Math 231b
Lecture 34

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34. Lecture 34: The image of J and the Adams conjecture

34.1. **The image of J.** The stable real J-homomorphism is a map

$$\pi_{k-1}O \to \pi^s_{k-1}(S^0) = \pi_{k-1}S^0.$$

We are interested in the case $k = 4n$ because in those degrees the homotopy groups of O provide the most interesting image in the stable homotopy groups. We saw in the previous lectures that if $x_{4n-1}O$ is a generator then

$$e(Jx_{4n}) = \pm B_{2n}/4n$$

where B_i is the ith Bernoulli number. Hence the order of the image of J in $\pi_{4n-1}S^0$ is divisible by the denominator of $B_{2n}/4n$. Today we want to explore the information of the J-homomorphism a bit further.

Let us denote the denominator of $B_{2n}/4n$ by $m(2n)$. We have a lower bound for the image of J, for the order of $\operatorname{Im} J$ is divisible by $m(2n)$. So what about an upper bound? Adams showed that there is actually an upper bound and thereby determined the image of J in $\pi_{4n-1}S^0$ completely. (Well, almost completely since he could not figure out a possible factor of 2 for $4n \equiv 0 \bmod 8$.) We want to follows Adams' great ideas and see how close he got to determine the image of J.

Adams proved the following result.

Theorem 34.1. *The image $J(\pi_{4n-1}O)$ of the stable J-homomorphism in $\pi_{4n-1}S^0$ is cyclic of order*

(i) *$m(2n)$ if $4n \equiv 4$ modulo 8*
(ii) *$m(2n)$ or $2m(2n)$ if $4n \equiv 0$ modulo 8.*

Remark 34.2. Mahowald showed later that the factor 2 in (ii) is not there. Adams could not settle this factor since he could prove his conjecture only for the complex K-theory and not for the real K-theory of S^{4n}. Adams' conjecture was then proven independently and in full generality by Quillen-Friedlander, Quillen, Sullivan and Becker-Gottlieb. We are going to sketch a proof in the next lecture.

Before we think about a proof, let us first note a consequence of Theorem 34.1. Let $j\colon \operatorname{Im} J \hookrightarrow \pi_{4n-1}S^0$ denote the inclusion. Adams shows that the image of

e in $\mathbb{Q}/\mathbb{Z}$ is precisely the subgroup of cosets $z/m(2n)$, $z \in \mathbb{Z}$. Hence we have a commutative diagram

$$\begin{array}{ccc} & \pi_{4n-1}S^0 & \\ {}^{j}\nearrow & & \searrow^{e} \\ \mathrm{Im}\, J & \xrightarrow{\;e\circ j\;} & \mathbb{Z}/m(2n). \end{array}$$

By Theorem 34.1 and its improvement we know that Im J is cyclic of order $m(2n)$. Therefore the diagram provides a direct sum splitting

$$\pi_{4n-1}S^0 \cong \mathrm{Im}\, J \oplus \mathrm{Ker}\, e.$$

Example 34.3. For $r = 4n-1$ let us take the generator in $\pi_r SO$ and let its image under $J\colon \pi_r SO \to \pi_r S^0$ be j_r. Then we have:

$$e(j_3) = 1/24,\ \ e(j_7) = -1/240,\ \ e(j_{11}) = 1/504,\ \ e(j_{15}) = -1/480,\ \ e(j_{19}) = 1/264.$$

For $r = 3, 7, 11$, we have

$$\pi_3 S^0 \cong \mathbb{Z}/24,\ \ \pi_7 S^0 \cong \mathbb{Z}/240,\ \ \pi_{11} S^0 \cong \mathbb{Z}/504.$$

Or in other words, the kernel of e is trivial in these cases. But for $r = 15, 19$, the kernel of e is $\mathbb{Z}/2$.

Remark 34.4. Since the numbers $m(2n)$ are unbounded we see that, even though the stable homotopy groups $\pi_r S^0$ are of finite, arbitrarily large orders can occur.

34.2. **Adams' upper bound for** Im J**.** We know that Im J is divisible by $m(2n)$. To prove Theorem 34.1 we need an argument in the opposite direction.

Let Y be an abelian group with Adams operations, i.e., an abelian group with endomorphisms ψ^k for every $k \in \mathbb{Z}$. A map between such groups is a homomorphism of abelian groups which is compatible with the operations.

Let e be a function that assigns to each pair $k \in \mathbb{Z}$, $y \in Y$ a non-negative integer $e(k, y)$. Then we define Y_e to be the subgroup of Y generated by the elements

$$k^{e(k,y)}(\psi^k - 1)y.$$

It is clear that if

$$e_1 \geq e_2, \text{ then } Y_{e_1} \subseteq Y_{e_2}.$$

Hence we can define

$$J''(X) := Y/\cap_e Y_e$$

where the intersection runs over all functions e.

Remark 34.5. If Y is finitely generated, it is easy to see that it suffices to let e run over the functions f which are independent of y and get the same quotient group $J''(X)$. For it is clear that

$$\cap_e Y_e \subseteq \cap_f Y_f.$$

For $y \in Y$, let $y_1, \dots, y_n$ generate y. For any function $e(k, y)$ define the corresponding function $f(k)$ by

$$f(k) := \mathrm{Max}_{1 \leq r \leq n} e(k, y_r).$$

It is clear that we have $Y_f \subseteq Y_e$ and hence

$$\cap_f Y_f \subseteq \cap_e Y_e.$$

Moreover, if Y_1 and Y_2 are finitely generated, then we have

$$(Y_1 \oplus Y_2)_f = (Y_1)_f \oplus (Y_2)_f$$

and hence

$$\cap_f (Y_1 \oplus Y_2)_f = \cap_f (Y_1)_f \oplus \cap_f (Y_2)_f.$$

As a consequence we get

$$J''(Y_1 \oplus Y_2) = J''(Y_1) \oplus J''(Y_2).$$

For $Y = K(X)$ we set $J''_{\mathbb{C}}(X) := J''(K(X))$ and for $Y = KO(X)$ we set $J''(X) := J''(KO(X))$. Let

$$r \colon K(X) \to KO(X)$$

be the canonical map. Since it is compatible with the Adams operations, it induces a map

$$J''_{\mathbb{C}}(X) \to J''(X).$$

Proposition 34.6. *a) Let P be a point. Then*

$$J''(P) = \mathbb{Z}.$$

b) If X is a finite cell complex, then

$$J''(X) = \mathbb{Z} + \tilde{J}''(X) \text{ with } \tilde{J}''(X) = J''(\tilde{KO}(X)).$$

Proof. a) We know $KO(P) = \mathbb{Z}$ and the operations are just given by $(\psi^k - 1)y = 0$ for all k and y.
b) We just need to apply part a) and the second part of the above remark. □

Here is the reason why we are interested in the groups $J''(Y)$ for real K-theory. Adams made the following important conjecture. The formulation of the conjecture and its proof require to give a different interpretation of $J(X)$ in terms of spherical fibrations. Since we will need some time to think about these fibrations in more detail, we postpone this interpretation for a moment.

Nevertheless we formulate the conjecture in its general form and think for now of the special case $X = S^m$.

The Adams conjecture 34.7. *If k is an integer, X a finite cell complex and $y \in KO(X)$, then there exists a non-negative integer $e = e(k)$ such that*

$$J(k^e(\psi^k - 1)y) = 0.$$

The consequence of the conjecture for our discussion is the following.

Proposition 34.8. *Suppose for S^{4n} Conjecture 34.7 holds for all k and y. Then $\tilde{J}''(S^{4n})$ is an upper bound for $\operatorname{Im} J$ in the sense that the surjective map $J\colon KO(S^{4n}) \to \operatorname{Im} J$ factors through an epimorphism $\tilde{J}''(S^{4n}) \to \operatorname{Im} J$.*

Example 34.9. Take X to be the sphere S^{4n}. We claim that the group $\tilde{J}''(S^{4n})$ is cyclic of order $m(2n)$. If $y \in \tilde{KO}(S^{4n})$, we have

$$k^{f(k)}(\psi^k - 1)y = k^{f(k)}(k^{2n} - 1)y$$

since ψ^k acts on the K-theory of S^{4n} by multiplication by k^{2n}. (We proved this only for complex K-theory, but the same argument shows it for real K-theory too.) Thus the subgroup Y_f of $\tilde{KO}(S^{4n}) = \mathbb{Z}$ consists of the multiples of $h(f, 2n)$ where $h(f, 2n)$ is the greatest common divisor of the integers

$$k^{f(k)}(k^{2n} - 1), \text{ for all } k \in \mathbb{Z}.$$

But this number is exactly $m(2n)$. Hence $\tilde{J}''(S^{4n}) = \tilde{KO}(S^{4n})/Y_f = \mathbb{Z}/m(2n)$.

34.3. **Adams' proof of Theorem 34.1.** Adams proved Conjecture 34.7 for the real K-theory of a sphere S^{2n} under the assumption that the map

$$r\colon \tilde{K}(S^{2n}) \to \tilde{KO}(S^{2n})$$

is an epimorphism.

For $4n \equiv 4$ modulo 8, the map

$$r\colon \tilde{K}(S^{4n}) \to \tilde{KO}(S^{4n})$$

is an epimorphism. Hence by Proposition 34.8 $\tilde{J}''_{\mathbb{R}}(S^{4n})$ is an upper bound for $\operatorname{Im} J$. By Example 34.9 this implies that $\tilde{J}_{\mathbb{R}}(S^{4n})$ divides $m(2n)$.

For $4n \equiv 0$ modulo 8 the proof would be the same except that in this case image of the map

$$r\colon \tilde{K}(S^{4n}) \to \tilde{KO}(S^{4n})$$

consists of the elements divisible by 2. For this case Adams could not prove his conjecture for S^{4n} and hence he could not settle the factor 2. We will investigate this further in the next lecture.