Math 231b
Lecture 15

G. Quick

15. Lecture 15: The cohomology of the Grassmannian

Our first goal is to show the following result.

Proposition 15.1. *The cellular boundary map*
$$d^{cell}\colon C_*^{cell}(\Omega_{\underline{a}}) \otimes \mathbb{Z}/2 \to C_{*-1}^{cell}(\Omega_{\underline{a}}) \otimes \mathbb{Z}/2$$
is zero.

Let $x_{\underline{a}}$ be the homology class corresponding to the cellular cycle given by the map $s_{\underline{a}}\colon D^{\underline{a}} \to \Omega_{\underline{a}}$ defined in the previous lecture. Then the above result implies the following fundamental fact.

Corollary 15.2. *The classes*
$$x_{\underline{a}'} \in H_{\underline{a}'}(\Omega_{\underline{a}}; \mathbb{Z}/2)$$
with $\underline{a}' \leq \underline{a}$ form a basis for the homology groups, where $|\underline{a}| = a_1 + \ldots + a_k$.

15.1. **The flag varieties.** The aim of this section is to prove Proposition 15.1. Therefore, we start with an observation. Suppose that X is a CW-complex, M is a closed manifold of dimension n, and $f\colon M \to X^{(n)}$ is a map form M to the n-skeleton of X. Let $\alpha_M \in H_n(M; \mathbb{Z}/2)$ be the fundamental class. The image of α_M under the map
$$H_n(M) \to H_n(X^{(n)}) \to H_n(X^{(n)}, X^{(n-1)}) = C_n^{cell}(X)$$
defines a cellular chain $c_M \in C_n^{cell}(X)$. In fact this chain is a cycle since it lies in the image of $H_n(X^{(n)})$ and so goes to zero under the first map in the factorization
$$H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{n-1}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)})$$
of the cellular boundary map. In this way, maps of manifolds give homology classes, and, in fact cycles in the complex of cellular chains.

We will need to be able to specify the cycle we constructed more precisely. If the map
$$f\colon M \to X' := X^{(n-1)} \cup D_\alpha^n \subset X^{(n)},$$
and that for some point x in the interior of D_α^n there is a neighborhood U of x, contained in the interior of D_α^n, with the property that the restriction of f is a homeomorphism
$$f^{-1}(U) \to U.$$

In that case, the diagram

$$\begin{array}{ccccc} H_n(M) & \xrightarrow{\approx} & H_n(M, M - f^{-1}(x)) & & \\ \downarrow & & \downarrow \approx & & \\ H_n(X') & \longrightarrow & H_n(X', X' - \{x\}) & & \\ & & \downarrow \approx & & \\ & & H_n(D^n_\alpha, S^{n-1}_\alpha) & \longrightarrow & C^{cell}_n(X) \end{array}$$

shows that the cellular cycle c_M is just the chain represented by the cell D^n_α. In particular, one learns in this case that the cellular represented by D^n_α is, in fact, a cycle. We will use these ideas to prove Proposition 15.1.

For each $\underline{a}$, let

$$F_{\underline{a}} \subset \mathrm{Gr}_1(H_1) \times \cdots \times \mathrm{Gr}_k(H_k)$$

be the subspace consisting of sequences $(V_1, \ldots, V_k)$ with

$$V_1 \subset V_2 \subset \cdots \subset V_k.$$

For some purposes it is useful to note that $F_{\underline{a}}$ can also be identified with the space

$$F_{\underline{a}} \subset \mathbb{P}(H_1) \times \cdots \times \mathbb{P}(H_k)$$

consisting of sequences of lines $(\ell_1, \ldots, \ell_k)$ which are pairwise orthogonal. There is an obvious homeomorphism between these, under which V_j corresponds to $\ell_1 \oplus \cdots \oplus \ell_j$, and ℓ_j to the orthogonal complement of V_{j-1} in V_j.

Proposition 15.3. *The space $F_{\underline{a}}$ is a manifold.*

Proof. The proof is very similar to the proof of Proposition 11.3. Let

$$\begin{array}{ccccccc} V_1 & \longrightarrow & V_2 & \longrightarrow & \cdots & \longrightarrow & V_k \\ \downarrow & & \downarrow & & & & \downarrow \\ H_1 & \longrightarrow & H_2 & \longrightarrow & \cdots & \longrightarrow & H_k \end{array} \tag{1}$$

be a point in $F_{\underline{a}}$, and write W_i for the orthogonal complement of V_i in H_i. By identifying W_i with the quotient space H_i/V_i, the W_i fit into a sequence

$$W_1 \to W_2 \to \cdots \to W_k.$$

(This sequence is not, in general, a sequence of monomorphisms.)

Let $U \subset F_{\underline{a}}$ be the open neighborhood of the point (1) consisting of sequences $(V'_1 \subset \cdots \subset V'_k)$ with the property that for all i, $V'_i \cap W_i = \{0\}$. For such a sequence, we may think of V'_i as the graph of a homomorphism $V_i \to W_i$. This correspondence gives a homeomorphism of U with the space of sequences of linear maps $V_i \to W_i$ fitting into a diagram

$$\begin{array}{ccccccc} V_1 & \longrightarrow & V_2 & \longrightarrow & \cdots & \longrightarrow & V_k \\ \downarrow & & \downarrow & & & & \downarrow \\ W_1 & \longrightarrow & W_2 & \longrightarrow & \cdots & \longrightarrow & W_k \end{array}$$

By choosing a basis $\{v_1, \ldots, v_k\}$ of V_k with $v_i \in V_i$ one can identify this space with

$$W_1 \oplus \cdots \oplus W_k.$$

Hence this is a vector space with of dimension

$$\dim W_1 + \cdots + \dim W_k = a_1 + \cdots + a_k.$$

□

Now let

$$f_{\underline{a}} \colon F_{\underline{a}} \to \Omega_{\underline{a}}$$

be the map sending a sequence $(V_1, \ldots, V_k)$ to V_k.

Proposition 15.4. *The map*

$$f_{\underline{a}}^{-1}(\Omega_{\underline{a}}^0) \to \Omega_{\underline{a}}^0$$

is a homeomorphism.

Proof. The inverse map sends $V \in \Omega_{\underline{a}}^0$ to the sequence $(V_1, \ldots, V_k)$ in which $V_i = V \cap H_i$. □

Now are finally ready to prove Proposition 15.1. The Schubert cell of $\Omega_{\underline{a}}$ has one cell of dimension $a_1 + \cdots + a_k$ and all other cells of lower dimension. We just proved that $F_{\underline{a}}$ is a manifold. Hence the argument described at the beginning of this section applied to the map

$$F_{\underline{a}} \to \Omega_{\underline{a}},$$

shows that the corresponding chain is a cycle. This shows that the boundary map d^{cell} vanishes on the one cell in dimension $|\underline{a}|$. All other elements in the cell complex are given by maps from cells $D^{\underline{a}'}$ for $\underline{a}' < \underline{a}$ to $\Omega_{\underline{a}}$. It follows from the ordering of the Schubert cells and the definition of Schubert varieties that the map $s_{\underline{a}'} \colon D^{\underline{a}'} \to \Omega_{\underline{a}}$ factors through the map $\Omega_{\underline{a}'} \to \Omega_{\underline{a}}$. This shows that the boundary map d^{cell} actually vanishes on all elements in $C_*^{cell}(\Omega) \otimes \mathbb{Z}/2$. This completes the proof of Proposition 15.1.

15.2. **The cohomology ring $H^*(\mathrm{Gr}_k;\mathbb{Z}/2)$.** We will finally determine the cohomology ring of the Grassmannian manifold Gr_k.

Theorem 15.5. *The cohomology ring $H^*(\mathrm{Gr}_k;\mathbb{Z}/2)$ is a polynomial algebra over $\mathbb{Z}/2$ freely generated by the Stiefel-Whitney classes $w_1(\gamma^k),\dots,w_k(\gamma^k)$.*

The idea of the proof is to show first that the Stiefel-Whitney classes of the canonical bundle over Gr_k freely generate a polynomial algebra over $\mathbb{Z}/2$ contained in $H^*(\mathrm{Gr}_k;\mathbb{Z}/2)$. Our knowledge about the cell structure of Gr_k then allows us to show that $H^*(\mathrm{Gr}_k;\mathbb{Z}/2)$ is actually equal to this polynomial algebra.

We start with the following lemma.

Lemma 15.6. *There are no polynomial relations among the $w_i(\gamma^k)$.*

Proof. Suppose that there is a relation of the form $p(w_1(\gamma^k),\dots,w_k(\gamma^k)) = 0$, where p is a polynomial in k variables over $\mathbb{Z}/2$. By the naturality of Stiefel-Whitney classes, for any k-dimensional bundle ξ over a paracompact base space there exists a bundle map $g\colon \xi \to \gamma^k$. If we denote the induced map on base spaces by $\bar{g}$ we get

$$w_i(\xi) = \bar{g}^*(w_i(\gamma^k)).$$

It follows that the cohomology classes $w_i(\xi)$ must satisfy the corresponding relation

$$p(w_1(\xi),\dots,w_k(\xi)) = \bar{g}^* p(w_1(\gamma^k),\dots,w_k(\gamma^k)) = 0.$$

Thus to prove the lemma it suffices to find some k-dimensional bundle ξ such that there are no polynomial relations among the classes $w_1(\xi),\dots,w_k(\xi)$.

Let γ^1 be the canonical line bundle over $\mathbb{P}^\infty = \mathrm{Gr}_1$. We know that $H^*(\mathbb{P}^\infty;\mathbb{Z}/2)$ is a polynomial algebra over $\mathbb{Z}/2$ with one generator a of dimension one and $w(\gamma^1) = 1 + a$. Taking the k-fold product

$$X := \mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty,$$

it follows that $H^*(X;\mathbb{Z}/2)$ is a polynomial algebra on k generators $a_1,\dots,a_k$ of dimension one. Here a_i can be defined as the image $\pi_i^*(a)$ induced by the projection map

$$\pi_i\colon X \to \mathbb{P}^\infty$$

to the ith factor. We define ξ to be the k-fold product

$$\xi = \gamma^1 \times \cdots \times \gamma^1 \cong (\pi_1^*\gamma^1) \oplus \cdots \oplus (\pi_k^*\gamma^1).$$

Then ξ is a k-dimensional bundle over X, and the total Stiefel-Whitney class

$$w(\xi) = \pi_1^*(w(\gamma^1)) \cdot \cdots \cdot \pi_k^*(w(\gamma^1)) = (1 + a_1)(1 + a_2)\cdots(1 + a_k).$$

Hence $w_i(\xi)$ is the ith elementary symmetric function of $a_1,\dots,a_k$. It is a well-known theorem in algebra that the k elementary symmetric functions in

k variables over a field do not satisfy any polynomial relations. Thus the classes $w_1(\xi), \ldots, w_k(\xi)$ are algebraically independent over $\mathbb{Z}/2$, and it follows that the $w_1(\gamma^k), \ldots, w_k(\gamma^k)$. □

Now let us turn to the proof of Theorem 15.5. By the previous lemma, we know that $H^*(\mathrm{Gr}_k; \mathbb{Z}/2)$ contains a polynomial algebra over $\mathbb{Z}/2$ freely generated by $w_1(\gamma^k), \ldots, w_k(\gamma^k)$. We will show that $H^*(\mathrm{Gr}_k; \mathbb{Z}/2)$ actually coincides with this sub-algebra.

We know from the discussion of the cell discussion of Gr_k is equal to the number of partitions of r into at most k integers. Hence the dimension of $H^r(\mathrm{Gr}_k; \mathbb{Z}/2)$ is at most equal to this number of partitions. On the other hand, we claim that the number of distinct monomials of the form

$$w_1(\gamma^k)^{r_1} \cdots w_k(\gamma^k)^{r_k}$$

in $H^r(\mathrm{Gr}_k; \mathbb{Z}/2)$ is also precisely equal to the number of partitions of r into at most k integers. For to each sequence $r_1, \ldots, r_k$ of non-negative integers with

$$r_1 + 2r_2 + \cdots + kr_k = r \tag{2}$$

we can associate the partition of r which is obtained from the k-tuple

$$r_k, r_k + r_{k-1}, \ldots, r_k + r_{k-1} + \cdots + r_1 \tag{3}$$

by deleting any zeros which may occur. Conversely, to a partition (3) corresponds a sequence $r_1, \ldots, r_k$ of non-negative integers satisfying (2).

Since $\mathbb{Z}/2[w_1(\gamma^k), \ldots, w_k(\gamma^k)]$ is a sub-algebra of $H^*(\mathrm{Gr}_k; \mathbb{Z}/2)$, comparing the degrees and dimensions proves the theorem.