Math 231b
Lecture 25

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25. Lecture 25: Adams operations in complex K-theory

There are very important ring homomorphisms in complex K-theory, called *Adams operations*. Today we are going to see how they can be defined and that they have the following properties:

Theorem 25.1. *For each non-zero integer k and each compact Hausdorff space X, there is a ring homomorphism*

$$\psi^k \colon K(X) \to K(X)$$

satisfying the following properties:

(1) $\psi^1 = \mathrm{id}$ *and* ψ^{-1} *is induced by conjugation of complex bundles.*
(2) $\psi^k f^* = f^* \psi^k$ *for all maps* $f \colon X \to Y$*, i.e., the* ψ^k *are natural homomorphisms.*
(3) $\psi^k(L) = L^k = L \otimes \cdots \otimes L$ *if* L *is a line bundle.*
(4) $\psi^k \circ \psi^\ell = \psi^{k\ell}$.
(5) $\psi^p(\alpha) \equiv \alpha^p$ *modulo* p *for a prime* p
(6) If X *is a based space, then, by the naturality property (2), each* ψ^k *restricts to an operation*

$$\psi^k \colon \tilde{K}(X) \to \tilde{K}(X),$$

since $\tilde{K}(X)$ *is the kernel of the homomorphism* $K(X) \to K(x_0)$.
For $2n$*-spheres, the Adams operations act as*

$$\psi^k(x) = k^n x \text{ for } x \in \tilde{K}(S^{2n}).$$

The proof of the theorem will occupy the rest of today's lecture.

First of all, if we impose property (4), $\psi^{-k} = \psi^k \psi^{-1}$, and use (1) to define ψ^{-1}, we only need to construct the ψ^k for $k > 1$.

By extending the construction from vector spaces to bundles we can form an exterior power $\lambda^k(E)$ which has the following properties:

(i) $\lambda^k(E_1 \oplus E_2) \cong \oplus_{i+j=k} \lambda^i(E_1) \otimes \lambda^j(E_2)$.
(ii) $\lambda^0(E) = 1$, the trivial line bundle.
(ii) $\lambda^1(E) = E$.
(iv) $\lambda^k(E) = 0$ for k greater than the maximum dimension of the fibers of E.

Lemma 25.2. *The λ^k extend to operations on K-theory*

$$\lambda^k \colon K(X) \to K(X).$$

Proof. Consider the multiplicative group G of power series with constant term 1 in the ring $K(X)[[t]]$ of formal power series in the variable t. We define a function from equivalence classes of vector bundles to this abelian group by setting

$$\Lambda(E) := 1 + \lambda^1(E)t + \cdots + \lambda^k(E)t^k + \cdots .$$

Property (i) above implies

$$\Lambda(E_1 \oplus E_2) = \Lambda(E_1)\Lambda(E_2).$$

This means that Λ is a morphism of monoids and hence induces a homomorphism of groups

$$\Lambda \colon K(X) \to G.$$

We define

$$\lambda^k(x) \text{ to be the coefficient of } t^k \text{ in } \Lambda(x).$$

□

Back to the Adams operations. Let us consider the special case of a vector bundle E which is a sum of line bundles L_i. Then properties (3) and (4) give us a formula

$$\psi^k(L_1 + \cdots + L_n) = L_1^k + \cdots + L_n^k.$$

The construction of the ψ^k will be based on showing that there is a polynomial Q_k with integral coefficients with

$$L_1^k + \cdots + L_n^k = Q_k(\lambda^1(E), \ldots, \lambda^k(E)).$$

This leads us to define

$$\psi^k(E) = Q_k(\lambda^1(E), \ldots, \lambda^k(E))$$

for arbitrary E.

So we need to find these polynomials Q_k. Therefor we consider the polynomial algebra $\mathbb{Z}[x_1, \ldots, x_n]$ and let

$$\sigma_i = x_1 x_2 \cdots x_i + \cdots$$

be the ith elementary symmetric function in the x_i's. The σ_i's form a subring

$$\mathbb{Z}[\sigma_1, \ldots, \sigma_n] \subset \mathbb{Z}[x_1, \ldots, x_n],$$

and satisfy

$$(1 + x_1) \cdots (1 + x_n) = 1 + \sigma_1 + \cdots + \sigma_n.$$

The crucial property for us is that every *symmetric* polynomial of degree k in $x_1, \dots, x_n$ can be expressed as a unique polynomial in $\sigma_1, \dots, \sigma_k$. In particular, there is a polynomial Q_k such that

$$Q_k(\sigma_1, \dots, \sigma_k) = x_1^k + \cdots + x_n^k. \tag{1}$$

Moreover, this Q_k is independent of n as long $k \leq n$, since we can pass from n to $n-1$ by setting $x_n = 0$.

Lemma 25.3. *The Q_k satisfy the recursive formula*

$$Q_k = \sigma_1 Q_{k-1} - \sigma_2 Q_{k-2} + \cdots + (-1)^{k-2}\sigma_{k-1}Q_1 + (-1)^{k-1}k\sigma_k.$$

Proof. This is an exercise. □

The lemma yields for example

$$Q_1 = \sigma_1,\ Q_2 = \sigma_1^2 - 2\sigma_2,\ Q_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

Lemma 25.4. *For $E = L_1 + \cdots + L_n$:*

$$L_1^k + \cdots + L_n^k = Q_k(\lambda^1(E), \dots, \lambda^k(E)).$$

Proof. The assumption on E implies

$$\Lambda(E) = \prod_i \Lambda(L_i) = \prod_i (1 + \lambda^1(L_i)t) = \prod_i (1 + L_i t).$$

When we compute the product we see that the coefficient $\lambda^i(E)$ of t^i in $\Lambda(E)$ satisfies

$$\lambda^i(E) = \sigma_i(L_1, \dots, L_n).$$

Substituting L_i for x_i in (1) now yields the assertion. □

Now we can define ψ^k.

Definition 25.5. For every element ξ in $K(X)$ we define

$$\psi^k(\xi) = Q_k(\lambda^1(\xi), \dots, \lambda^k(\xi)).$$

Now we need to show that the ψ^k's satisfy the properties of the theorem. To do this we will use the following fact, known as the Splitting Principle, which is very useful for proving all kinds of statements in $K(X)$.

Theorem 25.6. *Given a vector bundle $E \to X$ over a compact Hausdorff space X, there is a compact Hausdorff space $F(E)$ and a map $p\colon F(E) \to X$ such that the induced map $p^*\colon K^*(X) \to K^*(F(E))$ is injective and $p^*(E)$ splits as a sum of line bundles.*

Using Theorem 25.6 we finish the proof of Theorem 25.1:

(1) holds by definition for ψ^{-1} and follows from $Q_1 = \sigma_1$ and Theorem 25.6 for ψ^1.

(2) follows from the naturality of λ^k, i.e., $f^*(\lambda^i(E)) = \lambda^i(f^*(E))$.

(3) If $E = L$ is a line bundle, then $\lambda^1(L) = L$ and $\lambda^k(L) = 0$ for $k \geq 2$. Hence

$$\psi^k(L) = Q_k(L) = L^k.$$

For it follows from Lemma 25.3 that $Q_k \equiv \sigma_1^k$ modulo terms in the ideal generated by the σ_i's for $i > 1$.

Additivity: Let E and F be vector bundles over X. By (2) and Theorem 25.6 we take a pullback to split E and then take another pullback to split F as sums of line bundles. But then the identity

$$\psi^k(L_1 + \cdots + L_n) = L_1^k + \cdots L_n^k$$

shows us that ψ^k is additive for sums of line bundles. The injectivity statement of Theorem 25.6 implies that we have

$$\psi^k(E \oplus F) = \psi^k(E) + \psi^k(F).$$

This implies that ψ^k is an additive map $K(X) \to K(X)$.

Multiplicativity: Let E and F be vector bundles over X. By (2) and Theorem 25.6 we take a pullback to split E of line bundles L_i's and then take another pullback to split F as sums of line bundles M_j's. Then $E \otimes F$ is a sum of line bundles $L_i \otimes M_j$. Hence

$$\psi^k(E \otimes F) = \sum_{i,j} \psi^k(L_i \otimes M_j) = \sum_{i,j} (L_i \otimes M_j)^k = \sum_i L_i^k \sum_j M_j^k = \psi^k(E)\psi^k(F).$$

This implies that ψ^k is a multiplicative map $K(X) \to K(X)$.

(4) Theorem 25.6 and Additivity reduce us to the case $E = L$ a line bundle. But in this case we know

$$\psi^k(\psi^\ell(L)) = L^{k\ell} = \psi^{k\ell}(L).$$

(5) Once again we can assume $E = L_1 + \cdots + L_n$. Then

$$\psi^p(E) = L_1^p + \cdots + L_n^p \equiv (L_1 + \cdots + L_n)^p = E^p \text{ modulo } p.$$

(6) We know from before that $\tilde{K}(S^2)$ is generated by $1 - [H]$ with $(1 - [H])^2 = 0$. By additivity, we know

$$\psi^k(1 - [H]) = 1 - [H]^k.$$

By induction on k, one sees $1 - [H]^k = k(1 - [H])$. For

$$1 - [H]^k = (1 - [H]^{k-1})[H] + (1 - [H]) = (k-1)(1 - [H]) + (1 - [H]) = k(1 - [H]).$$

This shows the formula for S^2. Now we use that

$$S^{2n} = S^2 \wedge \cdots \wedge S^2$$

and $\tilde{K}(S^{2n})$ is generated by the k-fold tensor power

$$(1 - [H]) \otimes \cdots \otimes (1 - [H]).$$

Now (6) follows from the multiplicativity of ψ^k.