Math 231b
Lecture 13

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13. Lecture 13: Schubert cells and Schubert varieties

In this lecture we follow notes by Mike Hopkins which are not listed in the references mentioned at the beginning of the semester.

The interior of the i-cell in $\mathbb{P}^n$ is the space of lines contained in $\mathbb{R}^{i+1}$ but not in $\mathbb{R}^i$. There is an analogous cell decomposition of the Grassmannian. Each k-plane $V \subset \mathbb{R}^{n+k}$ determines a sequence of numbers

$$(\dim V \cap \mathbb{R}^1, \dim V \cap \mathbb{R}^2, \ldots).$$

Note that the dimension jumps in each step by at most one, since the following sequence is exact:

$$0 \to V \cap \mathbb{R}^{i-1} \to V \cap \mathbb{R}^i \xrightarrow{i\text{-th coordinate}} \mathbb{R}.$$

Moreover, he sequence contains exactly k jumps.

For instance, if V is the 3-plane in $\mathbb{R}^5$ spanned by the rows of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 2 \end{pmatrix}$$

then our sequence of numbers would be

$$(0, 1, 2, 2, 3).$$

Let us keep track of where the dimensions jump, and record these numbers as $(j_1, \ldots, j_k)$. In our example the sequence of j's would be

$$(2, 3, 5).$$

Finally, for reasons that will be clear in a moment, we decide to use the sequence $(a_1, \ldots, a_k)$ instead with

$$a_i = j_i - i.$$

In our example the sequence of a's is

$$(1, 1, 2).$$

Definition 13.1. A *Schubert symbol* is a sequence $\underline{a} = (a_1, \ldots, a_k)$, with

$$0 \leq a_1 \leq \ldots \leq a_k.$$

The associated *jump sequence* is the sequence $\underline{j} = (j_1, \ldots, j_k)$ with $j_i = a_i + i$.

Remark 13.2. One should be aware of that other authors also use the name "Schubert symbol" to refer to the sequence $\underline{j}$.

Now let $\underline{a} = (a_1, \ldots, a_k)$ with $a_k \leq n$ be a Schubert symbol, and let

$$H_i := \mathbb{R}^{j_i}$$

with $j_i = a_i + i$ as before. Then the H_i define a filtration of $\mathbb{R}^{n+k}$

$$0 \subset H_1 \subset H_2 \subset \cdots \subset H_k \subseteq \mathbb{R}^{n+k}.$$

We set

$$\Omega_{\underline{a}} = \{V \in \mathrm{Gr}_k(\mathbb{R}^{n+k}) |\, \dim V \cap H_i \geq i\}.$$

Definition 13.3. The space $\Omega_{\underline{a}}$ is called the *Schubert variety* associated to the Schubert symbol $\underline{a}$.

Example 13.4. When $k = 1$ the sequence $\underline{a}$ is just a number a. In that case the Schubert variety is $\mathbb{P}^a$.

As a next step we will make the set of Schubert symbols of a fixed length k into a partially ordered set by defining $\underline{a}' \leq \underline{a}$ if and only if

$$a_i' \leq a_i \text{ for } i = 1, \ldots, k.$$

We can use this ordering to make the set of *all* Schubert symbols into a partially ordered set by first filling the symbols on the left with 0's to make them have the same length, and then using the above partial ordering. Thus, with this convention

$$(1, 2, 2) \geq (1, 2),$$

since

$$(1, 2, 3) \geq (0, 1, 2).$$

Definition 13.5. The *Schubert cell* associated to the Schubert symbol $\underline{a}$ is the space

$$\Omega_{\underline{a}}^0 = \Omega_{\underline{a}} - \bigcup_{\underline{a}' < \underline{a}} \Omega_{\underline{a}'}.$$

Remark 13.6. Another warning: The Schubert cells are not quite "cells". They are merely the interiors of cells.

Remark 13.7. The space $\Omega_{\underline{a}}^0$ consists exatly of the $V \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ whose associated Schubert symbol is $\underline{a}$. In particular, each V lies in exactly one $\Omega_{\underline{a}}^0$ where $\underline{a}$ is the Schubert symbol corresponding to the dimension sequence of V.

Proposition 13.8. *The space* $\Omega_{\underline{a}}^0$ *is homeomorphic to* $\mathbb{R}^{|\underline{a}|}$*, where we denote* $|\underline{a}| = a_1 + \ldots + a_k$.

Proof. We show that each $V \in \Omega^0_{\underline{a}}$ has a canonical basis of a special form. Let $\{\epsilon_1, \ldots, \epsilon_{n+k}\}$ be the standard basis of $\mathbb{R}^{n+k}$. First, choose a non-zero $v_1 \in V \cap H_1$. This space is one-dimensional, so v_1 is determined up to a scalar mutliple. We can normalize v_1 by requiring $\langle \epsilon_{j_1}, v_1 \rangle = 1$. Now choose $v_2 \in V \cap H_2$ with the properties

$$\begin{aligned} \langle \epsilon_{j_2}, v_2 \rangle &= 1 \\ \langle \epsilon_{j_1}, v_2 \rangle &= 0. \end{aligned}$$

Since $V \cap H_2$ has dimension 2 these two equations characterize v_2 uniquely, provided they can be solved. But we know they can be solved. For the map

$$V \cap H_2 \to H_2 \to H_2/\mathbb{R}^{j_2-1}\mathbb{R} \cdot \epsilon_{j_2}$$

is non-zero since $\dim V \cap \mathbb{R}^{j_2-1} = 1$. Continuing, we find a unique basis $\{v_1, \ldots, v_k\}$ of V with the property that $v_i \in H_i$ for all i, and

$$\begin{aligned} \langle \epsilon_{j_s}, v_s \rangle &= 1 \text{ for all } s \text{ and} \\ \langle \epsilon_{j_s}, v_t \rangle &= 0 \text{ for } s \neq t. \end{aligned}$$

Now if we let V vary in $\Omega^0_{\underline{a}}$, we see that the space of all possible v_i's is a vector space of dimension $\dim H_i - i$, since v_i lies in H_i and has to satisfy i equations. □

Remark 13.9. Another way to think of the v_i is to consider them as the rows in a matrix. For example, in the case $\mathrm{Gr}_3(\mathbb{R}^{4+3})$, with $\underline{a} = (2, 3, 4)$ such a matrix takes the form

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 \\ * & * & 0 & * & 0 & * & 1 \end{pmatrix}$$

where the $*$'s denote arbitrary numbers as entries. The rows of this matrix are the vectors v_1, v_2, and v_3. Hence we see that the decomposition of $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ into Schubert cells corresponds to taking a matrix, reducing it to row echelon form, and recording the columns with the pivots.