Math 231b
Lecture 12

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12. Lecture 12: Representability of $\mathrm{Vect}^k(B)$

Our next goal is to prove the following fundamental result.

Theorem 12.1. *For a paracompact space B, the map $[B, \mathrm{Gr}_k] \to \mathrm{Vect}^k(B)$, $[f] \mapsto f^*(\gamma^k)$, is a bijection.*

Remark 12.2. The theorem justifies to call the infinite Grassmannian Gr_k is the *classifying space* and γ^k is the *universal bundle* for k-dimensional real vector bundles.

Example 12.3. Let τ be the tangent bundle to S^n in $\mathbb{R}^{n+1}$. It is given by the projection $p\colon E(\tau) \to S^n$ where

$$E(\tau) = \{(x,v) \in S^n \times \mathbb{R}^{n+1} | x \perp v\}.$$

Each fiber $p^{-1}(x)$ is an n-plane and hence defines a point in $\mathrm{Gr}_n(\mathbb{R}^{n+1})$. This defines a map

$$S^n \to \mathrm{Gr}_n(\mathbb{R}^{n+1}),\ x \mapsto p^{-1}(x).$$

Via the inclusion $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^\infty$ we can view this as a map

$$f\colon S^n \to \mathrm{Gr}_n(\mathbb{R}^\infty) = \mathrm{Gr}_n.$$

The bundle τ is exactly the pullback $f^*(\gamma^n)$. We check this on total spaces in the diagram

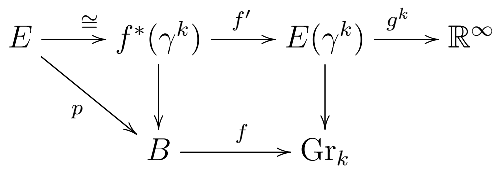
$$\begin{array}{ccc} E(\tau) \cong f^*(E(\gamma^n)) & \longrightarrow & E(\gamma^n) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ S^n & \xrightarrow{\ f\ } & \mathrm{Gr}_n. \end{array}$$

since we have

$$f^*(E(\gamma^n)) = \{(x,(V,v)) \in S^n \times E(\gamma^n) | f(x) = \pi(V,v)\} = \{(x,(V,v)) | p^{-1}(x) = V, \text{i.e. } x \perp v\}.$$

12.1. **Proof of Theorem 12.1.**

We first claim that, for a k-dimensional bundle $p\colon E = E(\xi) \to B$, an isomorphism $\xi \cong f^*(\gamma^k)$ is equivalent to a map $g\colon E \to \mathbb{R}^\infty$ which is linear and injective on each fiber. To prove this claim suppose we have a map $f\colon B \to \mathrm{Gr}_k$ and an isomorphism $\xi \cong f^*(\gamma^k)$. Then we have a commutative diagram



with $g^k(V,v) = v$. The composition along the top row is a map $g\colon E \to \mathbb{R}^\infty$ which is linear and injective on each fiber, since both f' and g^k have this property. Conversely, given a map $g\colon E \to \mathbb{R}^\infty$ which is linear and injective on each fiber, define $f\colon B \to \mathrm{Gr}_k$ by letting $f(b)$ be the k-plane $g(p^{-1}(b))$. This yields a commutative diagram as above.

Now we are ready to prove the theorem. We start with the surjectivity of the map $[B, \mathrm{Gr}_k] \to \mathrm{Vect}^k(B)$. Let ξ be a k-dimensional bundle given by the map $p\colon E \to B$. Since B is paracompact there is a countable open cover $\{U_j\}$ of B such that ξ is trivial over each U_j and there is a partition of unity $\{\varphi_j\}$ with φ_j supported on U_j. Let $g_j\colon \pi^{-1}(U_j) \to \mathbb{R}^n$ be the composition of a trivialization $p^{-1}(U_j) \to U_j \times \mathbb{R}^n$ with the projection onto $\mathbb{R}^n$. The map

$$(\varphi_j \circ p) \cdot g_j\colon p^{-1}(U_j) \to \mathbb{R}^n,\ v \mapsto \varphi_j(p(v)) \cdot g_j(v)$$

extends to a map $E \to \mathbb{R}^n$ that is zero outside $p^{-1}(U_j)$. Near each point of B only finitely many φ_j's are nonzero, and at least one φ_j is nonzero. Hence these extended maps $(\varphi_j \circ p) \cdot g_j$ are the coordinates of a map $g\colon E \to (\mathbb{R}^n)^\infty = \mathbb{R}^\infty$ that is a linear injection on each fiber. By our claim above this induces a map $f\colon B \to \mathrm{Gr}_k$ and the proof of surjectivity is complete.

For injectivity, let f_0, $f_1\colon B \to \mathrm{Gr}_k$ be two maps with isomorphisms $\xi \cong f_0^*(\gamma^k)$ and $\xi \cong f_1^*(\gamma^k)$. By our first claim these two maps induce maps g_0, $g_1\colon E \to \mathbb{R}^\infty$ which are linear and injective on each fiber. We will now show that g_0 and g_1 are homotopic through maps g_t which are linear and injective on each fiber. Then f_0 and f_1 are homotopic via

$$f_t(b) = g_t(p^{-1}(b)).$$

Therefore, let L_t be the homotopy

$$L_t\colon \mathbb{R}^\infty \to \mathbb{R}^\infty,\ L_t(x_1, x_2, \ldots) = (1-t)(x_1, x_2, \ldots) + t(x_1, 0, x_2, 0, \ldots).$$

For each t, this is a linear map. Its kernel is trivial, since if

$$L_t(x_1, \ldots, x_n) = ((1-t)x_1 + tx_1, (1-t)x_2, (1-t)x_3 + tx_2, \ldots) = 0$$

then we get $x_1 = 0$, $x_2 = 0, \ldots$. Hence L_t is injective. Composing L_t with g_0 moves the image of g_0 into the odd-numbered coordinates and we have a homotopy which is linear and injective on fibers

$$g_0 = L_0 \circ g_0 \sim L_1 \circ g_0 =: \tilde{g}_0.$$

Similarly, let M_t be the homotopy

$$M_t \colon \mathbb{R}^\infty \to \mathbb{R}^\infty,\ M_t(x_1, x_2, \dots) = (1-t)(x_1, x_2, \dots) + t(0, x_1, 0, x_2, 0, \dots).$$

For each t, this is a linear map. Its kernel is trivial, since if

$$M_t(x_1, \dots, x_n) = ((1-t)x_1, (1-t)x_2 + tx_1, (1-t)x_3, (1-t)x_4 + tx_2, \dots) = 0$$

then we get $x_1 = 0$, $x_2 = 0, \dots$. Hence M_t is injective. Composing M_t with g_1 moves the image of g_1 into the even-numbered coordinates and we have a homotopy which is linear and injective on fibers

$$g_1 = M_0 \circ g_1 \sim M_1 \circ g_1 =: \tilde{g}_1.$$

Then we let

$$\tilde{g}_t = (1-t)\tilde{g}_0 + t\tilde{g}_1.$$

The reason for composing with L_t and M_t is that $\tilde{g}_t$ is a map which is linear and injective on fibers for each t, since g_0 and g_1 are linear and injective on fibers. Overall we obtain homotopies which are linear and injective on fibers

$$g_0 \sim \tilde{g}_0 \sim \tilde{g}_1 \sim g_1$$

as desired. This completes the proof of Theorem 12.1.

12.2. **Universality reformulated.** The statement of Theorem 12.1 is closely related to the following two assertions which reformulate the universality of the canonical bundle γ^k.

Theorem 12.4. *For any k-dimensional bundle ξ over a paracompact base space B there exists a bundle map $f \colon \xi \to \gamma^k$.*

Proof. We have seen in the previous proof that there is a map

$$g \colon E(\xi) \to \mathbb{R}^\infty$$

which is linear and injective on the fibers of ξ and which is unique up to a homotopy which is linear and injective on the fibers. Then we can define the the bundle map f by

$$f(e) = (g(\text{fiber in which } e \text{ lies}), g(e)).$$

□

Two bundle maps $F, G\colon \xi \to \gamma^k$ are called *bundle-homotopic* if there exists a one-parameter family of maps

$$H_t\colon \xi \to \gamma^k,\ 0 \le t \le 1,$$

with $H_0 = F$, $H_1 = G$ such that

$$H\colon E(\xi) \times [0,1] \to E(\gamma^k)$$

is continuous as a function of both variables.

Theorem 12.5. *Any two bundle maps from a k-dimensional bundle ξ to γ^k are bundle-homotopic.*

Proof. Let ξ be given by the map $p\colon E \to B$. We know that a bundle map $F\colon \xi \to \gamma^k$ determines a map

$$g\colon E(\xi) \to \mathbb{R}^\infty$$

whose restriction to each fiber of ξ is linear and injective. Conversely, g determines F by the identity

$$F(e) = (g(\text{fiber in which } e \text{ lies}), g(e)).$$

Now suppose we have two bundle maps F_0, $F_1\colon \xi \to \gamma^k$ and let f_0, $f_1\colon B \to \mathrm{Gr}_k$ be the corresponding maps on base spaces. We have seen in Lecture 04 that the bundle maps F_0, F_1 come equipped with isomorphisms $\xi \cong f_0^*(\gamma^k)$ and $\xi \cong f_1^*(\gamma^k)$. Then we know from the proof of Theorem 12.1 that there is a homotopy g_t between g_0 and g_1 which induces a homotopy f_t between f_0 and f_1. But the homotopy g_t also induces a bundle homotopy F_t between F_0 and F_1 by defining

$$F_t(e) := (g_t(\text{fiber in which } e \text{ lies}), g_t(e)).$$

□

12.3. **Universal characteristic classes.** We can use the above results to reconsider the concept of characteristic classes. For a k-dimensional vector bundle ξ let $f_\xi\colon B \to \mathrm{Gr}_k$ be a representative of the homotopy class corresponding to ξ under the bijection of Theorem 12.1.

Now let R be any coefficient ring and let

$$c \in H^i(\mathrm{Gr}_k; R)$$

be any cohomology class. Then we get an induced class

$$c(\xi) := f_\xi^*(c) \in H^i(B; R).$$

Definition 12.6. The class $c(\xi)$ is called the *characteristic cohomology class* of ξ determined by c.

Note that the correspondence $\xi \mapsto c(\xi)$ is natural with respect to bundle maps, i.e., it commutes with pullbacks.

Conversely, given any correspondence

$$\xi \mapsto c(\xi) \in H^i(B; R)$$

which is natural with respect to bundle maps, then we must have

$$c(\xi) = f_\xi^* c(\gamma^k).$$

Thus the above construction is the most general one. In other words:

Corollary 12.7. *The ring consisting of all characteristic cohomology classes for k-dimensional bundles over paracompact base spaces with coefficient ring R is canonically isomorphic to the cohomology ring $H^*(\mathrm{Gr}_k; R)$.*

Hence it is a very important task to compute the cohomology ring $H^*(\mathrm{Gr}_k; R)$. For $R = \mathbb{Z}/2$, we will do this in the next lecture.