Math 231b
Lecture 26

G. Quick

26. Lecture 26: The Hopf invariant one problem via K-theory

We return to one of our initial problems and answer the question for which n there can be a division algebra structure on $\mathbb{R}^n$. The answer to this question will follow from the solution of a famous problem in algebraic topology, the *Hopf invariant one problem*.

26.1. **The Hopf invariant.** For $n \geq 2$, let S^n be an oriented n-sphere. Assume we are given a pointed map $f\colon S^{2n-1} \to S^n$. Considering S^{2n-1} as the boundary of an oriented $2n$-cell, we can form the cell complex $X = X_f = S^n \cup_f e^{2n}$, the *cofiber of* f. It is the complex formed from the disjoint union of S^n and e^{2n} by identifying each point in $S^{2n-1} = \mathring{e}^{2n}$ with its image under f. The cell complex X has a single vertex, a single n-cell and a single $2n$-cell.

Let
$$\pi\colon X \to X/S^n \cong S^{2n}$$
be the quotient map that collapses S^n. It fits into a sequence
$$S^{2n-1} \xrightarrow{f} S^n \xrightarrow{i} X \xrightarrow{\pi} S^{2n} \xrightarrow{\Sigma f} S^{n+1}.$$
Now we specialize to the case that n is *even* and form the long exact sequence in reduced K-theory of the pair (X, S^n). Since
$$\tilde{K}^1(S^{2n}) = \tilde{K}^1(S^n) = 0$$
we obtain a short exact sequence
$$0 \to \tilde{K}(S^{2n}) \xrightarrow{\pi^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(S^n) \to 0. \tag{1}$$

Let i_n be a generator of $\tilde{K}(S^n)$ and i_{2n} be a generator of $\tilde{K}(S^{2n})$. Choose an element
$$a \in \tilde{K}(X) \text{ such that } i^*(a) = i_n \text{ and let } b = \pi^*(i_{2n}) \in \tilde{K}(X).$$
The sequence (1) shows that $\tilde{K}(X)$ is a free abelian with generators a and b, since
$$\tilde{K}(S^{2n}) \cong \tilde{K}(S^n) \cong \mathbb{Z}.$$

Since any square in $\tilde{K}(S^n)$ vanishes we have $i_n^2 = 0$. Hence
$$a^2 = h(f) \cdot b \text{ for some integer } h(f).$$

Lemma 26.1. *The integer $h(f)$ is well-defined.*

Proof. We need to show that $h := h(f)$ does not depend on the choice of a. Because of the exactness of (1), a is unique up to adding a multiple of b. Moreover,

$$(a+mb)^2 = a^2 + 2m \cdot a \cdot b, \text{ since } b^2 = \pi^*(i_{2n}^2) = 0.$$

Hence it suffices to show $a \cdot b = 0$. Since b maps to 0 in $\tilde{K}(S^n)$, so does $a \cdot b$. Hence

$$a \cdot b = k \cdot b \text{ for some integer } k.$$

Multiplying the equation $k \cdot b = b \cdot a$ on the right by a gives

$$k \cdot b \cdot a = b \cdot a^2 = b \cdot h \cdot b = h \cdot b^2 = 0 \text{ since } b^2 = 0.$$

Thus $k \cdot b \cdot a = 0$, which implies $a \cdot b = 0$ since $a \cdot b$ lies in the image of $\tilde{K}(S^{2n})$ in $\tilde{K}(X)$ which is an infinite cyclic subgroup of $\tilde{K}(X)$. $\square$

Definition 26.2. The *Hopf invariant* of f is the integer $h(f)$.

Example 26.3. If n is 2, 4, or 8, there exists a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant one. For $n = 2$, f may be taken as the natural projection

$$f\colon S^3 \to S^2 = \mathbb{C}\mathrm{P}^1,$$

viewing S^3 as the unit sphere in the complex plane $\mathbb{C}^2$. Such an f is the attaching map in the complex projective plane

$$\mathbb{C}\mathrm{P}^2 = S^2 \cup_f e^4.$$

Then we have $h(f) = 1$, since $\tilde{K}(\mathbb{C}\mathrm{P}^2) \cong \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot a^2$, and hence the generator b is exactly a^2.

The cases $n = 4$ and $n = 8$ correspond to the quaternionic plane and the Cayley plane, respectively. We will get back to these examples later.

Remark 26.4. The Hopf invariant is usually defined using integral cohomology groups. But we will show later that both definitions yield the same number. Using the cohomological definition it is clear that, if n is odd, then $h(f) = 0$ for all f. So n even is the only interesting case and our initial reduction to that case is not really a restriction.

Remark 26.5. The homotopy type of X depends only on the homotopy class of the map f. Thus $h(f)$ only depends on the homotopy class of f. We may therefore speak of the Hopf invariant of a homotopy class and consider h as a function

$$h\colon \pi_{2n-1}(S^n) \to \mathbb{Z}.$$

The Hopf invariant has the following properties.

Proposition 26.6. *Let $n \geq 2$ be an even integer. The Hopf invariant has the following properties:*

(1) *If $g\colon S^{2n-1} \to S^{2n-1}$ has degree d, then $h(f \circ g) = d \cdot h(f)$.*
(2) *If $e\colon S^n \to S^n$ has degree d, then $h(e \circ f) = d^2 \cdot h(f)$.*
(3) *There exists a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant two.*
(4) *The Hopf invariant defines a homomorphism of groups $\pi_{2n-1}(S^n) \to \mathbb{Z}$.*

We will postpone the proof of the proposition. We just mention an immediate consequence for the structure of the homotopy groups of spheres.

Corollary 26.7. *If n is even, then $\pi_{2n-1}(S^n)$ contains an infinite cyclic subgroup as a direct summand.*

Proof. In fact, the cyclic subgroup generated by the homotopy class of a map of Hopf invariant two must be mapped isomorphically onto the even integers by the homomorphism h. □

The much more important and harder result is the following famous theorem of J. F. Adams. Adams' initial proof was based on cohomological methods. Using Adams operations in complex K-theory yields a much simpler proof due to Adams and Atiyah.

Theorem 26.8. *For an even integer $n \geq 2$, there exists a map $f\colon S^{2n-1} \to S^n$ with $h(f) = \pm 1$ only if $n = 2$, 4, or 8.*

Proof. We write $n = 2m$. Since we computed the effect of the kth Adams operation ψ^k on $\tilde{K}(S^{2m})$ we know

$$\psi^k(i_{2n}) = k^{2m} i_{2n} \text{ and } \psi^k(i_n) = k^m i_n.$$

Hence

$$\psi^k(b) = k^{2m} b \text{ and } \psi^k(a) = k^m a + \mu_k$$

for some integer μ_k. For $k = 2$ this is

$$2^m a + \mu_2 b = \psi^2(a) \equiv a^2 = h(f) \cdot b \bmod 2.$$

Thus $h(f) = \pm 1$ implies that μ_2 is odd.

Now, for any odd k,

$$\begin{aligned} \psi^k \psi^2(a) &= \psi^k(2^m a + \mu_2 b) \\ &= k^m 2^m a + (2^m \mu_k + k^{2m} \mu_2) b \end{aligned}$$

while

$$\begin{aligned} \psi^2 \psi^k(a) &= \psi^2(k^m a + \mu_k b) \\ &= 2^m k^m a + (k^m \mu_2 + 2^{2m} \mu_k) b. \end{aligned}$$

Since $\psi^k\psi^2 = \psi^{2k} = \psi^2\psi^k$, these two expressions must be equal. Moreover, since $\tilde{K}(X)$ is a free abelian group, the coefficients of b must agree

$$2^m(2^m - 1)\mu_k = k^m(k^m - 1)\mu_2.$$

Since μ_2 is odd, this implies that 2^m divides $k^m - 1$. Already with $k = 3$, the following elementary number theoretic lemma shows that this implies $m = 1$, 2, or 4. $\square$

Lemma 26.9. *If 2^m divides $3^m - 1$ then $m = 1$, 2, or 4.*

Proof. Write $m = 2^\ell k$ with k odd. It suffices to show that the highest power of 2 dividing $3^m - 1$ is 2 for $\ell = 0$ and $2^{\ell+2}$ for $\ell > 0$. Then the lemma follows, since if 2^n divides $3^m - 1$, then we deduce $m \leq \ell + 2$, hence $2^\ell \leq 2^\ell k = m \leq \ell + 2$. This implies $\ell \leq 2$ and $m \leq 4$. The cases $m = 1$, 2, 3, and 4 can then be checked individually.

We use induction on ℓ. For $\ell = 0$ we have

$$3^m - 1 = 3^k - 1 \equiv 2 \mod 4, \text{ since } 3 \equiv -1 \mod 4 \text{ and } k \text{ is odd.}$$

Hence the highest power of 2 dividing $3^m - 1$ is 2. In the next case $\ell = 1$, we have

$$3^m - 1 = 3^{2k} - 1 = (3^k - 1)(3^k + 1).$$

The highest power of 2dividing the first factor is 2 as we just showed and the highest power of 2 dividing the second factor is 2 since

$$3^k + 1 \equiv 4 \mod 8 \text{ because } 3^2 \equiv 1 \mod 8 \text{ and } m \text{ is odd.}$$

So the highest power of 2 dividing the product $(3^k - 1)(3^k + 1)$ is 8. For the inductive step of passing from ℓ to $\ell + 1$ with $\ell \geq 1$, or in other words from m to $2m$ with m even, write

$$3^{2m} - 1 = (3^m - 1)(3^m + 1).$$

Then $3^m + 1 \equiv 2$ mod 4 since m is even, so the highest power dividing $3^m + 1$ is 2. Thus the highest power of 2 dividing $3^{2m} - 1$ is twice the highest power of 2 dividing $3^m - 1$. $\square$