Math 231b
Lecture 19

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19. Lecture 19: Complex K-theory as a cohomology theory

19.1. **K-theory as a cohomology theory.** Let $\mathcal{C}$ be the category of compact Hausdorff spaces, $\mathcal{C}^+$ be the category of compact Hausdorff spaces with a distinguished basepoint, and $\mathcal{C}^2$ the category of pairs. We have defined K-theory as functors K on $\mathcal{C}$ and $\tilde{K}$ on $\mathcal{C}^+$. We extend it a functor on $\mathcal{C}^2$ by defining

$$K(X,Y) := \tilde{K}(X/Y)$$

for any pair of compact spaces (X,Y).

Definition 19.1. For $n \geq 0$, we define functors by

$$\begin{array}{lcll} \tilde{K}^{-n}(X) & = & \tilde{K}(S^n X) = \tilde{K}(S^n \wedge X) & \text{for } X \in \mathcal{C}^+ \\ K^{-n}(X,Y) & = & \tilde{K}^{-n}(X/Y) = \tilde{K}(S^n(X/Y)) & \text{for } (X,Y) \in \mathcal{C}^2 \\ K^{-n}(X) & = & \tilde{K}^{-n}(X,\emptyset) = \tilde{K}(S^n(X_+)) & \text{for } X \in \mathcal{C} \end{array}$$

which are contravariant on the appropriate categories.

Lemma 19.2. *For $(X,Y) \in \mathcal{C}^2$ we have an exact sequence*

$$K(X,Y) \xrightarrow{j^*} K(X) \xrightarrow{i^*} K(Y)$$

where $i\colon Y \to X$ and $j\colon (X,\emptyset \to (X,Y)$ are the inclusions.

Proof. We could apply the representability of K-theory of the previous lecture. But there is a very nice direct way to prove the lemma:
The composition i^*j^* is induced by the composition

$$j \circ i\colon (Y,\emptyset) \to (X,Y)$$

and so factors through the zero group $K(Y,Y)$. Thus $i^*j^* = 0$. Suppose now that $\alpha \in \mathrm{Ker}(i^*)$. We may represent α in the form $[\xi] - n$ where ξ is a vector bundle over X. Since $i^*(\alpha) = 0$ it follows that

$$[\xi|Y] = n \text{ in } K(Y).$$

This implies that for some integer m we have

$$(\xi \oplus \epsilon^m)|Y = \epsilon^n \oplus \epsilon^m,$$

i.e., we have a trivialization h of $(\xi \oplus \epsilon^m)|Y$. This defines a bundle $(\xi \oplus \epsilon^m)/h$ on X/Y in the following way. The total space is the quotient of the total space of $\xi \oplus \epsilon^m$ modulo the relation

$$h^{-1}(y,v) \sim h^{-1}(y',v) \text{ for } y, y' \in Y,$$

and the projection is just the induced quotient map. We omit the details to show that this projection map staisfies local triviality. So we can define an element

$$\alpha' = [(\xi \oplus \epsilon^m)/h] - [\epsilon^n \oplus \epsilon^m] \in \tilde{K}(X/Y) = K(X,Y).$$

Then

$$\begin{aligned} j^*(\alpha') &= [\xi \oplus \epsilon^m] - [\epsilon^n \oplus \epsilon^m] \\ &= [E] - n = \xi. \end{aligned}$$

Thus α is in the image of j^* and we have $\mathrm{Ker}(i^*) = \mathrm{Im}(j^*)$, which proves the exactness. □

Corollary 19.3. *For $(X,Y) \in \mathcal{C}^2$ and $Y \in \mathcal{C}^+$ (hence $X \in \mathcal{C}^+$ by taking the same basepoint $y_0 \in X$) the sequence*

$$K(X,Y) \xrightarrow{i^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(Y)$$

is exact.

Proof. This follows from the previous lemma and the natural isomorphisms

$$K(X) \cong \tilde{K}(X) \oplus K(y_0)$$

and

$$K(Y) \cong \tilde{K}(Y) \oplus K(y_0).$$

□

Proposition 19.4. *For $(X,Y) \in \mathcal{C}^2$ there is a natural exact sequence which extends infinitely to the left*

$$\cdots \to K^{-2}(Y) \xrightarrow{\delta} K^{-1}(X,Y) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*} K^{-1}(Y) \xrightarrow{\delta} K^0(X,Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y).$$

Proof. it suffices to show the exactness only for the sequence with terms of degree -1 and 0. Once we have done that we cann apply suspensions and extend the sequence to the left.

Let C and S denote cone and suspension respectively. Then we the following sequence of maps

$$\begin{array}{ccccccccc} Y & \hookrightarrow & X & \hookrightarrow & X \cup CY & \hookrightarrow & (X \cup CY) \cup CX & \hookrightarrow & ((X \cup CY) \cup CX) \cup C(X \cup CY) \\ & & & & \downarrow^{p} & & \downarrow & & \downarrow \\ & & & & X/Y & & SY & & SX \end{array}$$

The vertical maps are the quotient maps obtained by collapsing the most recently attached cone to a point. Now we successicely apply Corollary 19.3 to the pairs

$(X \cup CY, X)$, $((C \cup CY) \cup (CX), X \cup CY)$, and $(((X \cup CY) \cup CX), ((X \cup CY) \cup CX) \cup C(X \cup CY))$. We start with the pair $(X \cup CY, X)$. By Corollary 19.3 we get an exact sequence

$$K(X \cup CY, X) \xrightarrow{m^*} \tilde{K}(X \cup CY) \xrightarrow{k^*} \tilde{K}(X).$$

Since CY is contractible, this implies by Lemma 19.6 below that

$$p^*\colon \tilde{K}(X/Y) \to \tilde{K}(X \cup CY)$$

is an isomorphism. The composition $k^* p^*$ coincides with j^*. Let

$$\theta\colon K(X \cup CY, X) \to K^{-1}(Y) = K(S^1 \wedge Y_+)$$

be the isomorphism induced by the homeomorphisms

$$(X \cup CY)/X \approx CY/Y \approx S^1 \wedge Y_+.$$

Then defining

$$\delta\colon K^{-1}(Y) \to K(X, Y) \text{ by } \delta = m^* \theta^{-1}$$

we obtain a diagram

$$\begin{array}{ccccc}
\tilde{K}^{-1}(Y) & \xrightarrow{\quad\delta\quad} & K(X,Y) & \xrightarrow{\;j^*\;} & \tilde{K}(X) \\
\Big\downarrow{\scriptstyle \theta^{-1}} & & \Big\downarrow{\scriptstyle p^*} & & \Big\downarrow{\scriptstyle =} \\
K(X \cup CY, X) & \xrightarrow{\;m^*\;} & \tilde{K}(X \cup CY) & \xrightarrow{\;k^*\;} & \tilde{K}(X)
\end{array}$$

where the vertical maps are isomorphisms/identities. Hence we obtain the exact sequence

$$\tilde{K}^{-1}(Y) \xrightarrow{\delta} K(X, Y) \xrightarrow{j^*} \tilde{K}(X).$$

Applying the same sort of arguments to the remaining pairs yields the remaining exactness (though it is a bit more complicated than the previous case). □

Example 19.5. In particular, we see that if X is the wedge sum $A \vee B$, then $X/A = B$ and the sequence breaks up into split short exact sequences. This implies

$$\tilde{K}(X) \cong \tilde{K}(A) \oplus \tilde{K}(B).$$

Lemma 19.6. *Let $Y \subset X$ be closed contractible subspace. Then the quotient map $q\colon X \to X/Y$ induces a bijection*

$$q^*\colon \mathrm{Vect}_{\mathbb{C}}(X/Y) \to \mathrm{Vect}_{\mathbb{C}}(X).$$

Proof. Let $p\colon E \to X$ be a bundle over X. Since Y is contractible, $E|Y$ is trivial. Thus there is a trivialization h

$$h\colon E|Y \to Y \times \mathbb{C}^n.$$

Moreover, two such trivializations differ by an automorphism of $Y \times \mathbb{C}^n$, i.e., by a map $Y \to \mathrm{GL}_n(\mathbb{C})$. But $\mathrm{GL}_n(\mathbb{C})$ is connected and V is contractible. Thus h is unique up to homotopy and so the isomorphism class of E/h is uiquely determined by that of E. Thus we have constructed a map

$$\mathrm{Vect}_{\mathbb{C}}(X) \to \mathrm{Vect}_{\mathbb{C}}(X/Y)$$

and this is a two-sided inverse for q^*. □

This shows that the complex K-theory functor behaves very much like the singular cohomology functor. In fact, complex K-theory defines a complex oriented cohomology theory.

19.2. **Bott periodicity for $\tilde{K}$.** We want a version of the periodicity theorem for the reduced groups too. We start with the following observation.

Lemma 19.7. *For nondegenerately based spaces X and Y, the projections of $X \times Y$ on X and Y and the quotient map $X \times Y \to X \wedge Y$ induce a natural isomorphism*

$$\tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y) \cong \tilde{K}(X \times Y).$$

The group $\tilde{K}(X \wedge Y)$ is the kernel of the map

$$\tilde{K}(X \times Y) \to \tilde{K}(X) \oplus \tilde{K}(Y)$$

induced by the inclusions of X and Y into $X \times Y$.

Proof. The inclusions and projections make X and Y into retracts of $X \times Y$. This implies that the map

$$\tilde{K}(X \times Y) \to \tilde{K}(X) \oplus \tilde{K}(Y)$$

induced by the inclusions is a split surjection with splitting

$$\tilde{K}(X) \oplus \tilde{K}(Y) \to \tilde{K}(X \times Y),\ (a,b) \mapsto p_1^*(a) + p_2^*(b)$$

where p_1 and p_2 are the projections. The inclusion $X \vee Y \to X \times Y$ is a cofibration by our assumption on X and Y. The quotient of this map is $X \wedge Y$. This cofibration induces an exact sequence

$$\tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \to \tilde{K}(X \vee Y).$$

Since we have

$$\tilde{K}(X \vee Y) \cong \tilde{K}(X) \oplus \tilde{K}(Y)$$

this proves the lemma. □

Lemma 19.8. *The Künneth map*

$$\mu\colon K(X)\otimes K(Y)\to K(X\times Y)$$

defined by

$$\mu(a\otimes b)=(p_1^*a)(p_2^*b),$$

where p_1 and p_2 are the projections onto the two factors, induces a reduced map

$$\tilde{\mu}\colon \tilde{K}(X)\otimes\tilde{K}(Y)\to\tilde{K}(X\wedge Y).$$

Proof. For: Let $x_0\in X$ and $y_0\in Y$ be the basepoints, and let $a\in\tilde{K}(X)=\mathrm{Ker}(K(X)\to K(x_0))$ and $b\in\tilde{K}(Y)=\mathrm{Ker}(K(Y)\to K(y_0))$. Then p_1^*a restricts to zero in $K(Y)$ and p_2^*b restricts to zero in $K(X)$. Hence the product $(p_1^*a)(p_2^*b)\in K(X\times Y)$ restricts to zero in both $K(X)$ and $K(Y)$ and hence in $K(X\vee Y)$. In particular, $(p_1^*a)(p_2^*b)$ lies in $\tilde{K}(X\times Y)$. Now Lemma 19.7 implies that $(p_1^*a)(p_2^*b)$ pulls back to a unique element in $\tilde{K}(X\wedge Y)$. This defines the reduced Künneth map $\tilde{\mu}$. □

We have a reduced splitting

$$K(X)\otimes K(Y)\cong\tilde{K}(X)\otimes\tilde{K}(Y)\oplus\tilde{K}(X)\oplus\tilde{K}(Y)\oplus\mathbb{Z},$$

which is compatible with the splitting of Lemma 19.7 and shows that the reduced Künneth map is a ring homomorphism.

The unreduced version of the periodicity theorem of the previous lecture now implies the following reduced version.

Theorem 19.9. *For nondegenerately based compact spaces X, the map*

$$\tilde{\mu}\colon \tilde{K}(X)\otimes\tilde{K}(S^2)\to\tilde{K}(X\wedge S^2)$$

is an isomorphism.

Let H^* be the canonical line bundle over $\mathbb{C}\mathrm{P}^1=S^2$ and H be its dual. We know from the previous lecture

$$K(S^2)\cong\mathbb{Z}[H]/(([H]-1)^2),$$

and hence

$$\tilde{K}(S^2)\text{ is the ideal }\mathbb{Z}([H]-1).$$

Then Theorem 19.9 implies the following version of Bott periodicity.

Theorem 19.10 (Bott periodicity)**.** *For nondegenerately based compact spaces X, the map*

$$\beta\colon \tilde{K}(X)\to\tilde{K}(X\wedge S^2),\ a\mapsto\tilde{\mu}(a,[H]-1)$$

is an isomorphism.

Corollary 19.11. *We have* $\tilde{K}(S^{2n+1}) = 0$ *and* $\tilde{K}(S^{2n}) = \mathbb{Z}$*, generated by the* n*-fold recuced product* $([H]-1)^n$.