Math 231b
Lecture 27

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27. Lecture 27: Consequences of the Hopf invariant one problem

Last time we discussed the K-theoretical proof of the following fundamental result.

Theorem 27.1. *For an even integer $n \geq 2$, there exists a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant one only if $n = 2$, 4, or 8.*

Today we will see some consequences of this result.

27.1. H-space structures on S^{n-1}.

As an important consequence of the theorem we can determine for which n the sphere S^n admits an H-space structure, i.e., there is a continuous multiplication map

$$g\colon S^n \times S^n \to S^n$$

with a two-sided identity element.

Theorem 27.2. *If S^{n-1} is an H-space, then $n = 1$, 2, 4, or 8.*

Let us first deal with the case that n is *odd*. Write $n-1 = 2k$. Since the K-theory group $K(S^{2k})$ is isomorphic to $\mathbb{Z}[\alpha]/(\alpha^2)$, the Bott periodicity theorem implies

$$K(S^{2k} \times S^{2k}) \cong \mathbb{Z}[\alpha,b]/(\alpha^2,\beta^2)$$

where α and b denote the pullback of generators of $K(S^{2k})$ and $K(S^{2k})$ under the projections of $S^{2k} \times S^{2k}$ onto its two factors. An additive basis for $K(S^{2k} \times S^{2k})$is thus $\{1, \alpha, \beta, \alpha\beta\}$.

Now let us assume we had an H-space multiplication map

$$\mu\colon S^{2k} \times S^{2k} \to S^{2k}$$

and let e be the identity element. The induced homomorphism of K-rings has the form

$$\mu^*\colon \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2).$$

We claim

$$\mu^*(\gamma) = \alpha + \beta + m\alpha\beta \text{ for some integer } m.$$

For: the composition

$$S^{2k} \xrightarrow{i} S^{2k} \times S^{2k} \xrightarrow{\mu} S^{2k}$$

is the identity, where i is the inclusion onto either of the subspaces $S^{2k} \times \{e\}$ or $\{e\} \times S^{2k}$ (with e the identity element of the H-space structure). The map i^* for i the inclusion onto the first factor sends α to γ and b to 0, so the coefficient of α in $\mu^*(\gamma)$ must be 1. Similarly the coefficient of β in $\mu^*(\gamma)$ must be 1. This proves the claim.

But this leads to a contradiction, since it implies

$$\mu^*(\gamma^2) = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta \neq 0,$$

which is impossible since $\gamma^2 = 0$.

The strategy to prove Theorem 27.2 for n *even* is the following: given an H-space structure on S^{n-1}, we construct from it a map $f\colon S^{2n-1} \to S^n$ of Hopf invariant one.

Let $g\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ be a continuous map. Regard S^{2n-1} as

$$\partial(D^n \times D^n) = \partial D^n \times D^n \cup D^n \times \partial D^n,$$

and we consider S^n as the union of two disks D^n_+ and D^n_- with their boundaries identified. Then $f\colon S^{2n-1} \to S^n$ is defined by

$$f(x,y) = |y| g(x, y/|y|) \in D^n_+ \text{ on } \partial D^n \times D^n$$

and

$$f(x,y) = |x| g(x/|x|, y) \in D^n_- \text{ on } D^n \times \partial D^n.$$

Note that f is well-defined and continuous, even when $|x|$ or $|y|$ is zero, and f agrees with g on $S^{n-1} \times S^{n-1}$.

Lemma 27.3. *Let $n \geq 2$ be an even integer. If $g\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ is an H-space multiplication, then the associated map $f\colon S^{2n-1} \to S^n$ has Hopf invariant ± 1.*

Proof. Let $e \in S^{n-1}$ be the identity element for the H-space multiplication, and let f be the map constructed above. In view of the definition of f it is natural to view the characteristic map ϕ of the $2n$-cell of X_f as a map

$$\phi\colon (D^n \times D^n, \partial(D^n \times D^n)) \to (X_f, S^n).$$

In the following commutative diagram the horizontal maps are the product maps. The diagonal map is the external product, equivalent to the external product

$$\tilde{K}(S^n) \otimes \tilde{K}(S^n) \to \tilde{K}(S^{2n}),$$

which is an isomorphism since it is an iterate of the Bott periodicity isomorphism.

$$
\begin{array}{ccc}
\tilde{K}(X_f) \otimes \tilde{K}(X_f) & \longrightarrow & \tilde{K}(X_f) \\
\cong \big\uparrow & & \big\uparrow \\
\tilde{K}(X_f,D^n_-) \otimes \tilde{K}(X_f, D^n_+) & \longrightarrow & \tilde{K}(X_f, S^n) \\
\phi^* \otimes \phi^* \big\downarrow & & \phi^* \big\downarrow \cong \\
\tilde{K}(D^n \times D^n, \partial D^n \times D^n) \otimes \tilde{K}(D^n \times D^n, D^n \times \partial D^n) & \longrightarrow & \tilde{K}(D^n \times D^n, \partial(D^n \times D^n)) \\
\big\downarrow \cong & \overset{\cong}{\nearrow} & \\
\tilde{K}(D^n \times \{e\}, \partial D^n \times \{e\}) \otimes \tilde{K}(\{e\} \times D^n, \{e\} \times \partial D^n) & &
\end{array}
$$

By the definition of an H-space and the definition of f, the map ϕ restricts to a homeomorphism from $D^n \times \{e\}$ onto D^n_+ and from $\{e\} \times D^n$ onto D^n_-. It follows that the element $a \otimes a$ in the upper left group maps to a generator of the group in the bottom row of the diagram, since a maps to a generator of $\tilde{K}(S^n)$ by definition. Therefore by the commutativity of the diagram, the product map in the top row sends

$$a \otimes a \mapsto \pm b$$

since b was defined to be the image of a generator of $\tilde{K}(X_f,S^n)$. Thus we have

$$a^2 = \pm b,$$

which means that the Hopf invariant of f is ± 1. ☐

Theorem 27.2 is now an immediate consequence of the lemma.

27.2. **Division algebra structures on $\mathbb{R}^n$.** The determination of which spheres are H-spaces has the following important implications.

Theorem 27.4. *Let $\omega\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a map with two-sided identity element $e \neq 0$ and no zero-divisors. Then $n = 1$, 2, 4, or 8.*

Proof. The product restricts to give $\mathbb{R}^n - \{0\}$ an H-space structure. Since S^{n-1} is homotopy equivalent to $\mathbb{R}^n - \{0\}$, it inherits an H-space structure. Explicitly, we may assume that $e \in S^{n-1}$by rescaling the metric, and we give S^{n-1} the multiplication

$$\phi\colon S^{n-1} \times S^{n-1} \to S^{n-1}$$

defined by

$$\phi(x,y) = \omega(x,y)/|\omega(x,y)|.$$

This is well-defined, since ω has no zero divisors. ☐

Remark 27.5. Note that ω need not be bilinear, just continuous. it also need not have a strict unit. All we needed is that e is a two-sided unit up to homotopy for the restriction of ω to $\mathbb{R}^n - \{0\}$.

In Lecture 3, we showed that there are trivializations of the tangent bundle of the spheres S^1, S^3, and S^7. Now we can show that there are no other spheres with trivial tangent bundle.

Theorem 27.6. *If S^n is parallelizable, i.e., if the tangent bundle τ to S^n is trivial, then $n = 0$, 1, 3, or 7.*

Proof. The case $n = 0$ is trivial. So let $n \geq 1$ and assume that S^n is parallelizable. Let $v_1, \dots, v_n$ be a tangent vector field which are linearly independent at each point of S^n. By the Gram-Schmidt process we may make the vectors $x, v_1(x), \dots, v_n(x)$ orthonormal for all $x \in S^n$. We may assume also that at the first standard basis vector e_1, the vectors $v_1(e_1), \dots, v_n(e_1)$ are the standard basis vectors $e_2, \dots, e_{n+1}$. To achieve this we might have to change the sign of v_n to get the orientations right and then deform the vector fields near e_1.

Now let $\phi_x \in SO(n+1)$ send the standard basis to $x, v_1(x), \dots, v_n(x)$. Then the map

$$\phi\colon (x,y) \mapsto \phi_x(y)$$

defines an H-space structure on S^n with the identity element e_1 since ϕ_{e_1} is the identity map and $\phi_x(e_1) = x$ for all x. Hence $n = 1$, 3, or 7. $\square$