hatcher's book (Theorem 4D.1).