Math 231b
Lecture 18

G. Quick

18. Lecture 18: Complex K-theory as a representable functor

We postpone the proof of the periodicity theorem for a while and first workout more properties of the K-theory functor.

18.1. **Reduced K-theory.** Let X be a compact Hausdorff space. Recall that a vector bundle over X may have different dimensions on the connected components of X. If X is a based space, i.e., has a chosen base point $* \in X$, then we can define a function

$$d\colon \mathrm{Vect}(X) \to \mathbb{Z}$$

that sends a vector bundle to the dimension of its restriction to the component of the basepoint $*$. The function d is a homomorphism of semirings and hence induces a dimension function

$$d\colon K(X) \to \mathbb{Z},$$

which is a homomorphism of rings. Since d is an isomorphism when X is a point, d can be identified with the induced map

$$K(X) \to K(*).$$

This leads to the following definition.

Definition 18.1. The *reduced K-theory* $\tilde{K}(X)$ of a based space is the kernel of $d\colon K(X) \to \mathbb{Z}$.

Remark 18.2. $\tilde{K}(X)$ is an ideal of K(X) and thus a ring without identity. It clearly holds

$$K(X) \cong \tilde{K}(X) \times \mathbb{Z}.$$

If X does not have a base point yet, let

$$X_+ := X \amalg *$$

be X together with a disjoint base point. Then we have

$$K(X) \cong \tilde{K}(X_+).$$

We denote the stable equivalence class of a bundle ξ by $\{\xi\}$ and the set of stable equivalence classes of finite dimensional complex vector bundles over X by $EU(X)$. The set $EU(X)$ forms an abelian group under direct sums, since we know that for each bundle ξ there is bundle ξ' such that $\xi \oplus \xi'$ is trivial.

Proposition 18.3. *There is a natural isomorphism of groups $EU(X) \xrightarrow{\cong} \tilde{K}(X)$.*

Proof. Denote the class of the trivial n-dimensional bundle ϵ^n over X by n. Then we know that every element in $K(X)$ can be written in the form $[\xi] - q$ for some vector bundle ξ and some non-negative integer q. Then we can define the required homomorphism by

$$\{\xi\} \mapsto [\xi] - d(\xi).$$

It is clear that this map is surjective and it is injective, since we know from the previous lecture that $[\xi] = [\eta]$ if and only if $\{\xi\} = \{\eta\}$. □

18.2. **Complex K-theory as a representable functor.** Let $\mathrm{Gr}_n(\mathbb{C})$ be the infinite dimensional complex Grassmannian manifold of complex n-planes. It is also common to write

$$BU(n) := \mathrm{Gr}_n(\mathbb{C}).$$

We know from Lecture 16 that there is a natural bijection

$$\mathrm{Vect}^n_{\mathbb{C}}(X) \cong [X, BU(n)]$$

where $[-,-]$ denotes homotopy classes of maps. As we have just seen base points can play a role for studying K-theory (as for any other cohomology theory). Let $[-,-]_*$ denote the set of homotopy classes of basepoint preserving maps. Then we have

$$\mathrm{Vect}^n_{\mathbb{C}}(X) \cong [X_+, BU(n)]_*.$$

The map $V \mapsto \mathbb{C} \oplus V$ defines an inclusion

$$i_n \colon BU(n) \to BU(n+1),$$

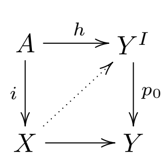
and we denote the colimit by

$$BU := \operatorname*{colim}_n BU(n)$$

with the direct limit (or union) topology.

Recall that a space is *nondegenerately based*, or *well-pointed*, if the inclusion of its basepoint is a cofibration.[1]

[1] A map $i \colon A \to X$ is a cofibration if for any commutative diagram of the form



there exists an $\tilde{h} \colon X \to Y^I$ that makes the diagram commute.

Theorem 18.4. *We endow $\mathbb{Z}$ with the discrete topology. For any compact space X, there is a natural isomorphism*

$$K(X) \cong [X_+, BU \times \mathbb{Z}]_*.$$

For a nondegenerately based compact space X, there is a natural isomorphism

$$\tilde{K}(X) \cong [X, BU \times \mathbb{Z}]_*.$$

Proof. When X is connected and ξ is an n-dimensional bundle over X with associated classifying map

$$f_\xi \colon X \to BU(n) \subset BU,$$

the first isomorphism sends

$$[\xi] - q \text{ to the pair } (f_\xi, n - q).$$

(Note that since $\mathbb{Z}$ is discrete, the map $X \to \mathbb{Z}$ must be constant.) Then we obtain also an isomorphism for non-connected spaces since both functors $K(-)$ and $[-,BU \times \mathbb{Z}]_*$ send disjoint unions to cartesian products.

For the second isomorphism follows from the first. For let $S^0 \to X_+$ be the cofibration induced by the basepoint and the disjoint basepoint. Then we can identify $d\colon K(X) \to \mathbb{Z}$ with the induced map

$$[X_+, BU \times \mathbb{Z}]_* \to [S^0, BU \times \mathbb{Z}]_*.$$

Hence we need to show that the kernel of this map is $[X, BU \times \mathbb{Z}]_*$. The cofibration $S^0 \to X_+$ with $X_+/S^0 = X$ induces an exact sequence

$$[S^1 \wedge S^0, BU \times \mathbb{Z}]_* \to [X, BU \times \mathbb{Z}]_* \to [X_+, BU \times \mathbb{Z}]_* \to [S^0, BU \times \mathbb{Z}]_*.$$

The left hand set is equal to $[S^1, BU \times \mathbb{Z}]_*$. Since we are looking at basepoint preserving maps, this is just $[S^1, BU]_+ = \pi_1(BU)$. Hence we need to show that $\pi_1(BU)$ is trivial or in other words that BU is simply connected. But $\pi_1(BU)$ is isomorphic to the set of isomorphism classes of complex vector bundles over S^1. We will show on the next problem set that this set is trivial. □

For more general, non-compact, spaces it is best to define K-theory to be the functor represented by the space $BU \times \mathbb{Z}$.

Definition 18.5. For a space X of the homotopy type of a CW-complex, we define

$$K(X) := [X_+, BU \times \mathbb{Z}]_*.$$

For a nondegenerately based space X of the homotopy type of a CW-complex, we define

$$\tilde{K}(X) \cong [X, BU \times \mathbb{Z}]_*.$$

When X is compact, we know that $K(X)$ is a ring. The following result shows that is also true for more general spaces.

Proposition 18.6. *The space $BU \times \mathbb{Z}$ is a ring space up to homotopy. This means that there are additive and multiplicative structures on $BU \times \mathbb{Z}$ such that the associativity, commutativity, and distributivity diagrams required of a ring commute up to homotopy.*

Idea of the proof. For the additive structure, note that taking direct sums induces maps for each m and n

$$\mathrm{Gr}_m(\mathbb{C}^\infty) \times \mathrm{Gr}_n(\mathbb{C}^\infty) \to \mathrm{Gr}_{m+n}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty.$$

After choosing an isomorphism $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$ we get a map

$$BU(m) \times BU(n) \to BU(m+n).$$

Taking colimits over m and n then yields a map

$$\oplus \colon BU \times BU \to BU.$$

This map is associative and commutative up to homotopy. The zero-dimensional plane provides a basepoint which is a zero element up to homotopy. Using the ordinary addition on $\mathbb{Z}$, we obtain the additive H-space structure on $BU \times \mathbb{Z}$. For multiplication, taking the tensor product of the canonical bundles induces a homotopy class of classifying maps

$$BU(m) \times BU(n) \to BU(mn).$$

With a lot more effort than for direct sums, one can show that these maps pass to colimits and define a multiplicative H-space structure on $BU \times \mathbb{Z}$. □