Math 231b
Lecture 32

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32. Lecture 32: The image of the J-homomorphism and Thom classes

We are still on the way to prove the following theorem on the complex J-homomorphism

$$J_{\mathbb{C}}\colon \pi_k(U) \to \pi_k(O) \to \pi_k(S^0).$$

Theorem 32.1. *If x_{2k} in $\pi_{2k}(BU)$ is a generator, then*

$$e(J_{\mathbb{C}} f) = \pm B_k/k$$

where B_k is the kth Bernoulli number defined by the power series

$$\frac{x}{e^x - 1} = \sum_k \frac{B_k x^k}{k!}.$$

Hence the image of J in $\pi_{2k-1}(S^0)$ has order divisible by the denominator of B_k/k (that is the denominator when we take B_k/k in reduced form).

32.1. Thom classes and the Thom isomorphism in K-theory.

We saw last time that if E is an n-dimensional complex vector bundle over S^{2n} classified by a map

$$f\colon S^{2k} \to BU$$

then the Thom complex of ξ is $S^{2n} \cup_{Jf} e^{2n+k}$.

Hence if we want to compute $eJ_{\mathbb{C}}(f)$ we need to compute $ch(a)$ for an element

$$a \in \tilde{K}(X_{Jf}) = \tilde{K}(T_\xi) \text{ which restricts to a generator in } \tilde{K}(S^{2n})$$

where S^{2n} is a fiber of $D(\xi)$ as in the previous proof. A class in $\tilde{K}(T(\xi))$ which restricts to a generator for each sphere S^n coming from a fiber of ξ is called a *Thom class* of ξ. Hence we need to understand the Chern character of Thom classes in K-theory.

We have seen Thom classes before. But let us briefly recall the basics theory. Let E be a complex vector bundle of dimension n over the compact Hausdorff space X. Let $X^E := T(E) = D(E)/S(E)$ denote the Thom space of E over X. The *Thom class* is an element

$$U \in \tilde{K}^0(X^E)$$

which restricts to a generator under the restriction map

$$\tilde{K}^0(X^E) \to \tilde{K}^0((X^E)_x) \cong \tilde{K}^0(E_x^+) \cong \mathbb{Z}$$

for every $x \in X$, where E_x^+ denotes the one-point compactification of the fiber E_x (it's a $2n$-sphere whence the last isomorphism). There are several natural ways to get such a Thom class. One construction uses the projective bundle formula.

First we remark that we can identify X^E with $\mathbb{P}(E \oplus 1)/\mathbb{P}(E)$. Let V be the vector space given by the fiber E_x over some $x \in X$. Given a line ℓ through the origin in $V \oplus 1$ which does not lie in V, there is a unique point v in V such that $(v, 1) \in V \oplus 1$. This defines a map $\mathbb{P}(V \oplus 1) \to V$. The lines that are in V correspond to the point at ∞ in the fiber of the Thom complex of V. Hence we have checked on each fiber that we have an isomorphism

$$X^E = \mathbb{P}(E \oplus 1)/\mathbb{P}(E).$$

Now it is easier to produce the Thom class on the right hand side, because we know that we have the tautological line bundle L over the projective space.

Let L be the canonical line bundle over $\mathbb{P}(E \oplus 1)$. We know that $K^*(\mathbb{P}(E \oplus 1))$ is the free $K^*(X)$-module with basis $1, L, \ldots, L^n$. Restricting to $\mathbb{P}(E) \subset \mathbb{P}(E \oplus 1)$, we see that $K^*(\mathbb{P}(E))$ is the free $K^*(X)$-module with basis (the restrictions to $\mathbb{P}(E)$ of) $1, L, \ldots, L^{n-1}$. So we have a short exact sequence

$$0 \to \tilde{K}^*(X^E) \to K^*(\mathbb{P}(E \oplus 1)) \xrightarrow{\rho} K^*(\mathbb{P}(E)) \to 0.$$

The map ρ sends L^n to L^n. But in $K^*(\mathbb{P}(E))$ we have the relation

$$\sum_i (-1)^i \lambda^i(E) L^{n-i} = 0$$

where the $\lambda^i(E)$ are the Chern classes of E in $K^*(X)$ by definition. The class $U_K \in \tilde{K}^0(X^E)$ that maps to the nonzero element

$$\sum_i (-1)^i \lambda^i(E) L^{n-i} \in K^0(\mathbb{P}(E \oplus 1))$$

is the *Thom class* of E that we were looking for.

Moreover, we get that multiplication by U_K gives the *Thom isomorphism*

$$U_K \colon K^0(X) \cong \tilde{K}^0(X^E)$$

and $\tilde{K}^0(X^E)$ is a free $K^0(X)$-module with one generator U_K.

Remark 32.2. We will also sometimes identify

$$U_K \text{ with } \sum_i (-1)^i \lambda^i(E) L^{n-i} \text{ in } \tilde{K}^0(\mathbb{P}(E \oplus 1)).$$

Note that all this makes sense for virtual bundles too, since it is an isomorphism of modules over $K^0(X)$.

Remark 32.3. The previous discussion applies to any cohomology theory with a projective bundle formula for complex vector bundles. In particular, it applies to $\widetilde{H}^{even}(-;\mathbb{Q})$. If $x = x(E) \in H^2(\mathbb{P}(E \oplus 1);\mathbb{Q})$ is an element that restricts to a generator of $H^2(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Q})$ in each fiber, then there is the relation

$$\sum_i (-1)^i c_i(E) x^{n-i} = 0 \text{ in } H^*(\mathbb{P}(E);\mathbb{Q}).$$

Hence the element $\sum_i (-1)^i c_i(E) x^{n-i} \in H^*(\mathbb{P}(E \oplus 1);\mathbb{Q})$ comes from an element $U_H \in H^{2n}(X^E;\mathbb{Q})$ (where we use that $x(E \oplus 1)$ restricts to $x(E)$). This is the Thom class in cohomology. In $H^*(\mathbb{P}(E \oplus 1);\mathbb{Q})$ we can identify U_H with $\sum_i (-1)^i c_i(E) x^{n-i}$. Then we get $U_H \cdot x = 0$ in $H^*(\mathbb{P}(E \oplus 1);\mathbb{Q})$, because we know $c_i(E \oplus 1) = c_i(E)$ and hence

$$0 = \sum_i (-1)^i c_i(E \oplus 1) x^{n+1-i} = \sum_i (-1)^i c_i(E) x^{n+1-i} = U_H \cdot x.$$

To prove the theorem we need to calculate $ch(U_K)$. By the splitting principle we may assume that $E = L_1 \oplus \cdots \oplus L_n$ splits as a sum of line bundles. The Thom class $U_H = \sum_i (-1)^i c_i(E) x^{n-i}$ in $\mathbb{P}(E \oplus 1)$ then factors as the product

$$U_H = \prod_i (x - x_i) \in H^*(\mathbb{P}(E \oplus 1);\mathbb{Q})$$

where $x_i = c_1(L_i)$. Similarly, the Thom class in K-theory becomes

$$U_K = \prod_i (L - L_i) \in \tilde{K}^0(\mathbb{P}(E \oplus 1)).$$

Therefore we have

$$ch(U_K) = \prod_i ch(L - L_i) = \prod_i (e^x - e^{x_i}) = U_H \cdot \prod_i (\frac{e^{x_i} - e^x}{x_i - x}).$$

Since $U_H \cdot x = 0$, we can set $x = 0$ and simplify this expression to

$$ch(U_K) = U_H \cdot \prod_i (\frac{e^{x_i} - 1}{x_i}).$$

Since the Thom isomorphism $\vartheta \colon H^*(X;\mathbb{Q}) \to H^*(X^E;\mathbb{Q})$ is given by multiplication with U_H, we get the formula

$$\vartheta^{-1} ch(U_K) = \prod_i (\frac{e^{x_i} - 1}{x_i}) \in H^*(X;\mathbb{Q}).$$

Dealing with such power series becomes easier when we take the logarithm. There is a power series expansion for $\log(\frac{e^y-1}{y})$ of the form $\sum_k c_k \frac{y^k}{k!}$ for some coefficients c_k since the function $\frac{e^y-1}{y}$ is nonzero at 0. Then we can have

$$\log \vartheta^{-1} ch(U_K) = \log(\prod_i (\frac{e^{x_i}-1}{x_i})) = \sum_i \log(\frac{e^{x_i}-1}{x_i}) = \sum_{i,k} c_k \frac{x_i^k}{k!} = \sum_k c_k ch^k(E)$$

where $ch^k(E)$ is the component of $ch(E)$ in dimension $2k$. The last equation uses the fact that E is the sum of line bundles and the definition of the Chern character for line bundles. The splitting principle then tells us that the formula also holds for arbitrary E.

We need to calculate the coefficients c_k. Therefor we differentiate both sides of

$$\sum_k c_k y^k/k! = \log(\frac{e^y-1}{y}) = \log(e^y-1) - \log y.$$

This yields

$$\begin{aligned} \textstyle\sum_k c_k y^{k-1}/(k-1)! &= \tfrac{e^y}{e^y-1} - y^{-1} \\ &= 1 + \tfrac{1}{e^y-1} - y^{-1} \\ &= 1 - y^{-1} + \textstyle\sum_{k\geq 0} B_k y^{k-1}/k! \\ &= 1 + \textstyle\sum_{k\geq 1} B_k y^{k-1}/k! \end{aligned}$$

where the last equation follows from the fact that $B_0 = 1$. Thus we obtain

$$c_k = B_k/k \text{ for } k > 1 \text{ and } 1 + B_1 = c_1.$$

Since $B_1 = -1/2$, we get $c_1 = 1/2$ and $c_1 = -B_1/1$.

32.2. **The proof of Theorem 32.1.** Now we apply the discussion to the n-dimensional bundle $E \to S^{2k}$ corresponding to the element $x_{2k} \in \pi_{2k}BU$. We choose $U_K \in \tilde{K}^0(X_{Jf}) = \tilde{K}^0((S^{2k})^E)$ as the element mapping to a generator in $\tilde{K}^0(S^{2k})$ (changing signs if necessary). We know

$$ch(U_K) = a + r \cdot b \in H^*(X_{Jf}; \mathbb{Q})$$

and hence

$$\vartheta^{-1} ch(U_K) = 1 + r \cdot s$$

where s is a generator of $H^{2k}(S^{2k}; \mathbb{Q})$ and $r = e(J_{\mathbb{C}}f)$ in $\mathbb{Q}/\mathbb{Z}$. Hence

$$\log \vartheta^{-1} ch(U_K) = r \cdot s$$

since $\log(1+z) = z - z^2/2 + \cdots$ and $s^2 = 0$. On the other hand, we have

$$\log \vartheta^{-1} ch(U_K) = c_k ch^k(E)$$

since $H^{2j}(S^{2k}; \mathbb{Q}) = 0$ for $j \neq k$. Moreover, we showed in Lecture 28 that

$$ch^k(E) = s \in H^{2k}(S^{2k}; \mathbb{Q}).$$

Thus, by comparing the two formulas for $\log \vartheta^{-1} ch(U_K)$ we get

$$e(J_{\mathbb{C}} f) = r = c_k = \pm B_k/k.$$

This finishes the prof of Theorem 32.1.