Math 231b
Lecture 23

G. Quick

23. Lecture 23: Proof of the Periodicity Theorem I

We still need to prove the periodicity theorem for complex K-theory. We will prove it in the following special form. The proof of the more general form of Lecture 17 is very similar. Let X be a compact Hausdorff space and H the canonical line bundle over $S^2 = \mathbb{C}\mathrm{P}^1$. We calculated ine one of the homework problems that we have the relation

$$(H \otimes H) \oplus 1 \cong H \oplus H,$$

or in other words, in $K(S^2)$ we have $(H^1 - 1) = 0$. This shows that there is a natural homomorphism of rings

$$\mathbb{Z}[H]/(H-1)^2 \to K(S^2).$$

Theorem 23.1. *The natural homomorphism*

$$\mu \colon K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2)$$

is an isomorphism of rings.

The proof of the theorem will occupy the rest of today's lecture and the next one. It is based on a careful analysis of the construction of complex vector bundles on $X \times S^2$ via clutching functions. In our exposition we follow Hatcher's notes. We encourage everyone to read Atiyah's original lecture notes as well.

23.1. Clutching functions.

We saw on Problem Set 4 that isomorphism classes of complex vector bundles over S^2 correspond to homotopy classes of maps

$$S^1 \to \mathrm{GL}_n(\mathbb{C}).$$

Such functions are called *clutching functions*. In the proof of Theorem 23.1 we make use of this idea to construct vector bundles over $X \times S^2$.

Let $p \colon E \to X$ be a vector bundle and let $f \colon E \times S^1 \to E \times S^1$ be an automorphism of the product vector bundle

$$p \times \mathrm{id} \colon E \times S^1 \to X \times S^1.$$

This means that for each $x \in X$ and $z \in S^1$, f specifies an isomorphism

$$f(x,z) \colon p^{-1}(x) \to p^{-1}(x).$$

From E and f we construct a vector bundle over $X \times S^2$ by taking two copies of $E \times D^2$ and identifying the subspaces $E \times S^1$ via f. We write this bundle as $[E,f]$, and call f a *clutching function* for $[E,f]$. If

$$f_t\colon E \times S^1 \to E \times S^1$$

is a homotopy of clutching functions, then we get an induced isomorphism

$$[E, f_0] \cong [E,f_1]$$

since from the homotopy f_t we can construct a vector bundle over $X \times S^2 \times I$ restricting to $[E,f_0]$ and $[E,f_1]$ over $X \times S^2 \times \{0\}$ and $X \times S^2 \times \{1\}$. It is also clear from the definitions that

$$[E_1,f_1] \oplus [E_2,f_2] \cong [E_1 \oplus E_2, f_1 \oplus f_2].$$

Let us have a look at some examples:

Example 23.2. For the identity map on S^1, $[E, \mathrm{id}]$ is just the pullback of E via the projection $X \times S^2 \to X$. As an element in $K(X \times S^2)$, $[E,\mathrm{id}]$ is equal to $\mu(E \otimes 1)$.

Example 23.3. Recall the clutching function for the canonical line bundle H over $\mathbb{C}\mathrm{P}^1$: We can write the elements $[Z_0,z_1]$ of $\mathbb{C}\mathrm{P}^1$ as ratios

$$z = z_0/z_1 \in \mathbb{C} \cup \{\infty\} = S^2.$$

Then we can write points in the disk D^2_0 inside the unit circle $S^1 \subset \mathbb{C}$ uniquely in the form

$$[z_0/z_1,1] = [z,1] \text{ with } |z| \leq 1,$$

and points in the disk D^2_∞ outside S^1 can be written uniquely in the form

$$[1,z_1/z_0] = [1,z^{-1}] \text{ with } |z^{-1}| \leq 1.$$

Over D^2_0 the map

$$[z,1] \mapsto (z,1)$$

defines a section of the canonical line bundle, and over D^2_∞ a section is

$$[1,z^{-1}] \mapsto (1,z^{-1}).$$

These sections determine trivializations of the canonical line bundle over these two disks, and over their common boundary S^1 we pass from the trivialization of D^2_∞ to the trivialization of D^2_0 by multiplying with z. Thus by taking D^2_∞ as D^2_+ and D^2_0 as D^2_- we see that the canonical line bundle has the clutching function

$$f\colon S^1 \to \mathrm{GL}_n(\mathbb{C}),\ f(z) = (z).$$

Example 23.4. a) Taking X to be a point in the previous example, we get

$$[1,z] \cong H,$$

where 1 is the trivial line bundle over the point and z means scalar multiplication by $z \in S^1 \subset \mathbb{C}$.
b) More generally, for $n \geq 0$ we have

$$[1,z^n] \cong H \otimes \cdots \otimes H = H^n.$$

Writing H^{-1} for the inverse of H with respect to the tensor product in $K(X)$, i.e., $H \otimes H^{-1} \cong 1$, we can extend this formula to negative n too. For $n \leq 0$, we have

$$[1, z^n] \cong H^{-1} \otimes \cdots \otimes H^{-1} = H^n.$$

Example 23.5. a) Now if E is a vector bundle over a compact space X, we deduce from the previous examples

$$[E,z^n] \cong \mu(E \otimes \hat{H}^n) \text{ for } n \in \mathbb{Z},$$

where $\hat{H}^n$ denotes the pullback of H^n via the projection $X \times S^2 \to S^2$.
b) More generally, if f is a clutching function we get

$$[E,z^n f] \cong [E,f] \otimes \hat{H}^n \text{ for } n \in \mathbb{Z}.$$

A key observation is that every bundle over $X \times S^2$ comes from a clutching function. More precisely:

Lemma 23.6. *Let $F \to X \times S^2$ be a vector bundle of dimension n. Then there is an n-dimensional bundle $E \to X$ and a clutching function $f\colon S^1 \to \mathrm{GL}_n(\mathbb{C})$ such that*

$$F \cong [E,f] \text{ over } X \times S^2.$$

Proof. As in Example 23.3, we consider the unit circle $S^1 \subset \mathbb{C} \cup \{\infty\} = S^2$ and decompose S^2 into the two disks D_0 and D_∞. Let F_α denote the restriction of F to $X \times D_\alpha$ for $\alpha = 0, \infty$. Now we define E to be the restriction of F to $X \times \{1\}$. Since D_α is a disk, the projection

$$X \times D_\alpha \to X \times \{1\}$$

is homotopic to the identity map of $X \times D_\alpha$, so the bundle F_α is isomorphic to the pullback of E by the projection map, and this pullback is $E \times D_\alpha$. This shows we have an isomorphism

$$h_\alpha\colon F_\alpha \to E \times D_\alpha.$$

Then we get

$$f = h_0 h_\infty^{-1} \text{ as a clutching function for } F.$$

□

Remark 23.7. We may assume that a clutching function f is *normalized* to be the identity over $X \times \{1\}$, since we may normalize any isomorphism of the form $h_\alpha \colon E_\alpha \to E \times D_\alpha$ by composing it over each $X \times \{z\}$ with the inverse of its restriction over $X \times \{1\}$.
Moreover, any two choices of normalized h_α are homotopic through normalized h_α's, since they differ by a map g_α from D_α to the automorphisms of E with $g_\alpha(1) = \mathrm{id}$, and such a g_α is homotopic to the constant map id by composing it with a deformation retraction of D_α to $*$.
Thus any two choices f_0 and f_1 of normalized clutching functions are joined by a homotopy of normalized clutching functions f_t.

We now know that clutching functions are a tool to understand all vector bundles over $X \times S^2$. The proof of Theorem 23.1 will require that we understand all possible clutching functions that are needed to construct all vector bundles over $X \times S^2$. The strategy will be to successively simplify the clutching functions.

23.2. **Laurent polynomial clutching functions.** The first step is to reduce to *Laurent polynomial* clutching functions, which have the form

$$\ell(x,z) = \sum_{|i| \leq n} a_i(x) z^i$$

where $a_i \colon E \to E$ is a map which restricts to a linear transformation $a_i(x)$ in each fiber $p^{-1}(x)$. Such an a_i will be called an *endomorphism* of E.

Note: The linear transformation $a_i(x)$ is not required to be invertible, hence the terminology. Nevertheless, the linear combination $\sum_{|i| \leq n} a_i(x) z^i$ must be invertible, since clutching functions are automorphisms.

Hence the first step is to prove the following simplification.

Proposition 23.8. *Every vector bundle $[E,f]$ is isomorphic to $[E,\ell]$ for some Laurent polynomial clutching function ℓ. Laurent polynomial clutching functions ℓ_0 and ℓ_1 which are homotopic through clutching functions are homotopic by a Laurent polynomial clutching function homotopy*

$$\ell_t(x,z) \sum_{|i| \leq n} a_i(x,t) z^i.$$

The proof is based on the fact that on a compact space X, we can approximate continuous functions $f \colon X \times S^1 \to \mathbb{C}$ by Laurent polynomial functions of the form

$$\sum_{|n| \leq N} a_n(x) z^n = \sum_{|n| \leq N} a_n(x) e^{in\theta},$$

where $z = e^{i\theta} \in S^1$ and each a_n is a continuous function $X \to \mathbb{C}$. Motivated by Fourier series, we set

$$a_n(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x,e^{i\theta}) e^{-in\theta} d\theta.$$

For positive real r, consider the series

$$u(x,r,\theta) = \sum_{n \in \mathbb{Z}} a_n(x) r^{|n|} e^{in\theta}.$$

For fixed $r < 1$, this series converges absolutely and uniformly as (x,θ) ranges over $X \times [0,2\pi]$. This follows from the fact that the geometric series

$$\sum_n r^n$$

converges, and, since $X \times S^1$ is compact,

$$|f(x, e^{i\theta})| \text{ is bounded and hence also } |a_n(x)|.$$

Now we need to show that $u(x,r,\theta)$ approaches $f(x,e^{i\theta})$ uniformly in x and θ as r goes to 1. For then sums of finitely many terms in the series for $u(r,x,\theta)$ with r near 1 will give the desired approximations to f by Laurent polynomial functions. Hence we need the following lemma.

Lemma 23.9. *As $r \to 1$, $u(r,x,\theta) \to f(x,e^{i\theta})$ uniformly in x and θ.*

Proof. For $r < 1$ we have

$$\begin{aligned} u(x,r,\theta) &= \textstyle\sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} r^{|n|} e^{in(\theta - t)} f(x,e^{it}) dt \\ &= \textstyle\int_0^{2\pi} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - t)} f(x,e^{it}) dt \end{aligned}$$

where the order of summation and integration can be interchanged since the series in the latter formula converges uniformly, by comparison with the geometric series $\sum_n r^n$. Define the Poisson kernel

$$P(r,\varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\varphi} \text{ for } 0 \le r \le 1 \text{ and } \varphi \in \mathbb{R}.$$

Then we have

$$u(r,x,\theta) = \int_0^{2\pi} P(r, \theta - t) f(x,e^{it}) dt.$$

By summing the two geometric series for positive and negative n in the formula for $P(r,\varphi)$, one computes that

$$P(r,\varphi) = \frac{1}{2\pi}[1 - \frac{1}{1 - re^{i\varphi}} + \frac{1}{1 - re^{-i\varphi}}] = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\varphi + r^2},$$

where one uses the formula

$$e^{i\varphi} + e^{-i\varphi} = 2\cos\varphi.$$

We will need three facts about $P(r,\varphi)$:

(a) As a function of φ, $P(r,\varphi)$ is even, of period 2π, and monotone decreasing on $[0,\pi]$, since the same is true for $\cos\varphi$ which appears in the denominator of $P(r,\varphi)$ with a minus sign. In particular, we have

$$P(r,\varphi) \geq P(r,\pi) > 0 \text{ for all } r < 1.$$

(b) $\int_0^{2\pi} P(r,\varphi)d\varphi = 1$ for each $r < 1$. This follows from integrating the series for

$$P(r,\varphi) = \frac{1}{2\pi}[1 + 2\sum_{n=1}^{\infty}\cos(n\varphi)]$$

term by term (the integral over all terms in the sum yield 0 and the integral over 1 yields 2π).

(c) For fixed $\varphi \in (0,\pi)$, $P(r,\varphi) \to 0$, since the numerator of $P(r,\varphi)$ approaches 0 and the denominator approaches $2 - 2\cos\varphi \neq 0$.

Now to show uniform convergence of $u(r,x,\theta)$ to $f(x,e^{i\theta})$ we first observe that, using (b), we have

$$\begin{aligned} |u(x,r,\theta) - f(x,e^{i\theta})| &= |\textstyle\int_0^{2\pi} P(r,\theta-t)f(x,e^{it})dt - \int_0^{2\pi} P(r,\theta-t)f(x,e^{i\theta})dt| \\ &\leq \textstyle\int_0^{2\pi} P(r,\theta-t)|f(x,e^{it}) - f(x,e^{i\theta})|dt. \end{aligned}$$

Given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x,e^{it}) - f(x,e^{i\theta})| < \epsilon \text{ for } |t-\theta| < \delta \text{ and all } x,$$

since f is uniformly continuous on the compact space $X \times S^1$. Let I_δ denote the integral

$$\int_0^{2\pi} P(r,\theta-t)|f(x,e^{it}) - f(x,e^{i\theta})|dt \text{ over the interval } |t-\theta| \leq \delta,$$

and let I'_δ denote this integral over the complement of the interval $|t-\theta| \leq \delta$ in an interval of length 2π. Then we have

$$I_\delta \leq \int_{|t-\theta|\leq\delta} P(r,\theta-t)\epsilon dt \leq \epsilon\int_0^{2\pi} P(r,\theta-t)dt = \epsilon.$$

By (a) the maximum value of $P(r,\theta-t)$ on $|t-\theta| \geq \delta$ is $P(r,\delta)$. Hence

$$I'_\delta \leq P(r,\delta)\int_0^{2\pi} |f(x,e^{it}) - f(x,e^{i\theta})|dt.$$

The integral here as a uniform bound for all x and θ since f is bounded. Thus by (c) we can make

$$I'_\delta \leq \epsilon \text{ by taking } r \text{ close enough to 1.}$$

Therefore

$$|u(x,r,\theta) - f(x,\theta)| \leq I_\delta + I'_\delta \leq 2\epsilon.$$

□

Now we are ready for the proof of the proposition.

Proof of Proposition 23.8. Choosing a Hermitian inner product on E, the endomorphisms of $E \times S^1$ form a vector space $\mathrm{End}(E \times S^1)$ with a norm

$$\|\alpha\| = \sup_{|v|=1} |\alpha(v)|.$$

Note that the triangle inequality holds for the sup-norm, so balls in $\mathrm{End}(E \times S^1)$ are convex. The subspace $\mathrm{Aut}(E \times S^1)$ of automorphisms is open in the topology defined by this norm since it is the preimage of $(0,\infty)$ under the continuous map

$$\mathrm{End}(E \times S^1) \to [0,\infty),\ \alpha \mapsto \inf_{(x,z)\in X\times S^1} |\det(\alpha(x,z))|.$$

Hence in order to prove the first statement of the proposition it will suffice to show that the Laurent polynomials are dense in $\mathrm{End}(E \times S^1)$, since a sufficiently close Laurent polynomial approximation ℓ to f will then be homotopic to f via the linear homotopy

$$t\ell + (1-t)f \text{ through clutching functions}$$

which is in $\mathrm{Aut}(E \times S^1)$ for all $0 \leq t \leq 1$. Hence f is homotopic to ℓ in $\mathrm{Aut}(E \times S^1)$ and

$$[E,f] \cong [E,\ell].$$

The second statement follows similarly by approximating a homotopy from ℓ_0 to ℓ_1, viewed as an automorphism of $E \times S^1 \times I$ by a Laurent polynomial homotopy ℓ'_t. Then we can combine these approximations with linear homotopies from ℓ_0 to ℓ'_0 and ℓ_1 to ℓ'_1 to obtain a homotopy ℓ_t from ℓ_0 to ℓ_1.

Hence we need to show that every $f \in \mathrm{End}(E \times S^1)$ can be approximated by Laurent polynomial endomorphisms. Therefor we choose open sets U_i covering X together with isomorphisms

$$h_i \colon p^{-1}(U_i) \to U_i \times \mathbb{C}^{n_i}.$$

We may assume that h_i takes the chosen inner product in $p^{-1}(U_i)$ to the standard inner product in $\mathbb{C}^{n_i}$, by applying the Gram-Schmidt process to h_i^{-1} of the standard basis vectors.

Let $\{\phi_i\}$ be a partition of unity subordinate to $\{U_i\}$ an let $\{X_i\}$ be the support of ϕ_i. Since X is compact, we cann choose $\{\phi_i\}$ such that each X_i is a compact subset in U_i. Via h_i, the linear maps $f(x,z)$ for $x \in X_i$ can be viewed as matrices. The entries of these matrices define functions

$$X_i \times S^1 \to \mathbb{C}.$$

Applying Lemma 23.9 to each entry of the matrices, we can find Laurent polynomial matrices $\ell_i(x,z)$ whose entries uniformly approximate those of $f(x,z)$ for $x \in X_i$. It follows that ℓ_i approximates f in the $\|\cdot\|$-norm, since the entries are uniformly approximated. From the Laurent polynomial approximations ℓ_i over X_i we form the convex linear combination

$$\ell = \sum_i \phi_i \ell_i,$$

which is a Laurent polynomial approximating f over all of $X \times S^1$. □