Math 231b
Lecture 24

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24. Lecture 24: Proof of the Periodicity Theorem II

We continue the sketch of the proof of the periodicity theorem for complex K-theory.

Theorem 24.1. *The natural homomorphism*

$$\mu\colon K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2)$$

is an isomorphism of rings.

The proof on a careful analysis of the construction of complex vector bundles on $X \times S^2$ via clutching functions. We conclude the proof today with an outline of the ideas.

24.1. Laurent polynomial clutching functions.

The first step is to reduce to *Laurent polynomial* clutching functions, which have the form

$$\ell(x,z) = \sum_{|i| \le n} a_i(x) z^i$$

where $a_i\colon E \to E$ is a map which restricts to a linear transformation $a_i(x)$ in each fiber $p^{-1}(x)$. Such an a_i will be called an *endomorphism* of E.

Note: The linear transformation $a_i(x)$ is not required to be invertible, hence the terminology. Nevertheless, the linear combination $\sum_{|i|\le n} a_i(x)z^i$ must be invertible, since clutching functions are automorphisms.

Hence the first step is to prove the following simplification.

Proposition 24.2. *Every vector bundle $[E,f]$ is isomorphic to $[E,\ell]$ for some Laurent polynomial clutching function ℓ. Laurent polynomial clutching functions ℓ_0 and ℓ_1 which are homotopic through clutching functions are homotopic by a Laurent polynomial clutching function homotopy*

$$\ell_t(x,z) \sum_{|i| \le n} a_i(x,t) z^i.$$

The proof is based on the fact that on a compact space X, we can approximate continuous functions $f\colon X\times S^1 \to \mathbb{C}$ by Laurent polynomial functions of the form

$$\sum_{|n|\le N} a_n(x)z^n = \sum_{|n|\le N} a_n(x)e^{in\theta},$$

where $z = e^{i\theta} \in S^1$ and each a_n is a continuous function $X \to \mathbb{C}$. Motivated by Fourier series, we set

$$a_n(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x,e^{i\theta})e^{-in\theta}d\theta.$$

For positive real r, consider the series

$$u(x,r,\theta) = \sum_{n\in\mathbb{Z}} a_n(x)r^{|n|}e^{in\theta}.$$

For fixed $r < 1$, this series converges absolutely and uniformly as (x,θ) ranges over $X \times [0,2\pi]$. This follows from the fact that the geometric series

$$\sum_n r^n$$

converges, and, since $X \times S^1$ is compact,

$$|f(x, e^{i\theta})| \text{ is bounded and hence also } |a_n(x)|.$$

Now we need to show that $u(x,r,\theta)$ approaches $f(x,e^{i\theta})$ uniformly in x and θ as r goes to 1. For then sums of finitely many terms in the series for $u(r,x,\theta)$ with r near 1 will give the desired approximations to f by Laurent polynomial functions. The proof of the following lemma can be found in the notes of the previous lecture (and of course in Hatcher's lecture notes).

Lemma 24.3. *As $r \to 1$, $u(r,x,\theta) \to f(x,e^{i\theta})$ uniformly in x and θ.*

Now we are ready for the proof of the proposition.

Proof of Proposition 24.2. Choosing a Hermitian inner product on E, the endomorphisms of $E \times S^1$ form a vector space $\mathrm{End}(E \times S^1)$ with a norm

$$\|\alpha\| = \sup_{|v|=1} |\alpha(v)|.$$

Note that the triangle inequality holds for the sup-norm, so balls in $\mathrm{End}(E\times S^1)$ are convex. The subspace $\mathrm{Aut}(E\times S^1)$ of automorphisms is open in the topology defined by this norm since it is the preimage of $(0,\infty)$ under the continuous map

$$\mathrm{End}(E \times S^1) \to [0,\infty),\ \alpha \mapsto \inf_{(x,z)\in X\times S^1} |\det(\alpha(x,z))|.$$

Hence in order to prove the first statement of the proposition it will suffice to show that the Laurent polynomials are dense in $\mathrm{End}(E \times S^1)$, since a sufficiently close Laurent polynomial approximation ℓ to f will then be homotopic to f via the linear homotopy

$$t\ell + (1-t)f \text{ through clutching functions}$$

which is in $\mathrm{Aut}(E\times S^1)$ for all $0 \leq t \leq 1$. Hence f is homotopic to ℓ in $\mathrm{Aut}(E\times S^1)$ and

$$[E,f] \cong [E,\ell].$$

The second statement follows similarly by approximating a homotopy from ℓ_0 to ℓ_1, viewed as an automorphism of $E \times S^1 \times I$ by a Laurent polynomial homotopy ℓ'_t. Then we can combine these approximations with linear homotopies from ℓ_0 to ℓ'_0 and ℓ_1 to ℓ'_1 to obtain a homotopy ℓ_t from ℓ_0 to ℓ_1.

Hence we need to show that every $f \in \mathrm{End}(E \times S^1)$ can be approximated by Laurent polynomial endomorphisms. Therefor we choose open sets U_i covering X together with isomorphisms

$$h_i \colon p^{-1}(U_i) \to U_i \times \mathbb{C}^{n_i}.$$

We may assume that h_i takes the chosen inner product in $p^{-1}(U_i)$ to the standard inner product in $\mathbb{C}^{n_i}$, by applying the Gram-Schmidt process to h_i^{-1} of the standard basis vectors.

Let $\{\phi_i\}$ be a partition of unity subordinate to $\{U_i\}$ an let $\{X_i\}$ be the support of ϕ_i, which is a compact subset in U_i. Via h_i, the linear maps $f(x,z)$ for $x \in X_i$ can be viewed as matrices. The entries of these matrices define functions $X_i \times S^1 \to \mathbb{C}$. Applying Lemma 24.3 to each entry of the matrices, we can find Laurent polynomial matrices $\ell_i(x,z)$ whose entries uniformly approximate those of $f(x,z)$ for $x \in X_i$. It follows that ℓ_i approximates f in the $\|\cdot\|$-norm, since the entries are uniformly approximated. From the Laurent polynomial approximations ℓ_i over X_i we form the convex linear combination

$$\ell = \sum_i \phi_i \ell_i,$$

which is a Laurent polynomial approximating f over all of $X \times S^1$. □

Now we are reduced to Laurent polynomial clutching functions. In fact, we are reduced to polynomial clutching functions, since if ℓ is a Laurent polynomial we can write it as

$$\ell = z^{-m} q \text{ for a polynomial function q and some } m.$$

Then we get

$$[E,\ell] \cong [E,q] \otimes \hat{H}^{-m}.$$

The next step is to simplify from polynomials to linear clutching functions.

Proposition 24.4. *If q is a polynomial clutching function of degree at most n, then*

$$[E,q] \oplus [nE,\mathrm{id}] \cong [(n+1)E, L^n q] \text{ for a linear clutching function } L^n q.$$

Proof. Let

$$q(x,z) = a_n(x)z^n + \cdots + a_0(x).$$

Each of the matrices

$$A = \begin{pmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & q \end{pmatrix}$$

defines an endomorphism of $(n+1)E$ by interpreting the (i,j)-entry of the matrix as a linear map from the jth summand of $(n+1)E$ to the ith summand, with the entries 1 denoting the identity $E \to E$ and z denoting z times the identity, for $z \in S^1$.

Now we define the sequence $q_r(z) = q_r(x,z)$ inductively by

$$q_0 = q, \; zq_{r+1}(z) = q_r(z) - q_r(0).$$

Then we have the following matrix identity:

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ q_1 & 1 & 0 & \cdots & 0 & 0 \\ q_2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ q_{n-1} & 0 & 0 & \cdots & 1 & 0 \\ q_n & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & q \end{pmatrix} \begin{pmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

We can rewrite this identity as

$$A = (1 + N_1)B(1 + N_2) \tag{1}$$

where N_1 and N_2 are nilpotent. If N is nilpotent, then $1 + tN$ is an invertible matrix for $0 \le t \le 1$. Since matrix B defines a clutching function for

$$[E,q] \oplus [nE, \mathrm{id}],$$

it is invertible in each fiber. Hence (1) shows that A is invertible in each fiber. Thus A defines an automorphism of $(n+1)E$ for each $z \in S^1$ and therefore a clutching function which we denote by $L^n q$. Since $L^n q$ has the form

$$L^n q(x,z) = a(x)z + b(x),$$

Moreover, it follows from (1) that A and B define homotopic clutching functions. Hence we obtain an isomorphism of vector bundles:

$$[E,q] \oplus [nE, \mathrm{id}] \cong [(n+1)E, L^n q].$$

□

24.2. **Linear clutching functions.** For linear clutching functions we have the following key fact:

Proposition 24.5. *Let $a, b \in \mathrm{End}(E)$ and assume we are given a bundle $[E, a(x)z + b(x)]$. Then there is a splitting $E \cong E_- \oplus E_+$ with*

$$[E, a(x)z + b(x)] \cong [E_+, z] \oplus [E_-, \mathrm{id}] (\cong E_+ \otimes H \oplus E_-).$$

To prepare the proof of the proposition, we start with a brief side discussion. Let T be an endomorphism of a finite dimensional vector space E, and let S be a circle in the complex plane which does not pass through any eigenvalue of T. Then

$$Q = \frac{1}{2\pi i} \int_S (z - T)^{-1} dz$$

is a projection operator in E, i.e., $Q^2 = Q$, which commutes with T. This induces a decomposition

$$E = E_+ \oplus E_-, \; E_+ = QE \text{ and } E_- = (1 - Q)E,$$

which is invariant under T. Hence T can be written as

$$T = T_+ \oplus T_-.$$

Moreover, the eigenvalues of T_+ are all inside S, while the eigenvalues of T_- are all outside of S.

Sketch of a proof of Proposition 24.5. For $a,b \in \mathrm{End}(E)$, write $p(x) = a(x)z + b(x)$. Since $a(x)z + b(x)$ is invertible for all x, $b(x)$ has no eigenvalues on the unit circle S^1. We define an endomorphism of E by

$$Q = \frac{1}{2\pi i} \int_{|z|=1} (az + b)^{-1} a dz.$$

(Hence Q defines a linear transformation on each fiber E_x of E.) It is even a projection operator. Moreover, Q commutes with a and b. Now one defines

$$E_+ = QE \text{ and } E_- = (1-Q)E.$$

Now one has to check that E_+ and E_- inherit a vector bundle structure from E. Once this is done, we get a decomposition

$$E \cong E_+ \oplus E_-$$

and our endomorphisms induce endomorphisms

$$p_+ = a_+z + b_+ \in \mathrm{End}(E_+ \times S^1) \text{ and } p_- = a_-z + b_- \in \mathrm{End}(E_- \times S^1).$$

Moreover, a_+ and b_- are isomorphisms (and so are a_- and b_+.) Setting

$$p^t = p_+^t + p_-^t, \text{ where } p_+^t = a_+z + tb_+,\ p_-^t = ta_-z + b_-,\ 0 \le t \le 1,$$

we obtain isomorphisms

$$\begin{aligned} [E,p] &\cong [E, a_+z + b_-] \text{ from the homotopies above} \\ &\cong [E_+,a_+z] \oplus [E_-,b_-] \\ &\cong [E_+,z] \oplus [E_-,\mathrm{id}] \text{ since } a_+z \sim z \text{ and } b_- \sim \mathrm{id} \end{aligned}$$

□

24.3. **Proof of the Periodicity Theorem.** As a consequence of the previous discussion we obtain that for every vector bundle F over $X \times S^2$ there is an integer $n \ge 0$ and bundles E_1, E_2 and E_3 over X such that

$$F \otimes H^n \oplus \pi^*E_1 \cong \pi^*E_2 \otimes H \oplus \pi^*E_3,$$

where $\pi\colon X \times S^2 \to X$ is the projection.

Moreover, the homotopy of clutching functions

$$\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

implies

$$H^2 \oplus 1 = H \oplus H.$$

Hence we have

$$([H]-1)^2 = ([H]^{-1}-1)^2 = 0 \text{ in } K(X \times S^2).$$

Finally, this implies that every element ξ in $K(X \times S^2)$ can be written as

$$\xi = \pi^*\xi_1 + \pi^*\xi_2^2 \cdot ([H]-1)$$

with $\xi_1, \xi_2 \in K(X)$. This shows the surjectivity statement of the Periodicity Theorem.

The injectivity can then be proved by showing that the elements ξ_1 and ξ_2 are in fact unique in $K(X)$. One has to check that all the choices we made during the constructions did not matter. We omit the careful analysis that is necessary to do this. We refer to Atiyah's book orHatcher's lecture notes for more details.

In the end, the Periodicity Theorem tells us that $K(X \times S^2)$ is a free $K(X)$-module with generators 1 and $[H] - 1$. The ring structure on $K(X \times S^2)$ is determined by the single relation $([H] - 1)^2 = 0$.