Math 231b
Lecture 16

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16. Lecture 16: Chern classes for complex vector bundles

16.1. **Orientations.** From now on we will shift our focus to complex vector bundles. Much of the theory for real vector bundles carries over to the complex case. But there are a couple of important features of complex bundles. The first one is that the complex structure induces an orientation of the underlying real bundle.

Lemma 16.1. *Let ω be a complex vector bundle. Then the underlying real vector bundle $\omega_{\mathbb{R}}$ has a canonical preferred orientation.*

Proof. Let V be a finite dimensional complex vector space. Choosing a basis $a_1, \ldots, a_n$ for V over $\mathbb{C}$, gives us a real basis for the underlying real vector space $V_{\mathbb{R}}$:

$$a_1, ia_1, a_2, ia_2, \ldots, a_n, ia_n.$$

We claim that this ordered basis determines the required orientation for $V_{\mathbb{R}}$. For if $b_1, \ldots, b_n$ is any other complex basis of V, then there is a matrix $A \in \mathrm{GL}_n(\mathbb{C})$ which transforms the first basis into the second. This deformation does not alter the orientation of the real vector space, since if $A \in \mathrm{GL}_n(\mathbb{C})$ is the coordinate change matrix, then the underlying real matrix $A_{\mathbb{R}} \in \mathrm{GL}_{2n}(\mathbb{R})$ has determinant

$$\det A_{\mathbb{R}} = |\det A|^2 > 0.$$

Hence $A_{\mathbb{R}}$ preserves the orientation of the underlying real vector space. Another way to see this is to note that $GL_n(\mathbb{C})$ is connected. Hence we can pass from any given complex basis to any other basis by a continuous deformation, and this continuous deformation cannot alter the orientation.

Now if ω is a complex vector bundle, then applying this construction to every fiber of ω yields the required orientation for $\omega_{\mathbb{R}}$, since overlapping trivializations determine a section in $\mathrm{GL}_n(\mathbb{C})$. $\square$

Remark 16.2. As a consequence, every complex manifold is oriented, since an orientation of the tangent bundle of a manifold induces an orientation of the manifold itself.

16.2. **Chern classes.** Chern classes for complex vector bundles can be characterized by almost the same set of axioms as Stiefel-Whitney classes.

Theorem 16.3. *There is a unique sequence of functions $c_1, c_2, \dots$ assigning to each complex vector bundle $E \to B$ over a a space B a class $c_i(E) \in H^{2i}(B; \mathbb{Z})$, depending only on the isomorphism type of E, such that*

a) $c_i(f^*E) = f^*(c_i(E))$ for a pullback along a map $f\colon B' \to B$ which is covered by a bundle map.
b) $c(E_1 \oplus E_2) = c(E_1)c(E_2)$ where $c = 1 + c_1 + c_2 + \ldots \in H^*(B; \mathbb{Z}/2)$.
c) $c_i(E) = 0$ if $i > \dim E$.
d) For the canonical complex line bundle γ_1^1 on $\mathbb{C}\mathrm{P}^\infty$, $c_1(\gamma_\infty^1)$ is a specified generator of $H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$.

Proof. The proof is almost the same as for the existence and uniqueness of Stiefel-Whitney classes with $\mathbb{Z}$-coefficients and $H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) = \mathbb{Z}[\alpha]$. The bundle E induces a map $g\colon E \to \mathbb{C}^\infty$ which is linear and injective on fibers. Define $x \in H^2(E; \mathbb{Z})$ to be the element $\mathbb{C}\mathrm{P}(g)^*(\alpha)$. The Leray-Hirsch theorem applied to the fiber bundle $\mathbb{C}\mathrm{P}(E) \to B$ then implies that the elements $1, x, \dots, x^{n-1}$ form a basis of $H^*(\mathbb{C}\mathrm{P}(E); \mathbb{Z})$ as an $H^*(B; \mathbb{Z})$-module. Since we are using $\mathbb{Z}$ coefficients instead of $\mathbb{Z}/2$ signs do matter now. We modify the defining relation for the Chern classes to be

$$x^n - c_1(E)x^{n-1} + \cdots + (-1)^n c_n(E) = 0$$

with alternating signs. The sign change does not affect the proofs of properties a)-c). For d), the sign convention turns the defining relation of $c_1(\gamma^1)$ into

$$x - c_1(\gamma^1) = 0$$

with $x = \alpha$. Thus $c_1(\gamma^1)$ is the chosen generator of $H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ (and not minus the generator). □

Proposition 16.4. *Regarding an n-dimensional complex vector bundle $E \to B$ as a $2n$-dimensional real vector bundle, then $w_{2i+1}(E) = 0$ and $w_{2i}(E)$ is the image of $c_i(E)$ under the homomorphism $H^{2i}(B; \mathbb{Z}) \to H^{2i}(B; \mathbb{Z}/2)$.*

Proof. There is a natural map $p\colon \mathbb{R}\mathrm{P}(E) \to \mathbb{C}\mathrm{P}(E)$ sending a real line to the complex line containing it. This projection fits into a commutative diagram

$$\begin{array}{ccccc}
\mathbb{R}\mathrm{P}^{2n-1} & \longrightarrow & \mathbb{R}\mathrm{P}(E) & \xrightarrow{\mathbb{R}\mathrm{P}(g)} & \mathbb{R}\mathrm{P}^\infty \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\
\mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & \mathbb{C}\mathrm{P}(E) & \xrightarrow{\mathbb{C}\mathrm{P}(g)} & \mathbb{C}\mathrm{P}^\infty
\end{array}$$

where the left vertical map is the restriction of p to a fiber of E and the maps $\mathbb{RP}(g)$ and $\mathbb{CP}(g)$ are the projectivizations of a map $g\colon E \to \mathbb{C}^\infty$ which is injective and $\mathbb{C}$-linear on the fibers of E. All three vertical maps are fiber bundles with fiber $\mathbb{RP}^1$, the real lines in a complex line (using $\mathbb{C} \cong \mathbb{R}$). The Leray-Hirsch theorem applies to the bundle $\mathbb{RP}^\infty \to \mathbb{CP}^\infty$, so if α is the generator of $H^2(\mathbb{CP}^\infty;\mathbb{Z})$, the $\mathbb{Z}/2$-reduction $\bar{\alpha} \in H^2(\mathbb{CP}^\infty;\mathbb{Z}/2)$ pulls back to a generator of $H^2(\mathbb{RP}^\infty;\mathbb{Z}/2)$. This generator is β^2, the square of the generator $\beta \in H^1(\mathbb{RP}^\infty;\mathbb{Z}/2)$. Hence the $\mathbb{Z}/2$-reduction

$$\bar{x}_{\mathbb{C}}(E) = \mathbb{CP}(g)^*(\bar{\alpha}) \in H^2(\mathbb{CP}(E);\mathbb{Z}/2)$$

of the class $x_{\mathbb{C}}(E) = \mathbb{CP}(g)^*(\alpha)$ pulls back to the square of the class

$$x_{\mathbb{R}}(E) = \mathbb{RP}(g)^*(\alpha) \in H^1(\mathbb{RP}(E);\mathbb{Z}/2).$$

Thus the $\mathbb{Z}/2$-reduction of the defining relation for the Chern classes of E, which is

$$\bar{x}_{\mathbb{C}}(E)^n + \bar{c}_1(E)\bar{x}_{\mathbb{C}}(E)^{n-1} + \cdots + \bar{c}_n(E) = 0,$$

(signs do not matter here since we are over $\mathbb{Z}/2$) pulls back to the relation

$$x_{\mathbb{R}}(E)^{2n} + \bar{c}_1(E)x_{\mathbb{R}}(E)^{2(n-1)} + \cdots + \bar{c}_n(E) = 0,$$

which is the defining relation for the Stiefel-Whitney classes of E. Hence we must have

$$w_{2i+1}(E) = 0 \text{ and } w_{2i}(E) = \bar{c}_i(E).$$

□

16.3. **The complex Grassmannian and its cohomology.** The complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^{n+k})$ is the space of complex k-planes in $\mathbb{C}^{n+k}$. We can topologize this space just as in the real case and we obtain a complex manifold of complex dimension kn or real dimension $2kn$. For $k = 1$, we get $\mathrm{Gr}_1(\mathbb{C}^{n+1}) = \mathbb{CP}^n$.

Moreover, the inclusions $\mathbb{C}^{n+k} \subset \mathbb{C}^{n+1+k} \subset \ldots$ induce inclusions

$$\mathrm{Gr}_k(\mathbb{C}^{n+k}) \subset \mathrm{Gr}_k(\mathbb{C}^{n+1+k}) \subset \ldots$$

The infinite complex Grassmannian manifold is the union

$$\mathrm{Gr}_k(\mathbb{C}) := \mathrm{Gr}(\mathbb{C}^\infty) = \bigcup_n \mathrm{Gr}_k(\mathbb{C}^{n+k}).$$

This is the set of all k-dimensional complex vector subspaces of $\mathbb{C}^\infty$. The topology of $\mathrm{Gr}_k(\mathbb{C})$ is the direct limit topology. We have $\mathrm{Gr}_1(\mathbb{C}) = \mathbb{CP}^\infty$.

The complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^{n+k})$ is equipped with a canonical k-dimensional complex vector bundle $\gamma^k(\mathbb{C}^{n+k})$ defined as in the real case. The total space

$$E = E(\gamma^k(\mathbb{C}^{n+k}))$$

is the set of all pairs

$$(\text{complex } k\text{-plane in } \mathbb{C}^{n+k}, \ \text{vector in that } k\text{-plane}).$$

The topology on E is the topology as a subset of $\mathrm{Gr}_k(\mathbb{C}^{n+k}) \times \mathbb{C}^{n+k}$. The projection map

$$\pi\colon E \to \mathrm{Gr}_k(\mathbb{C}^{n+k}), \text{ is defined by } \pi(V, v) = V,$$

and the vector space structure is defined by

$$t_1(V, v_1) + t_2(V, v_2) = (V, t_1 v_1 + t_2 v_2).$$

Over the infinite complex Grassmannian $\mathrm{Gr}_k(\mathbb{C})$, there is also a canonical bundle $\gamma^k_{\mathbb{C}}$ whose total space is

$$E(\gamma^k_{\mathbb{C}}) \subset \mathrm{Gr}_k(\mathbb{C}) \times \mathbb{C}^\infty$$

the set of all pairs

$$(\text{complex } k\text{-plane in } \mathbb{C}^\infty, \ \text{vector in that } k\text{-plane})$$

topologized as a subset of the product $\mathrm{Gr}_k(\mathbb{C}) \times \mathbb{C}^\infty$. The projection

$$\pi\colon E(\gamma^k_{\mathbb{C}}) \to \mathrm{Gr}_k(\mathbb{C})$$

is given by $\pi(V, v) = V$.

The crucial result is again the following theorem.

Theorem 16.5. *For a paracompact space B, the map $[B, \mathrm{Gr}_k(\mathbb{C})] \to \mathrm{Vect}^k_{\mathbb{C}}(B)$, $[f] \mapsto f^*(\gamma^k)$, is a bijection from the set of homotopy classes of maps $B \to \mathrm{Gr}_k(\mathbb{C})$ and the set of isomorphism classes of k-dimensional complex vector bundles.*

The proof is the same as for real bundles. The theorem justifies to call the infinite complex Grassmannian $\mathrm{Gr}_k(\mathbb{C})$ the *classifying space* and $\gamma^k_{\mathbb{C}}$ the *universal bundle* for k-dimensional complex vector bundles.

The complex Grassmannian $\mathrm{Gr}_k(\mathbb{C})$ is a CW-complex with one cell of dimension $2n$ corresponding to each partition of n into at most k integers.

Theorem 16.6. *The cohomology ring $H^*(\mathrm{Gr}_k(\mathbb{C}); \mathbb{Z})$ is a polynomial algebra over $\mathbb{Z}$ freely generated by the Chern classes $c_1(\gamma^k_{\mathbb{C}}), \ldots, c_k(\gamma^k_{\mathbb{C}})$.*

Proof. Just work out the proof for the real Grassmannian in the complex case. □