Math 231b
Lecture 14

G. Quick

14. Lecture 14: A cell decomposition for the Grassmannian

Recall from the previous lecture:

- *Schubert symbols*: sequences $\underline{a} = (a_1, \ldots, a_k)$, with $0 \leq a_1 \leq \ldots \leq a_k$. The associated *jump sequence* is the sequence $\underline{j} = (j_1, \ldots, j_k)$ with $j_i = a_i + i$.
- For given $\underline{a}$, filtration $0 \subset H_1 \subset H_2 \subset \cdots \subset H_k \subseteq \mathbb{R}^{n+k}$ with $H_i := \mathbb{R}^{j_i}$.
- The *Schubert variety* $\Omega_{\underline{a}} = \{V \in \mathrm{Gr}_k(\mathbb{R}^{n+k}) | \dim V \cap H_i \geq i\}$ associated to $\underline{a}$.
- The *Schubert cell* $\Omega^0_{\underline{a}} = \Omega_{\underline{a}} - \bigcup_{\underline{a}' \leq \underline{a}} \Omega_{\underline{a}'}$ associated to $\underline{a}$.
- We proved that the space $\Omega^0_{\underline{a}}$ is homeomorphic to $\mathbb{R}^{|\underline{a}|}$, where we denote $|\underline{a}| = a_1 + \ldots + a_k$. We did this by showing that each $V \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ has a special basis and the space of choices of those bases is a vector space of dimension $|\underline{a}|$.

We will use these notions and the above result to define a CW-decomposition of the Grassmannian manifold. We still follow the notes by Mike Hopkins.

14.1. A CW-decomposition.

To see that the Schubert cells serve as the cells of a CW-decomposition, we need to define the characteristic maps. For each $\underline{a}$ let $D^{\underline{a}} \subset V_k(\mathbb{R}^{n+k})$ be the set of orthonormal sequences $(v_1, \ldots, v_k)$ satisfying

$$\begin{aligned} v_i &\in H_i \\ \langle \epsilon_i, v_i \rangle &\geq 0. \end{aligned}$$

We define a map

$$s_{\underline{a}} \colon D^{\underline{a}} \to \Omega_{\underline{a}}$$

by sending $(v_1, \ldots, v_k)$ to the plane it spans.

Lemma 14.1. *The map $s_{\underline{a}}$ restricts to a homeomorphism of the interior of $D^{\underline{a}}$ with $\Omega^0_{\underline{a}}$.*

Proof. Let $s^0_{\underline{a}}$ be the restriction of $s_{\underline{a}}$ to the interior of $D^{\underline{a}}$. Let $(v_1, \ldots, v_k)$ be an orthonormal frame on the boudnary of $D^{\underline{a}}$. Then

$$V := s^0_{\underline{a}}((v_1, \ldots, v_k))$$

does not belong to $\Omega^0_{\underline{a}}$, for one of the vectors v_i must have $j_i - 1$th component equal to 0. This implies

$$\dim(V \cap \mathbb{R}^{j_i - 1}) \geq i,$$

since we have $\dim(V \cap \mathbb{R}^{j_i}) \geq i$. Hence V does not lie in $\Omega^0_{\underline{a}}$, since for a k-plane in $\Omega^0_{\underline{a}}$ the number j_i is exactly the first dimension where $V \cap \mathbb{R}^m$ has dimension i. The construction of the previous lecture of the special basis for the planes in $\Omega^0_{\underline{a}}$ then shows that $s^0_{\underline{a}}$ is a bijection. It remains to show that $s^0_{\underline{a}}$ and its inverse are continuous. We leave this to the reader. □

The next result shows that the $s_{\underline{a}}$ serve as characteristic maps for the cells in the Grassmannian.

Proposition 14.2. *The space $D^{\underline{a}}$ is homeomorphic to the product*

$$D_0^{a_1} \times D_0^{a_2} \times \ldots \times D_0^{a_k},$$

in which each $D_0^{a_i}$ is the disk consisting of the unit vectors $v \in H_i$ with the properties

$$\begin{aligned} \langle v, \epsilon_{j_i} \rangle &\geq 0 \\ \langle v, \epsilon_{j_t} \rangle &= 0 \text{ for } t < i. \end{aligned}$$

Hence $D^{\underline{a}}$ is homeomorphic to the disk $D^{a_1 + \cdots + a_k}$.

Proof. For each unit vector $v \in H_1$ with $\langle \epsilon_{j_i}, v \rangle \geq 0$, let $T_v \in SO(n+k)$ be the orthogonal transformation which rotates v to ϵ_{j_1} in the plane spanned by v and ϵ_{j_1}, and which is the identity on the orthogonal complement of this plane. Note that T_v restricts to an orthogonal transformation of H_i to itself since both v and ϵ_{j_i} are in H_i (H_1 is a subspace of H_i), and has the property that $T_v(\epsilon_{j_i}) = \epsilon_{j_i}$ for $i > 1$, since both v and ϵ_{j_1} are orthogonal to ϵ_{j_i}. We now use this transformation T to define a homeomorphism

$$D^{\underline{a}} \to D_0^{a_1} \times D_1^{\underline{a}'}, \tag{1}$$

in which $D_1^{\underline{a}'}$ is the space of orthonormal sequences

$$(v_2', \ldots, v_k')$$

with $v_i' \in H_i \cap \{\epsilon_{j_1}\}^\perp$, and

$$\langle \epsilon_i, v_i' \rangle \geq 0.$$

In other words, $D_1^{\underline{a}'}$ is the cell in $\mathrm{Gr}_{k-1}(\mathbb{R}^{n+k-1})$ associated to the sequence

$$\underline{a}' = (a_2, \ldots, a_k),$$

in which we are regarding $\mathbb{R}^{n+k-1}$ as the Euclidean space with basis

$$\{\epsilon_t | t \neq j_1\}.$$

Once we establish the homeomorphism (1), we are done by induction on k.

The homeomorphism (1) is the map whose first component is the projection

$$(v_1, \ldots, v_k) \mapsto v_1,$$

and whose second component is

$$(T_{v_1} v_2, \ldots, T_{v_1} v_k),$$

so that

$$v_i' = T_{v_1} v_i.$$

Since T_{v_1} is orthogonal, the sequence $(v_2', \ldots, v_k')$ is orthonormal. To verify the conditions that the sequence be in $D_1^{\underline{a}'}$, first note that for $i > 1$, we have

$$0 = \langle v_1, v_i \rangle = \langle T_{v_1} v_1, T_{v_1} v_i \rangle = \langle \epsilon_{j_1}, T_{v_1} v_i \rangle,$$

and also

$$0 \leq \langle \epsilon_{j_1}, v_i \rangle = \langle T_{v_1} \epsilon_{j_1}, T_{v_1} v_i \rangle = \langle \epsilon_{j_1}, T_{v_1} v_i \rangle,$$

since ϵ_i is orthogonal to both ϵ_{j_1} and v_1. The inverse homeomorphism is

$$(v_1, v_2', \ldots, v_k') \mapsto (v_1, T_{v_1}^{-1} v_2', \ldots, T_{v_1}^{-1} v_k').$$

Reversing the above computations which checked the conditions shows that it carries $D_0^{a_1} \times D_1^{\underline{a}'}$ to $D^{\underline{a}}$. ☐

Remark 14.3. a) There are $\binom{n+k}{k}$ cells in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$. This is the number of ways of choosing k distinct numbers j_i with $j_i \leq n + k$.
b) In particular, the number of r-cells in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is equal to the number of partitions of r into at most k integers a_i each of which is $\leq n$.
c) If k and n are $\geq r$ then the number of r-cells in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is equal to the number of partitions of r into at most k integers (zeroes in the beginning of the sequence $\underline{a}$ are allowed).
d) The number of r-cells in Gr_k is equal to the number of partitions of r into at most k integers.

Corollary 14.4. *The maps*

$$s_{\underline{a}'} \colon D^{\underline{a}'} \to \Omega_{\underline{a}}$$

with $\underline{a}' \leq \underline{a}$ are the characteristic maps of the cells in a CW-decomposition of the Schubert variety $\Omega_{\underline{a}}$.

In the next lecture we will prove the following result.

Proposition 14.5. *The cellular boundary map*

$$d^{cell} \colon C_*^{cell}(\Omega_{\underline{a}}) \otimes \mathbb{Z}/2 \to C_{*-1}^{cell}(\Omega_{\underline{a}}) \otimes \mathbb{Z}/2$$

is zero.

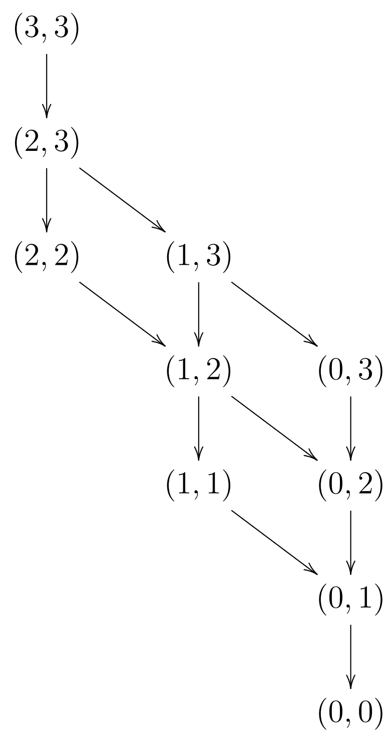
Let $x_{\underline{a}}$ be the homology class corresponding to the cellular cycle given by the map $s_{\underline{a}}$. Then the above result implies the following fundamental fact.

Corollary 14.6. *The classes*

$$x_{\underline{a}'} \in H_{\underline{a}'}(\Omega_{\underline{a}}; \mathbb{Z}/2)$$

with $\underline{a}' \leq \underline{a}$ *form a basis for the homology groups, where* $|\underline{a}| = a_1 + \ldots + a_k$.

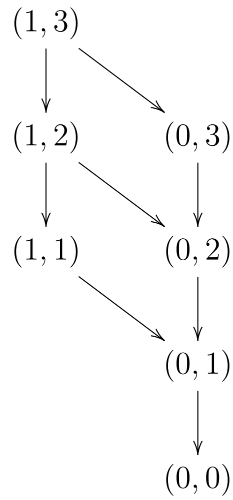
Before we prove these results, we look at some consequences. The picture below lists the sequences $\underline{a}$ occurring in the cell decomposition of $\mathrm{Gr}_2(\mathbb{R}^{3+2})$. The reverse of the partial ordering is indicated by an arrow, and the height corresponds to the dimension of the cell: (Recall: The dimension of $\mathrm{Gr}_2(\mathbb{R}^{3+2})$ is 6, the Schubert symbol $(3, 3)$ has associated the maximal jump sequence (4,5) and corresponds to a cell in dimension $3 + 3 = 6$. The cell $(0, 0)$ is in dimension zero.)



By looking at this diagram we see that the homology satisfies Poincaré duality in the sense that

$$\dim H_i(\mathrm{Gr}_2(\mathbb{R}^5)) = \dim H_{6-i}(\mathrm{Gr}_2(\mathbb{R}^5)).$$

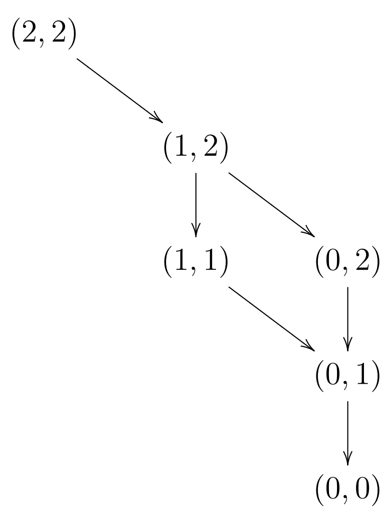
For instance, if we want the homology of $\Omega_{(1,3)}$ we look at the position labeled $(1,3)$, and everything below it



We can see from the diagram that $\Omega_{(1,3)}$ cannot satisfy Poincaré duality,

$$\dim H_i(\Omega_{\underline{a}}) = \dim H_{|\underline{a}|-i}(\Omega_{\underline{a}}). \tag{2}$$

Hence $\Omega_{(1,3)}$ cannot be a manifold. Looking at the diagram, the only Schubert varieties in $\mathrm{Gr}_2(\mathbb{R}^5)$ which might be manifold are $\Omega_{(2,2)}$ with



and $\Omega_{(0,i)}$ with $i \leq 3$ and

$$\begin{array}{c} (0,3) \\ \downarrow \\ (0,2) \\ \downarrow \\ (0,1) \\ \downarrow \\ (0,0) \end{array}$$

In fact, one can show that if the homology of $\Omega_{\underline{a}}$ satisfies Poincaré duality in the sense of (2) then $\Omega_{\underline{a}}$ is homeomorphic to $\mathrm{Gr}_\ell(\mathbb{R}^{m+\ell})$ for some pair (ℓ, m) and so is in fact a manifold. The point is that the Poincaré duality condition implies that the Schubert symbol $\underline{a}$ must have exactly one immediate predecessor. (You will be asked to prove this on the next Problem Set.)