Math 231b
Lecture 30

G. Quick

30. Lecture 30: The e-invariant and the J-homomorphism

We are trying to detect interesting maps between spheres. Last time we defined the e-invariant and showed that we should think of it as an element in some Ext group of abelian groups with Adams operations. This group is finite and cyclic and we saw a criterion for determining its order. But we still need to determine this order. The reason why this is so interesting is that the order will tell us something about the size of some of the stable homotopy groups of spheres.

Let us recall the setup. For $m, n \geq 1$, let

$$f\colon S^{2n+2m-1} \to S^{2n}$$

be a pointed map,

$$X = X_f = S^{2n} \cup_f e^{2n+2m}$$

be the mapping cone of f, $i\colon S^{2n} \hookrightarrow X$ be the inclusion, and

$$\pi\colon X \to X/S^{2n} \cong S^{2n+2m}$$

the map that collapses S^{2n}. This gives us a short exact sequence

$$0 \to \tilde{K}(S^{2n+2m}) \xrightarrow{\pi^*} \tilde{K}(S^{2n} \cup_f e^{2n+2m}) \xrightarrow{i^*} \tilde{K}(S^{2n}) \to 0. \tag{1}$$

Let i_{2n} be a generator of $\tilde{K}(S^{2n})$ and i_{2n+2m} be a generator of $\tilde{K}(S^{2n+2m})$. Choose an element

$a \in \tilde{K}(S^{2n} \cup_f e^{2n+2m})$ such that $i^*(a) = i_{2n}$ and let $b = \pi^*(i_{2n+2m}) \in \tilde{K}(S^{2n} \cup_f e^{2n+2m})$.

Then for any k, we have

$$\psi^k(a) = k^n \cdot a + \mu_k \cdot b.$$

Since the Adams operations commute, we must have

$$k^n(k^m - 1)\mu_\ell = \ell^n(\ell^m - 1)\mu_k$$

for any k and ℓ. This shows us that the rational number

$$e(f) := \frac{\mu_k}{k^n(k^m-1)} \in \mathbb{Q}.$$

is independent of k. But it might depend on our choice of a. If we change a by a multiple of b, then $e(f)$ is changed by an integer. Thus $e(f)$ is well-defined as an element of $\mathbb{Q}/\mathbb{Z}$.

The e-invariant defines a map

$$e\colon \pi_{2n+2m-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}.$$

An alternative description of the e-invariant can be given using the Chern character. The Chern character gives us a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{K}(S^{2n+2m}) \arrow[r, "\pi^*"] \arrow[d, "ch"] & \tilde{K}(X_f) \arrow[r, "i^*"] \arrow[d, "ch"] & \tilde{K}(S^{2n}) \arrow[r] \arrow[d, "ch"] & 0 \\
0 \arrow[r] & \tilde{H}^*(S^{2n+2m};\mathbb{Q}) \arrow[r, "\pi^*"'] & \tilde{H}^*(X_f;\mathbb{Q}) \arrow[r, "i^*"'] & \tilde{H}^*(S^{2n};\mathbb{Q}) \arrow[r] & 0
\end{tikzcd} \tag{2}$$

whose rows are exact.

Let $y = \pi^*(ch(i_{2n+2m})) \in \tilde{H}^{2n+2m}(X_f;\mathbb{Q})$ and x be an element in $\tilde{H}^{2n}(X_f;\mathbb{Q})$ that maps to the generator $ch(i_{2n})$. Then we have $ch(b) = y$. Let $r(f) \in \mathbb{Q}$ be such that

$$ch(a) = x + r(f)\cdot y \in \tilde{H}^{2n}(X_f;\mathbb{Q}) \oplus \tilde{H}^{2n+2m}(X_f;\mathbb{Q}).$$

Lemma 30.1. *$r(f) = e(f) \in \mathbb{Q}/\mathbb{Z}$.*

Proof. We calculate

$$ch(\psi^k(a)) = ch(k^n\cdot a + \mu_k\cdot b) = k^n\cdot ch(a) + \mu_k\cdot ch(b) = k^n\cdot x + (k^n\cdot r(f) + \mu_k)\cdot y.$$

On the other hand, we have seen above that ψ^k acts on $\tilde{H}^{2n}$ by multiplication by k^n. Hence

$$\psi^k(ch(a)) = k^n ch^n(a) + k^{n+m} ch^{n+m}(a) = k^n\cdot x + k^{n+m}\cdot r(f)\cdot y.$$

Comparing the coefficients of y in both formulas gives

$$\mu_k = r(f)\cdot(k^n(k^m-1)).$$

□

Lemma 30.2. *The map e is a group homomorphism.*

Proof. Let $X_{f,g}$ be obtained from S^{2n} by attaching two $2n+2m$-cells by f and g, so $X_{f,g}$ contains both X_f and X_g. There is a quotient map

$$Q\colon X_{f+g} \to X_{f,g}$$

collapsing a sphere $S^{2n+2m-1}$ that separates the $2n+2m$-cell of $X_{f,g}$ into a pair of $2n+2m$-cells. (This is also called the "pinching map".) It induces a commutative diagram

$$\begin{array}{ccc}\tilde{K}(X_{f,g}) & \xrightarrow{Q^*} & \tilde{K}(X_{f+g}) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ \tilde{H}^*(X_{f,g};\mathbb{Q}) & \xrightarrow{Q^*} & \tilde{H}^*(X_{f+g};\mathbb{Q}).\end{array}$$

In the upper row, the generators b_f and b_g are mapped to b_{f+g} and $a_{f,g}$ is mapped to a_{f+g}. Similarly, in the lower row, the generators y_f and y_g are mapped to y_{f+g} and $x_{f,g}$ is mapped to x_{f+g}. Using the previous lemma it now suffices to work with r and to look at

$$ch(a_{f,g}) = x_{f,g} + r(f)y_f + r(g)y_g$$

and hence

$$ch(a_{f+g}) = x_{f+g} + (r(f) + r(g))y_{f+g}.$$

□

Remark 30.3. The e-invariant is in fact a stable invariant. We know that the mapping cone satisfies $X_{S^2\wedge f} = S^2 \wedge X_f$ and we noticed in the proof of Proposition 28.5 of Lecture 28 that ch commutes with double suspension. This shows that we have a commutative diagram

$$\begin{tikzcd} \pi_{2n+2m-1}(S^{2n}) \arrow[rr, "S^2\wedge -"] \arrow[dr, "e"'] & & \pi_{2n+2+2m-1}(S^{2n+2}) \arrow[dl, "e"] \\ & \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

Hence we can view e also as a homomorphism

$$e\colon \pi^s_{2m-1}(S^0) \to \mathbb{Q}/\mathbb{Z}$$

from the $(2m-1)$-stable homotopy group of the sphere spectrum.

Now we should start to calculate the e-invariant. The maps for which we get the most important results are in the image of the J-homomorphism.

30.1. **The J-homomorphism.** The J-homomorphism is a natural way to construct maps between spheres. Let us first look at the idea of the construction.

Let $O(n)$ be the group of orthogonal $n \times n$-matrices. It acts on the Euclidean n-space $\mathbb{R}^n$ by linear isometries. A linear isometry of $\mathbb{R}^n$ extends to the one-point compactification S^n. Hence there is a natural map

$$J\colon O(n) \to \mathrm{LinIso}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{Map}_*(S^n, S^n) = \Omega^n S^n$$

where $\mathrm{Map}_*(-,-)$ denotes the space of basepoint preserving continuous maps (with the compact-open topology). This induces a homomorphism

$$J\colon \pi_k(O(n)) \to \pi_k(\Omega^n S^n) = \pi_{k+n}(S^n).$$

Remark 30.4. There is a little subtlety concerning the above construction of J. For the basepoint of $\Omega^n S^n$ is the constant map at the basepoint. The space $\Omega^n S^n$ has many path components, one for each degree. The image of $O(n)$ lies in the path components $\Omega^n_1 S^n$ and $\Omega^n_{-1} S^n$ of paths of degree ± 1 (remembering that $O(n)$ has two components). The basepoint of $O(n)$, the identity map, goes to the identity map of S^n. Hence the map $O(n) \to \Omega^n S^n$, as described above, is not basepoint preserving. So we should modify the map by "subtracting off" (in some group model for $\Omega^n S^n$) the identity map. Hence we should use

$$J\colon O(n) \to \Omega^n_1 S^n \xrightarrow{-1} \Omega^n_0 S^n.$$

Here is a more concrete way to define the J-homomorphism. Let $k \geq 1$. An element $[f] \in \pi_k(O(n))$ is represented by a family of isometries

$$f_x \in O(n),\ x \in S^k \text{ with } f_x = \mathrm{id} \text{ when } x \text{ is the basepoint of } S^k.$$

Writing

$$S^{n+k} = \partial(D^{k+1} \times D^n) = S^k \times D^n \cup D^{k+1} \times S^{n-1} \text{ and } S^n = D^n/\partial D^n,$$

let

$$Jf(x,y) = f_x(y) \text{ for } (x,y) \in S^k \times D^n \text{ and } Jf(D^{k+1} \times S^{n-1}) = \partial D^n,$$

where we think of the latter ∂D^n as the basepoint of $D^n/\partial D^n$.

It is easy to see that if $f \simeq g$ then $Jf \simeq Jg$. Hence we obtain a map

$$J\colon \pi_k(O(n)) \to \pi_{k+n}(S^n).$$

Lemma 30.5. *J is a homomorphism.*

Proof. Exercise. □

It is easy to check that if we embed $O(n)$ into $O(n+1)$ this corresponds to taking suspension. Since both groups $\pi_k(O(n))$ and $\pi_{k+n}(S^n)$ are independent of n for $n-1 > k$, we can pass to the limit in n and get the stable J-homomorphism

$$J\colon \pi_k(O) \to \pi_k^s(S^0) = \pi_k(S^0).$$

The image of the J-homomorphism in $\pi_k(S^0)$ is the main part of the stable homotopy groups which is accessible to direct computations.

30.2. **The complex J-homomorphism.** In our computations we will focus on the following complex version of J. We can compose J with the map

$$\pi_k(U) \to \pi_k(O) \text{ induced by the natural inclusions } U(n) \subset O(2n).$$

This defines the stable complex J-homomorphism

$$J_{\mathbb{C}} \colon \pi_k(U) \to \pi_k(S^0).$$

On the groups $\pi_k(S^0)$ we have defined the e-invariant. Our goal now is to compute the e-invariant on the image of $J_{\mathbb{C}}$, i.e., we want to compute the composition

$$e \circ J_{\mathbb{C}} \colon \pi_k(U) \to \mathbb{Q}/\mathbb{Z}.$$

There is the following great result.

Theorem 30.6. *Let $f \colon S^{2k-1} \to U(n)$ represent a generator in $\pi_{2k-1}(U)$. Then*

$$e(J_{\mathbb{C}} f) = \pm \beta_k / k$$

where β_k is defined by the power series

$$\frac{x}{e^x - 1} = \sum_k \frac{\beta_k x^k}{k!}.$$

Hence the image of J in $\pi_{2k-1}(S^0)$ has order divisible by the denominator of β_k/k (that is the denominator when we take β_k/k in reduced form).