Math 231b
Lecture 33

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33. Lecture 33: Clifford algebras and vector fields on spheres

And now turn the page...

Clifford Algebras and vector fields on spheres

Problem: Determine the maximum number of linearly independent vector fields on S^{n-1}.

Let $V_k(\mathbb{R}^n)$ be the Stiefel manifold of k-frames in $\mathbb{R}^n$.

$$V_k(\mathbb{R}^n) = \{[v_1, \dots, v_k] \mid v_i v_j = \delta_{ij}\}$$

$$= O(n)/O(n-k).$$

Have a map

$$\begin{tikzcd} V_k(\mathbb{R}^n) \arrow[d] & {[v_1,\dots,v_k]} \arrow[d, mapsto] \\ S^{n-1} \arrow[u, bend left, "\exists"] & v_k \end{tikzcd}$$

$\Updownarrow$

S^{n-1} has $(k-1)$ linearly independent vector fields.

Can we lift this?:

$$\begin{tikzcd} S^{n-k-1} \arrow[r] \arrow[d, phantom, "\parallel"] & V_{k+1}(\mathbb{R}^n) \arrow[r] & V_k(\mathbb{R}^n) \arrow[d] \arrow[l, dashed, bend right, "?"'] \\ \text{fiber of} \arrow[ur] & & S^{n-1} \arrow[u, bend right] \end{tikzcd}$$

Obstruction to going further is an elt of $\pi_{n-2} S^{n-k-1}$.

This is the setup.

Let us look at examples:

- We know even spheres have no vector fields ("hairy ball theorem")

- $S^{2n-1} \subset \mathbb{C}^n$
  $v \longmapsto iv$ gives a vector field

$S^{4n-1} \subset \mathbb{H}^n$
$v \longmapsto iv, jv, kv$ give 3 v. fields

$S^{8n-1} \subset \mathbb{O}^n \rightsquigarrow 7$ vector fields

This led to the expectation that S^{15} has 15 vector fields...

This is not true. So let us see why:

First a construction:

$$\begin{tikzcd} V_{k+1}(\mathbb{R}^n) \arrow[d, shift right] \\ S^{n-1} \arrow[u, dashed, shift right] \end{tikzcd} \Leftrightarrow k \text{ vector fields} \qquad \begin{array}{c} S^{n-1} \times V \hookrightarrow TS^{n-1} \\ \dim V = k \end{array}$$

get a map $S^{n-1} \times V \xrightarrow{\varphi} S^{n-1}$ st $\varphi(x,v) \perp x$

or a map $\mathbb{R}^n \times V \longrightarrow \mathbb{R}^n$, think of it as $T_v(x) = \varphi(x,v)$ as a transformation $\mathbb{R}^n \to \mathbb{R}^n \ \forall v$.

- some simplifying assumptions on this map:
  1) bilinear
  2) $T_v^2 = -1$ if $|v| = 1$, or $T_v^2 = -|v|^2$.

<u>Definition</u>: V vector space with $\langle , \rangle$.

$$Cl(V) = \text{free associative algebra on } V \big/ {v^2 = -|v|^2}$$

"Clifford algebra"

<u>E.g.</u>: $Cl_k = Cl(\mathbb{R}^k, \langle , \rangle)$

$Cl_k' = Cl(\mathbb{R}^k, -\langle , \rangle)$.

· Cl_k = free assoc. algebra on $e_1, \dots, e_k$ modulo

· $e_i^2 = -1$ for $i \neq j$: $(e_i + e_j)^2 = -|e_i + e_j|^2 = -2$

· $e_i e_j = -e_j e_i$ hence $e_i^2 + e_j^2 + e_i e_j + e_j e_i \Rightarrow e_i e_j = -e_j e_i = 0$

have $\dim Cl_k = 2^k$

· Cl'_k = gen. by $e'_1, \dots, e'_k$ mod $e_i'^2 = +1$, $e'_i e'_j = -e'_j e'_i$

Fact: If Cl_k acts on $\mathbb{R}^n$ then S^{n-1} has k vector fields

	Cl_k	Cl'_k
$k = 0$	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R} \leadsto \frac{e_1 - 1}{\sqrt{2}}, \frac{e_1 + 1}{\sqrt{2}}$
2	$\mathbb{H}$	$\mathbb{R}(2)$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$	
6	$\mathbb{R}(8)$	
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	
8	$\mathbb{R}(16)$	
9		

Notes:

· where $A(n) = \{n \times n$-matrices over $A\}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

· Think of $Cl(V)$ as a $\mathbb{Z}/2$ - graded algebra
$v \in V$ has degree 1, $v \cdot v = -|v|^2$

then $Cl(V \oplus W) \cong Cl(V) \hat{\otimes} Cl(W)$ = $\mathbb{Z}/2$-graded tensor product!

Prop: · $Cl_k \otimes Cl'_2 = Cl'_{k+2}$

· $Cl'_k \otimes Cl_2 = Cl_{k+2}$

This tells us how to complete the table!

Proof of prop: Have generators $e_1, \dots, e_k, e'_1, e'_2$ with $e_i^2 = -1$, $e_i e_j = -e_j e_i$, $e_1'^2 = e_2'^2 = 1$

now need to figure out how they "commute":   · $e'_1 e'_2 = -e'_2 e'_1$

e.g. $e_1 e'_1 e'_2 e_1 e'_1 e'_2 = e_1^2 e'_1 e'_2 e'_1 e'_2$

$= -e_1^2 e_1'^2 e_2'^2$

$= 1$.

$Cl'_{k+2} \to Cl_k \otimes Cl'_2$ sends

$e_1 \longmapsto e_1 e'_1 e'_2$

$e_k \longmapsto e_k e'_1 e'_2$

$e_{k+1} \longmapsto e'_1$

$e_{k+2} \longmapsto e'_2$ □

Some algebra facts: $\cdot\ \mathbb{H}\otimes\mathbb{C}=\mathbb{C}(2)$ (follows from the theory of central simple algebras)
(that we need) $\cdot\ \mathbb{H}\otimes\mathbb{H}=\mathbb{R}(4)$

We read off from the table and the prop.

$\cdot\ Cl_k\otimes Cl_2\otimes Cl_2' = Cl_k\otimes\mathbb{H}(2)$

$\cdot\ Cl_k\otimes\mathbb{R}(16)=Cl_{k+8}$ "Periodicity of Clifford algebras".

k	Cl_k	dim. of smallest real representation	
0	$\mathbb{R}$	1	
1	$\mathbb{C}$	2	
2	$\mathbb{H}$	4	
3	$\mathbb{H}\oplus\mathbb{H}$	4	$\leftarrow S^{4k-1}$ has 3 vector fields
4	$\mathbb{H}(2)$	8,	
5	$\mathbb{C}(4)$	8	
6	$\mathbb{R}(8)$	8	
7	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	8	$\leftarrow S^{8k-1}$ has 7 vector fields
8	$\mathbb{R}(16)$	16	and 8 vector fields on S^{15}
9	$\mathbb{C}(16)$	32	

This method produces a formula:

write $n=m\cdot 2^{r}$, $(m,2)=1$, $r=4c+d$, $\rho(n)=2^d+8c$:

then there are $\rho(n)-1$ vector fields on S^{n-1}.

$\rho(n)$ is called the <u>nth Radon Hurwitz number</u>.

Let us look at S^{15}:

$$
\begin{tikzcd}
S^6 \arrow[r] & V_{10}(\mathbb{R}^{16}) \arrow[r] & V_9(\mathbb{R}^{16}) \arrow[d] \\
& & S^{15} \arrow[u, bend right]
\end{tikzcd}
$$

The obstruction is in $\pi_{14} S^6 \cong \pi_8^{st}(S^0)$

This obstruction is $J_{\mathbb{R}}$(generator of $\pi_9 BO = \mathbb{Z}/2$).

Adams then showed that this obstruction is nonzero and thereby determined the number of vector fields on S^{15} (and on all spheres).