Math 231b
Lecture 28

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28. Lecture 28: The Chern character

We have seen that singular cohomology and K-theory enjoy similar properties. The splitting principle implies a direct connection between them which we will describe in today's lecture.

28.1. **The Chern character.** Let X be a compact Hausdorff space. We want to define a ring homomorphism, called *Chern character*, from K-theory to cohomology.

Before we define this homomorphism we think of an assignment that sends vector bundles to cohomology classes, the Chern classes. We need to understand how the tensor product of line bundles behaves under Chern classes. Recall

$$\mathbb{C}\mathrm{P}^\infty \simeq K(\mathbb{Z},2)$$

and that line bundles are classified by their Chern classes regarded as elements of

$$[X,\mathbb{C}\mathrm{P}^\infty] \cong H^2(X;\mathbb{Z}).$$

The tensor product of two line bundles is represented by a product map

$$\phi\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$$

which gives $\mathbb{C}\mathrm{P}^\infty$ an H-space structure. We may think of ϕ as an element of

$$H^2(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \oplus H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$$

and this element is the sum of the Chern classes in the two copies of $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$

This shows that for two line bundles L_1 and L_2 over X, we have

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

Now we would like to define a ring homomorphism $ch\colon K(X) \to H^*(X;\mathbb{Q})$. We start with the case of a line bundle $L \to X$. We want ch to send the tensor product to products in in cohomology. So we set

$$ch(L) = e^{c_1(L)} = 1 + c_1(L) + c_1(L)/2! + \cdots \in H^*(X;\mathbb{Q}),$$

because then

$$ch(L_1 \otimes L_2) = e^{c_1(L_1\otimes L_2)} = e^{c_1(L_1)+c_1(L_2)} = ch(L_1)\cdot ch(L_2).$$

(If the sum defining $ch(L)$ has infinitely many terms, it will not lie in the direct sum but rather in the direct product of the groups $H^*(X;\mathbb{Q})$. But in the main examples, $H^n(X;\mathbb{Q})$ will be zero for n sufficiently large.)

For a direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_n$ we define

$$ch(E) = \sum_i ch(L_i) = \sum_i e^{t_i} = n + (t_1 + \cdots + t_n) + \cdots + (t_1^k + \cdots + t_n^k)/k! + \cdots$$

where $t_i = c_1(L_i)$. The total Chern class $c(E)$ is then

$$c(E) = (1 + t_1) \cdots (1 + t_n) = 1 + c_1(E) + \cdots c_n(E)$$

and $c_j(E) = \sigma_j$ is the jth elementary symmetric polynomial in the t_i's.

As we saw in Lecture 25, there is a polynomial Q_k with

$$Q_k(\sigma_1, \ldots, \sigma_k) = t_1^k + \cdots + t_n^k.$$

Hence the above formula reads

$$ch(E) = \dim E + \sum_{k>0} Q_k(c_1(E), \ldots, c_k(E))/k!.$$

For general E, we define $ch(E)$ by this formula.

Remark 28.1. In fact, if we want to define ch as a natural ring homomorphism which sends "generators for spheres to generators" then we have only one chance to do this. For, assume ch is such a map. Then for $X = S^2 = \mathbb{C}\mathrm{P}^1$

$$ch\colon K(S^2) \to H^*(S^2;\mathbb{Q})$$

the generator $H - 1$ is sent to a generator x in $H^2(S^2;\mathbb{Q})$, hence H is sent to $1 + x$ in $H^*(S^2;\mathbb{Q})$. For $\mathbb{C}\mathrm{P}^\infty$ this implies

$$ch\colon K(\mathbb{C}\mathrm{P}^\infty) \to H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Q}), H \mapsto 1 + x + \cdots = f(x)$$

where $f(x)$ is some power series in x. Now looking at the commutative diagram

$$\begin{array}{ccc} K(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) & \longrightarrow & K(\mathbb{C}\mathrm{P}^\infty) \\ \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} \\ H^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty;\mathbb{Q}) & \longrightarrow & H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Q}) \end{array}$$

we see that the series f must satisfy $f(x + y) = f(x) \cdot f(y)$, where y is the label for the generator of the cohomology of the other copy of $\mathbb{C}\mathrm{P}^\infty$. But there is only one power series that does the job, namely $f(x) = e^x$.

28.2. **A more formal description of** ch**.** Let R be a any commutative ring and consider a formal power series

$$f(t) = \sum_i a_i t^i \in R[[t]].$$

Given an element $x \in H^n(X; R)$, we let

$$f(x) = \sum a_i x^i \in H^{**}(X; R),$$

where $H^{**}(X; R) = \prod_i H^i(X; R)$ whose elements are considered as formal sums $\sum_i y_i$ with $\deg(y_i) = i$.

Via the splitting principle we can use f to construct a natural homomorphism of abelian monoids

$$\hat{f} \colon \mathrm{Vect}(X) \to H^{**}(X; R)$$

For a line bundle L over X, we set

$$\hat{f}(L) = f(c_1(L)).$$

For a sum $E = L_1 \oplus \cdots \oplus L_n$ of line bundles over X, we set

$$\hat{f}(E) = \sum_{i=1}^{n} f(c_1(L)).$$

For a general n-plane bundle E over X, we let $\hat{f}(E)$ be the unique element of $H^{**}(X; R)$ such that

$$p^*(\hat{f}(E)) = \hat{f}(p^*(E)) \in H^{**}(F(E); R).$$

More explicitly, writing $p^*E = L_1 \oplus \cdots L_n$, we know by the definition of Chern classes

$$\prod_{1 \le k \le n} (x - c_1(L_k)) = 0.$$

This implies that

$$c_k(p^*E) = p^*(c_k(E)) = \sigma_k(c_1(L_1), \ldots, c_1(L_n))$$

is the kth elementary symmetric polynomial in the $c_1(L_k)$. Likewise, we see that $\hat{f}(p^*E)$ is a symmetric polynomial in the $c_1(L_i)$ and can therefore be written as a polynomial in the elementary symmetric polynomials. Applying this polynomial to the $c_k(E)$ gives the element $\hat{f}(E) \in H^{**}(X; R)$. For a vector bundle E over a non-connected space X, we add the elements obtained by restricting E to the components of X. By the naturality property of $K(X)$, $\hat{f}$ extends to a homomorphism

$$\hat{f} \colon K(X) \to H^{**}(X; R).$$

There is also an analogous multiplicative extension $\bar{f}$ of f that starts from the definition

$$\bar{f}(E) = \prod_{i=1}^{n} f(c_1(L_i))$$

on a sum $E = L_1 \oplus \cdots \oplus L_n$ of line bundles.

As an example, we look at the following special case.

Lemma 28.2. *For any R, if $f(t) = 1 + t$, then $\bar{f}(E) = c(E)$ is the total Chern class of E.*

Proof. For a line bundle, we have $\bar{f}(L) = 1 + c_1(L) = c(L)$, and for a sum $E = L_1 \oplus \cdots \oplus L_n$ of line bundles we get

$$\bar{f}(E) = \prod_i (1 + c_1(L_i)) = 1 + c_1(E) + \cdots + c_n(E)$$

since $c_k(E)$ is equal to the kth elementary symmetric function in the $c_1(L_i)$'s. Hence if E is an arbitrary bundle, then

$$\bar{f}(E) = 1 + c_1(E) + \cdots + c_n(E) = c(E).$$

□

The example we are interested in is the Chern character which gives rise to an isomorphism between rationalized K-theory and rational cohomology.

Definition 28.3. For $R = \mathbb{Q}$ and $f(t) = e^t = \sum_i t^i/i!$, we define the *Chern character*

$$ch(E) \in H^{**}(X; \mathbb{Q}) \text{ by } ch(E) = \hat{f}(E).$$

It is clear that both descriptions of ch agree.

28.3. Properties of ch.

This allows us to prove the following result.

Proposition 28.4. *The Chern character is a ring homomorphism*

$$ch\colon K(X) \to H^{**}(X; \mathbb{Q}).$$

Proof. By the splitting principle and the construction of ch it suffices to check this when E_1 and E_2 are sums of line bundles. In this case we have

$$ch(E_1 \oplus E_2) = ch(\oplus_{i,j} L_{ij}) = \sum e^{c_1(L_{ij})} = ch(E_1) + ch(E_2)$$

and

$$ch(E_1 \otimes E_2) = ch(\oplus_{j,k}(L_{1j} \otimes L_{2k})) = \sum ch(L_{1j} \otimes L_{2k}) = \sum ch(L_{1j}) \cdot ch(L_{2k}) = ch(E_1) \cdot ch(E_2).$$

□

Proposition 28.5. *For $n \geq 1$, the Chern character maps $\tilde{K}(S^{2n})$ isomorphically onto the image of $H^{2n}(S^{2n};\mathbb{Z})$ in $H^{2n}(S^{2n};\mathbb{Q})$.*

Proof. Since $ch(x \otimes (H-1)) = ch(x) \cdot ch(h-1)$ we have the commutative diagram

$$\begin{array}{ccc} \tilde{K}(X) & \xrightarrow{\cong} & \tilde{K}(S^2 \wedge X) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ \tilde{H}^*(X;\mathbb{Q}) & \xrightarrow{\cong} & \tilde{H}^{*+2}(S^2 \wedge X;\mathbb{Q}) \end{array}$$

where the upper map is the external tensor product with $H-1$, and the lower map is the product with

$$ch(H-1) = ch(H) - ch(1) = 1 + c_1(H) - 1 = c_1(H),$$

which is a generator of $H^2(S^2;\mathbb{Z})$. Hence the lower map is an isomorphism too and even restricts to an isomorphism with $\mathbb{Z}$-coefficients. Taking $X = S^{2n}$, the result follows by induction on n, starting with the trivial case $n = 0$. □

Corollary 28.6. *A class in $H^{2n}(S^{2n};\mathbb{Z})$ occurs as a Chern class $c_n(E)$ if and only if it is divisible by $(n-1)!$.*

Proof. For vector bundles $E \to S^{2n}$ we have $c_1(E) = \cdots = c_{n-1}(E) = 0$, so

$$ch(E) = \dim E + Q_n(c_1, \ldots, c_n)/n! = \dim E \pm n c_n(E)/n! = \dim E \pm c_n(E)/(n-1)!$$

by the recursive formula for Q_n we mentioned in Lecture 25

$$Q_n = \sigma_1 Q_{n-1} - \sigma_2 Q_{n-2} + \cdots + (-1)^{n-2}\sigma_{n-1}Q_1 + (-1)^{n-1} n\sigma_n.$$

□

Now since Chern classes are in even degrees, the image of ch lies in the sum of the even degree elements in $H^{**}(X;\mathbb{Q})$ which we denote by $H^{even}(X;\mathbb{Q})$. We define $H^{odd}(X;\mathbb{Q})$ to be the sum of the odd degree elements. Then we can extend ch to $\mathbb{Z}/2$-graded reduced cohomology by defining ch on $\tilde{K}^1(X)$ to be the composite

$$\tilde{K}^1(X) \cong \tilde{K}(\Sigma X) \xrightarrow{ch} \tilde{H}^{even}(\Sigma X;\mathbb{Q}) \cong \tilde{H}^{odd}(X;\mathbb{Q}).$$

Then we can prove the following fundamental result.

Theorem 28.7. *For any pointed finite CW-complex X, ch induces an isomorphism*

$$\tilde{K}^*(X) \otimes \mathbb{Q} \xrightarrow{\cong} \tilde{H}^{**}(X;\mathbb{Q}).$$

Sketch of the proof. We think of both the source and the target as $\mathbb{Z}/2$-graded. The Proposition 28.5 implies the conclusion when $X = S^n$ for any n. The crucial point is that the map of the theorem is part of a natural transformation of cohomology theories. Then the assertion follows from the result for $X = S^n$, the five lemma and induction on the number of cells of X.
More explicitely, the case of a cell complex with a single cell is trivial. Then if X is obtained from a subcomplex A by attaching a cell, then we get a sequence

$$X/A \to S^1 \wedge A \to S^1 \wedge X \to (S^1 \wedge X)/(S^1 \wedge A) \to S^2 \wedge A.$$

Applying the Chern character to this sequence yields a commutative diagram of five-term exact sequence (tensoring with $\mathbb{Q}$ is exact). Now the spaces X/A and $(S^1 \wedge X)/(S^1 \wedge A)$ are spheres, and both $S^1 \wedge A$ and $S^2 \wedge A$ are both cell complexes with the same number of cells as A (we collapse the suspension or double suspension of a 0-cell). The five-lemma gives us the result for $S^1 \wedge X$. Then we obtain the result for X by replacing X with $S^1 \wedge X$ in the above argument and using that ch commutes with double suspension. □