Math 231b
Lecture 11

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11. Lecture 11: The Grassmannian manifold and the universal bundle

11.1. **Representability of** $\mathrm{Vect}^k(B)$**.** In the previous lecture we used the fact that the canonical line bundle γ^1 over the (infinite) real projective space is universal among all line bundles in the sense that given a line bundle ξ there is a bundle map $\xi \to \gamma^1$ which is unique up to homotopy. This bundle map comes equipped with a homotopy class of maps $B \to \mathbb{P}^\infty$ where B denotes the base space of ξ. In fact, there is a bijection

$$\mathrm{Vect}^1(B) \cong [B, \mathbb{P}^\infty]$$

between the set of isomorphism classes of real line bundles over B and the set of homotopy classes of maps $B \to \mathbb{P}^\infty$.

We still need to prove this statement. In fact, we would like to show a generalization to k-dimensional bundles. For each k there is a real manifold, called the Grassmannian manifold and denoted by Gr_k, with a k-dimensional real vector bundle γ^k on Gr_k such that for any paracompact base space B the set of isomorphism classes of k-dimensional bundles over B is in bijection with the set of homotopy classes of maps $B \to \mathrm{Gr}_k$:

$$\mathrm{Vect}^k(B) \cong [B, \mathrm{Gr}_k].$$

The bundle γ^k is called the universal k-dimensional vector bundle.

11.2. **The Grassmannian.** The Grassmannian manifold $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is the space of k-planes in $\mathbb{R}^{n+k}$. It can be identified with the quotient of the Stiefel manifold $V_k(\mathbb{R}^{n+k})$ of orthonormal sequences

$$[v_1, \ldots, v_k]$$

of vectors $v_i \in \mathbb{R}^{n+k}$, modulo the equivalence relation

$$[v_1, \ldots, v_k] \sim [v_1, \ldots, v_k] \cdot T,$$

with T any orthogonal $k \times k$-matrix.

Remark 11.1. The topology of the Stiefel manifold is given as follows. We can consider $V_k(\mathbb{R}^{n+k})$ as a subspace of the product $\mathbb{R}^{n+k} \times \ldots \times \mathbb{R}^{n+k}$ of k copies of $\mathbb{R}^{n+k}$. More precisely, $V_k(\mathbb{R}^{n+k})$ is the subspace of $S^{n+k-1} \times \ldots \times S^{n+k-1}$ of k copies of spheres S^{n+k-1} given by all orthogonal k-tuples. It is a closed subspace since orthogonality of two vectors can be expressed by an algebraic equation. In particular, $V_k(\mathbb{R}^{n+k})$ is compact, since the product of spheres is compact.

Now there is a natural surjective map

$$V_k(\mathbb{R}^{n+k}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k})$$

sending an orthonormal sequence to the subspace it spans. We equip $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ with the quotient topology with respect to this surjection. In particular, $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is compact as well.

Example 11.2. We already know one example of a Grassmannian. The Grassmannian $\mathrm{Gr}_1(\mathbb{R}^{n+1})$ is just $\mathbb{P}^n$, and the presentation described above is just the representation of $\mathbb{P}^n$ as the quotient space of S^n by the antipodal action.

Proposition 11.3. *The space* $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ *is a manifold of dimension* $k \cdot n$.

Proof. Let $V \subset \mathbb{R}^{n+k}$ be a k-plane, and let W be the orthogonal complement of V. Then the subspace $U \subset \mathrm{Gr}_k(\mathbb{R}^{n+k})$ consisting of k-planes V' with the property that $V' \cap W = \{0\}$ is an open neighborhood of V.

Note: To see that U is open it suffices to show that its preimage $\tilde{U}$ in $V_k(\mathbb{R}^{n+k})$ is open. The set $\tilde{U}$ consists of all orthonormal frames $[v_1, \ldots, v_k]$ such that the $p(v_1), \ldots, p(v_k)$ are linearly independent where p is the projection

$$p\colon \mathbb{R}^{n+k} \to V.$$

Writing M for the $k \times k$-matrix with column vectors $p(v_1), \ldots, p(v_k)$ we see that $\tilde{U}$ consists of all frames such that the resulting M has non-zero determinant. Hence $\tilde{U}$ is an open subset.

Thinking of $V' \in U$ as the graph of a linear map $V \to W$, gives a bijection

$$T\colon U \to \mathrm{Hom}(V{,}W)$$

of U with $\mathrm{Hom}(V, W)$, which is a real vector space of dimension $k \cdot n$.

The correspondence T is in fact a homeomorphism. For let

$$p\colon V \oplus W \to V$$

be the orthogonal projection and let $x_1, \ldots, x_n$ be a fixed orthonormal basis for V. Then each V' in U has a unique basis $y_1, \ldots, y_n$ such that

$$p(y_1) = x_1, \ldots, p(y_n) = x_n.$$

The orthonormal frame $[y_1, \ldots, y_n]$ depends continuously on V'. Moreover, the $y_1, \ldots, y_n$ satisfy the identity

$$y_i = x_i + T(V')x_i \tag{1}$$

by definition of T and the choice of the y_i's. Hence, since y_i depends continuously on V', it follows that the image $T(V')x_i \in W$ depends continuously on V'. Therefore the linear transformation depends continuously on V'.

On the other hand the identity (1) shows that the n-frame $[y_1, \ldots, y_n]$ depends continuously on $T(V')$, and hence that V' depends continuously on $T(V')$. Thus the function T^{-1} is also continuous and T is a homeomorphism. $\square$

The inclusions $\mathbb{R}^{n+k} \subset \mathbb{R}^{n+1+k} \subset \ldots$ induce inclusions

$$\mathrm{Gr}_k(\mathbb{R}^{n+k}) \subset \mathrm{Gr}_k(\mathbb{R}^{n+1+k}) \subset \ldots$$

The infinite Grassmannian manifold is the union

$$\mathrm{Gr}_k := \mathrm{Gr}(\mathbb{R}^\infty) = \bigcup_n \mathrm{Gr}_k(\mathbb{R}^{n+k}).$$

This is the set of all k-dimensional vector subspaces of $\mathbb{R}^\infty$. The topology of Gr_k is the direct limit topology, i.e., a subset of Gr_k is open (or closed) if and only if its intersection with $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is open (or closed) as a subset of $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ for each n.

Once again, Gr_1 is just the infinite real projective space $\mathbb{P}^\infty$.

11.3. **The canonical bundle.** The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is equipped with a canonical k-dimensional vector bundle $\gamma^k(\mathbb{R}^{n+k})$ defined as follows. Let

$$E = E(\gamma^k(\mathbb{R}^{n+k}))$$

be the set of all pairs

$$(k\text{-plane in } \mathbb{R}^{n+k},\ \text{vector in that } k\text{-plane}).$$

The topology on E is the topology as a subset of $\mathrm{Gr}_k(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$. The projection map

$$\pi \colon E \to \mathrm{Gr}_k(\mathbb{R}^{n+k}), \text{ is defined by } \pi(V, v) = V,$$

and the vector space structure is defined by

$$t_1(V, v_1) + t_2(V, v_2) = (V, t_1 v_1 + t_2 v_2).$$

Over the infinite Grassmannian Gr_k, there is also a canonical bundle γ^k whose total space is

$$E(\gamma^k) \subset \mathrm{Gr}_k \times \mathbb{R}^\infty$$

the set of all pairs

$$(k\text{-plane in } \mathbb{R}^\infty,\ \text{vector in that } k\text{-plane})$$

topologized as a subset of the product $\mathrm{Gr}_k \times \mathbb{R}^\infty$. The projection $\pi \colon E(\gamma^k) \to \mathrm{Gr}_k$ is given by $\pi(V, v) = V$.

Note that the bundles $\gamma^1(\mathbb{R}^{n+1})$ and γ^1 are of course just the bundles γ_n^1 on $\mathbb{P}^n$ and γ^1 on $\mathbb{P}^n$ respectively.

Lemma 11.4. *The just constructed bundles $\gamma^k(\mathbb{R}^{n+k})$ and γ^k satisfy the local triviality condition.*

Proof. We start with $\gamma^k(\mathbb{R}^{n+k})$. Let $V \subset \mathbb{R}^{n+k}$ be a k-plane, and let U be the open neighborhood of V constructed in the proof of Proposition 11.3. The coordinate homeomorphism

$$h\colon U \times \mathbb{R}^k \cong U \times V \to \pi^{-1}(U)$$

is defined as follows. Set $h(V', x) := (V', y)$ where y denotes the unique vector in V' which is carried into x by the orthogonal projection

$$p\colon \mathbb{R}^{n+k} \to V.$$

The identities

$$h(V', x) = (V', x + T(V')x) \text{ and } h^{-1}(V', y) = (V', p(y))$$

show that h and h^{-1} are continuous.

For γ^k it suffices to note that an open neighborhood U for a k-plane V in Gr_k is just the union of the neighborhoods of V in the $\mathrm{Gr}_k(\mathbb{R}^{n+k})$. Hence the coordinate homeomorphisms just constructed fit together to give a coordinate homeomorphism over U. The continuity follows from the fact that we use the direct limit topology on Gr_k. □

Our next goal is to prove the following fundamental result.

Theorem 11.5. *For a paracompact space B, the map $[B, \mathrm{Gr}_k] \to \mathrm{Vect}^k(B)$, $[f] \mapsto f^*(\gamma^k)$, is a bijection.*

Remark 11.6. The infinite Grassmannian Gr_k is called the *classifying space* and γ^k is called the *universal bundle* for k-dimensional real vector bundles.