Math 231b
Lecture 31

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31. Lecture 31: The image of the J-homomorphism

The stable J-homomorphism $J\colon \pi_k(O) \to \pi_k(S^0)$ is an important tool to produce interesting maps between spheres. Last time we also considered its complex analogue

$$J_{\mathbb{C}}\colon \pi_k(U) \to \pi_k(O) \to \pi_k(S^0)$$

which is a little bit easier to handle. Today we start to prove the following great result:

Theorem 31.1. *If $f\colon S^{2k} \to BU$ represents a generator x_{2k} in $\pi_{2k}(BU)$, then*

$$e(J_{\mathbb{C}}f) = \pm B_k/k$$

where B_k is the kth Bernoulli number defined by the power series

$$\frac{x}{e^x-1} = \sum_k \frac{B_k x^k}{k!}.$$

Hence the image of J in $\pi_{2k-1}(S^0)$ has order divisible by the denominator of B_k/k (that is the denominator when we take B_k/k in reduced form).

Before we start let us think a bit more about the maps in question. We can rewrite $J_{\mathbb{C}}$ as

$$\pi_{m-1}U = \pi_m BU \cong \tilde{K}^0(S^m) \to \pi_{m-1}(S^0)$$

and it factors through the real J-homomorphism

$$\pi_{m-1}O = \pi_m BO \cong \tilde{KO}^0(S^m) \to \pi_{m-1}(S^0)$$

where $\tilde{KO}^0(S^m)$ denotes the real K-theory of S^m.

The groups $\pi_{m-1}U$ alternate between being $\mathbb{Z}$ and 0: if m is even, then we get $\mathbb{Z}$; if m is odd, then we get 0:

m	1	2	3	4	5	6	7	8
$\pi_{m-1}U$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$.

The homotopy groups $\pi_{m-1}O$ of O show an 8-fold periodicity:

m	1	2	3	4	5	6	7	8
$\pi_{m-1}O$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$.

The map

$$\mathbb{Z} \cong \pi_{m-1}U \to \pi_{m-1}O \cong \mathbb{Z}$$

is an isomorphism when $m \equiv 4 \bmod 8$ and is multiplication by 2 when $m \equiv 0 \bmod 8$. (One can see this by looking at the composite $\pi_m BU \to \pi_m BO \to \pi_m BU$.)

We formulated our theorem in terms of the complex J-homomorphism, because it makes things easier. But from the table of $\pi_{m-1}BO$ we see immediately that the J-homomorphism is zero when m is odd.

Moreover, the cost of working with complex rather than real K-theory is an overall factor of two, i.e., by computing

$$\pi_{2k-1}U \xrightarrow{J_{\mathbb{C}}} \pi_{2k-1}S^0 \xrightarrow{e_{\mathbb{C}}} \mathbb{Q}/\mathbb{Z}$$

we get twice the value of the real e-invariant $e_{\mathbb{R}}$ of the real J-homomorphism of the generator of real K-theory.

The theorem tells us that if $x_{2n} \in \pi_{2n}BU$ is a generator, then $e_{\mathbb{C}}(J_{\mathbb{C}}(x_{2n})) = \frac{B_n}{n}$. Then one can deduce from the above discussion the following result.

Corollary 31.2. *If $y_{4n} \in \pi_{4n}BO$ is a generator, then $e_{\mathbb{R}}(J_{\mathbb{R}}(y_{4n})) = \frac{B_{2n}}{4n}$.*

31.1. **Thom complexes and the J-homomorphism.** The initiating idea for the proof of the theorem is based on the following very important fact. If we want to show that a map of spheres is nontrivial, we have to make computations in the mapping cone. When a map is in the image of J, we have a lot of information about this mapping cone: it is actually a Thom complex.

Proposition 31.3. *Let ξ be an n-dimensional complex vector bundle over S^{2k} classified by a map*

$$\xi\colon S^{2k} \to BU(n).$$

The Thom complex of ξ is $S^{2n} \cup_{J\xi} e^{2n+k}$.

Proof. Since $\pi_{2k}(BU(n)) \cong \pi_{2k-1}(U(n))$, there is a map

$$f\colon S^{2k-1} \to U(n).$$

We consider f as a clutching function for ξ. In fact, we can identify ξ with the bundle ξ_f obtained from $D^{2k} \times \mathbb{C}^n \amalg \mathbb{C}^n$ by identifying

$$(x,v) \sim f_x(v) \text{ for } x \in \partial D^{2k}.$$

Restricting to the unit disk bundle $D(\xi_f)$ we have $D(\xi_f)$ expressed as a quotient of $D^{2k} \times D^{2n} \amalg D^{2n}$ by the same relation. The quotient $T(\xi_f) = D(\xi_f)/S(\xi_f)$ contains a sphere $S^{2n} = D^{2n}/\partial D^{2n}$, coming from the second copy of D^{2n}, and $T(\xi_f)$ is obtained from S^{2n} by attaching a cell e^{2k+2n} with characteristic map the quotient map

$$D^{2k} \times D^{2n} \to D(\xi_f) \to T(\xi_f).$$

The attaching map of the cell is precisely $J(f)$, since it is given by

$$(x, v) \mapsto f_x(v) \in D^{2n}/\partial D^{2n} \text{ on } \partial D^{2k} \times D^{2n}$$

and maps all of $D^{2k} \times \partial D^{2n}$ to the point $\partial D^{2n}/\partial D^{2n}$. □

If we want to compute $eJ_{\mathbb{C}}(f)$ we need to compute $ch(a)$ for an element

$$a \in \tilde{K}(X_{Jf}) = \tilde{K}(T_\xi) \text{ which restricts to a generator in } \tilde{K}(S^{2n})$$

where S^{2n} is a fiber of $D(\xi)$ as in the previous proof. A class in $\tilde{K}(T(\xi))$ which restricts to a generator for each sphere S^n coming from a fiber of ξ is called a *Thom class* of ξ. Hence we need to understand the Chern character of Thom classes in K-theory.