Math 231b
Lecture 29

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29. Lecture 29: The e-invariant

Today we are going to elaborate a little bit more on the construction we used for the Hopf invariant one problem. It turns out that this picture contains much more information.

29.1. **Getting information about maps between spheres.** Let us look at a slight variation of the way we defined the Hopf invariant using K-theory. For $m, n \geq 1$, let

$$f\colon S^{2n+2m-1} \to S^{2n}$$

be a pointed map. Let

$$X = X_f = S^{2n} \cup_f e^{2n+2m}$$

be the mapping cone of f, $i\colon S^{2n} \hookrightarrow X$ be the inclusion, and

$$\pi\colon X \to X/S^{2n} \cong S^{2n+2m}$$

be the map that collapses S^{2n}. We would like to measure the extend to which f is not null, i.e., not homotopic to a constant map. Therefor we would like to use our favorite (at least for the moment) cohomology theory, complex K-theory.

As in Lecture 26, the sequence

$$S^{2n+2m-1} \xrightarrow{f} S^{2n} \xrightarrow{i} S^{2n} \cup_f e^{2n+2m} \xrightarrow{\pi} S^{2n+2m}$$

(or rather the pair (X, S^{2n})) induces a long exact sequence in reduced K-theory. Since the K-theory of spheres is concentrated in even degrees, the K-theory degree of f, i.e., $\tilde{K}(f)$, is zero. For our goal to measure the extend to which f is not null this is bad news. But there is still some more information to exploit.

Since $\tilde{K}(f) = 0$, we obtain a short exact sequence

$$0 \to \tilde{K}(S^{2n+2m}) \xrightarrow{\pi^*} \tilde{K}(S^{2n} \cup_f e^{2n+2m}) \xrightarrow{i^*} \tilde{K}(S^{2n}) \to 0. \tag{1}$$

We know that the outermost groups are the integers and the group in the middle is an extension. We would like to understand how far from the trivial extension the sequence (1). In order to make this more precise we need to think a little bit more about what kind of groups we are talking about.

We have already noticed that the outermost groups in (1) are the integers. But we also know that the Adams operation ψ^k acts on $\tilde{K}(S^{2n})$ by k^n and it acts on $\tilde{K}(S^{2n+2m})$ by k^{n+m}. So let us write $\mathbb{Z}(n)$ for the first group and $\mathbb{Z}(n+m)$ for the second. We want to consider them in some category of "abelian groups with Adams operations".

Let us make an informal definition:

Definition 29.1. An *abelian group with Adams operations* is an abelian group A together with morphisms $\psi^k \colon A \to A$, for $k \in \mathbb{Z}$, which commute with each other and satisfy $\psi^\ell \psi^k = \psi^{k\ell}$.

But we can say even a little bit more about the K-theory groups. In the previous lecture we defined the Chern character

$$ch \colon K(Y) \to \oplus_n H^{2n}(Y; \mathbb{Q})$$

which becomes an isomorphism after tensoring $K(Y)$ with $\mathbb{Q}$ (assuming Y is a finite cell complex). The splitting principle now tells us that the Adams operations on cohomology are given by

$$\psi^k = k^n \text{ on } H^{2n}(Y; \mathbb{Q}).$$

To check this, write a bundle E as a sum of line bundles. Then we only need to compute the effect of ψ^k on the $2n$th component ch^n of $ch(L)$ for a line bundle. Then we have $\psi^k(L) = L^k$, and hence

$$ch^n(\psi^k(L)) = ch^n(L^k) = (c_1(L^k))^n/n! = (kc_1(L))^n/n! = k^n c_1(L)^n/n! = k^n ch^n(L).$$

Hence the action of the Adams operations is *semisimple* on rational K-theory. In other words, if A is in the image of the K-theory functor, then $A \otimes \mathbb{Q}$ is a big sum of copies of $\mathbb{Q}(n)$.

29.2. **The e-invariant as an extension.** Now let us get back to the geometric situation. The short exact sequence (1) corresponds to an element $e(f)$ ("e" for *extension*) in

$$\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+m))$$

where the Ext is in the category of abelian groups together with Adams operations.

What can we say about this group $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+m))$? The short exact sequence

$$0 \to \mathbb{Z}(n+m) \to \mathbb{Q}(n+m) \to \mathbb{Q}/\mathbb{Z}(n+m) \to 0$$

induces a long exact sequence of Ext-groups

$$\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}(n+m)) \to \mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}/\mathbb{Z}(n+m)) \to \mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+m)) \to \mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Q}(n+m)).$$

Lemma 29.2. *For $m \neq 0$, the two outermost groups* $\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}(n+m))$ *and* $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Q}(n+m))$ *are zero.*

Proof. We only prove the first assertion. If there is a non-trivial homomorphism $\mathbb{Z}(n) \to \mathbb{Q}(n+m)$, then $1 \in \mathbb{Z}(n)$ is sent to some element $\alpha \in \mathbb{Q}(n+m)$, and thus k^n would have to be sent to $k^{n+m}\alpha$ which is a contradiction. Hence $\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}(n+m)) = \{0\}$. The second assertion requires a little bit more work. Since the discussion is more philosophical for the moment, we skip the proof. □

As a consequence of the lemma we get an isomorphism

$$\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}/\mathbb{Z}(n+m)) \cong \mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+m)).$$

The group $\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}/\mathbb{Z}(n+m))$ is a subgroup of $\mathbb{Q}/\mathbb{Z}$ and consists of things compatible with the Adams operations.

In order to understand this group a bit more, let us spell out what we know. A homomorphism

$$\mathbb{Z}(n) \to \mathbb{Q}/\mathbb{Z}(n+m)$$

is determined by where it sends $1 \in \mathbb{Z}(n)$. Let us call the image $x \in \mathbb{Q}/\mathbb{Z}(n+m)$. Then x has to satisfy a condition in order to make the map a homomorphism of abelian groups with Adams operations. Namely, for all k, we must have

$$(k^{n+m} - k^n) \cdot x = 0 \in \mathbb{Q}/\mathbb{Z},$$

because this expresses the compatibility with ψ^k. This means that the denominator of x must divide all the numbers $(k^{n+m} - k^n)$ for all k.

In other words, the group $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+m))$ is cyclic of order

the greatest common divisor of $k^n(k^m - 1)$ for all k.

Hence we should calculate this greatest common divisor. There is a nice answer for it. But before we do this let us make things a bit more concrete. We should also think about the specific element in $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+m))$ that sequence (1) produces.

29.3. **The e-invariant as an element in $\mathbb{Q}/\mathbb{Z}$.** Let i_{2n} be a generator of $\tilde{K}(S^{2n})$ and i_{2n+2m} be a generator of $\tilde{K}(S^{2n+2m})$. Choose an element $a \in \tilde{K}(S^{2n} \cup_f e^{2n+2m})$ such that $i^*(a) = i_{2n}$ and let $b = \pi^*(i_{2n+2m}) \in \tilde{K}(S^{2n} \cup_f e^{2n+2m})$. Then for any k, we have

$$\psi^k(a) = k^n \cdot a + \mu_k \cdot b.$$

Since the Adams operations commute, we must have

$$\psi^k(\psi^\ell(a)) = \psi^k(\ell^n a+\mu_\ell b) = \ell^n k^n a+\ell^n \mu_k b+k^{n+m}\mu_\ell b = \ell^n k^n a+k^n \mu_\ell b+\ell^{n+m}\mu_k b = \psi^\ell(\psi^k(a))$$

and hence

$$k^n(k^m-1)\mu_\ell = \ell^n(\ell^m-1)\mu_k$$

for any k and ℓ. This shows us that the rational number

$$e(f) := \frac{\mu_k}{k^n(k^m-1)} \in \mathbb{Q}.$$

is independent of k. But it might depend on our choice of a. If we change a by a multiple of b, then $e(f)$ is changed by an integer. (For $a' = a + p \cdot b$, we get $e'(f) = e(f) + p$.) Thus $e(f)$ is well-defined as an element of $\mathbb{Q}/\mathbb{Z}$.

Finally, recalling where we started we see that we have produced an assignment

$$(f\colon S^{2n+2m-1} \to S^{2n}) \mapsto e(f) \in \mathbb{Q}/\mathbb{Z}.$$

Remark 29.3. 1. The map e is called the *e-invariant*. It plays an important role in understanding the structure of the (stable) homotopy groups of the sphere. To get further into this story we introduce in the next lecture the J-homomorphism.
2. That $e(f)$ is an element in $\mathbb{Q}/\mathbb{Z}$ fits well with our discussion above. To determine an element in $\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}/\mathbb{Z}(n+m))$ we needed to determine the image of 1 in $\mathbb{Q}/\mathbb{Z}(n+m)$.

Lemma 29.4. *If $f \sim g$, then $e(g) = e(f)$, i.e., e induces a map*

$$e\colon \pi_{2n+2m-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}.$$

Proof. This follows from applying the functor $\tilde{K}$ to the diagram

$$\begin{array}{ccccccccc}
S^{2n+2m-1} & \xrightarrow{f} & S^{2n} & \xrightarrow{i} & S^{2n} \cup_f e^{2n+2m} & \xrightarrow{\pi} & S^{2n+2m} & \xrightarrow{\Sigma(f)} & S^{2n+1} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
S^{2n+2m-1} & \xrightarrow{g} & S^{2n} & \xrightarrow{i'} & S^{2n} \cup_g e^{2n+2m} & \xrightarrow{\pi'} & S^{2n+2m} & \xrightarrow{\Sigma(g)} & S^{2n+1}.
\end{array}$$

□