Math 231b
Lecture 35

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35. Lecture 35: Sphere bundles and the Adams conjecture

35.1. **Sphere bundles.** Let X be a connected finite cell complex. We saw that the J-homomorphism could be defined by sending an automorphism of $\mathbb{R}^n$ to the induced automorphism of the one-point compactification. Today we want to generalize this construction and study J as a construction on vector bundles as follows.

Let $E \to X$ be an n-dimensional real vector bundle. By taking the *fiberwise one-point compactification* we get an associated fiber bundle $S(E) \to X$ whose fibers are all n-spheres S^n. We call such a bundle a sphere bundle.

We will say that a map $f\colon S(E) \to S(E')$ of bundles is a *fiber homotopy equivalence* if there is a bundle map $g\colon S(E') \to S(E)$ such that $f \circ g$ and $g \circ f$ are homotopic through bundle maps to the respective identities.

Taking the associated sphere bundle of a vector bundle respects direct sums in the sense that

$$S(E \oplus E') \cong S(E) \wedge_X S(E')$$

where $\wedge_X$ denotes the fiberwise smash product.

Definition 35.1. We denote by $\mathcal{SF}(X)$ the Grothendieck group of pointed sphere bundles over X modulo fiber homotopy equivalence. The group law is given by the fiberwise smash product.

Remark 35.2. A fiber bundle whose fibers who are all of the *homotopy type* of a sphere is called a *pointed spherical fibration*. Hence we could have defined $\mathcal{SF}(X)$ also as the Gorthendieck group of (pointed) spherical fibrations.

Sending a vector bundle to its fiberwise one-point compactification defines a homomorphism

$$KO(X) \to \mathcal{SF}(X).$$

Example 35.3. We want to understand this map for X a sphere. A vector bundle over X is determined by its clutching function. This can be expressed as an isomorphism

$$\tilde{KO}(S^n) \cong \pi_{n-1}O.$$

Similarly, a sphere bundle is determined by a clutching function

$$f\colon S^{n-1} \to \mathrm{Homeo}(S^k, S^k).$$

Since we are only interested in sphere bundles modulo fiber homotopy equivalence, it suffices to specify the clutching function up to homotopy equivalence. Hence a function

$$f\colon S^{n-1} \to \mathrm{Equiv}(S^k, S^k)$$

to the monoid of homotopy self-equivalences of S^k determines a spherical fibration over X or a sphere bundle up to fiber homotopy equivalence . Let us denote this topological monoid by $G(k) = \mathrm{Equiv}(S^k, S^k)$. If we choose k large enough, we have an isomorphism

$$\mathcal{SF}(S^n) \cong \pi_{n-1}G(k) \text{ for } k \gg 0.$$

But we can say a bit more. An element of $G(k)$ is a map $S^k \to S^k$. Now we observe that $G(k)$ is a subset of maps of degree ± 1

$$\Omega^k_{\pm 1}S^k \subset \Omega^k S^k = \mathrm{Map}_*(S^k, S^k).$$

Therefore, if we subtract the identity, we get an isomorphism

$$\pi_{n-1}G(k) \cong \pi_{n-1+k}(S^k) \text{ for } k \gg 0.$$

Thus, the group $\mathcal{SF}(S^n)$ is equal to the $(n-1)$st stable homotopy group of the sphere

$$\mathcal{SF}(S^n) \cong \pi^s_{n-1}(S^0).$$

Hence, for $X = S^n$, the map

$$KO(S^n) \to \mathcal{SF}(S^n)$$

defined by taking fiberwise one-point compactifications *is* the J-homomorphism.

Motivated by this example, we will call the map

$$J\colon KO(X) \to \mathcal{SF}(X)$$

the *J-homomorphism* for any finite cell complex X. As a consequence of the discussion in Example 35.3 we also get the following finiteness result of Atiyah's.

Proposition 35.4. *If X is a connected finite cell complex, the group $\mathcal{SF}(X)$ is finite.*

Sketch of a proof. We can argue just as in Example 35.3 that every element in $\mathcal{SF}(X)$ is classified by a homotopy class of a map

$$X \to BG(k) \text{ for } k \gg 0$$

where $BG(k)$ denotes the classifying space of the monoid $G(k)$ (such a classifying space construction exists). Since X is a finite cell complex we can use induction on the number of cells and are reduced to show that $\pi_n BG(k)$ is finite. But the latter group is equal to $\pi_{n-1}G(k)$ and we have seen in Example 35.3 that this group is equal to $\pi^s_{n-1}(S^0)$. The stable homotopy groups of the sphere spectrum are finite by Serre. □

35.2. **The Adams conjecture.** Adams conjectured the following property for the J-homomorphism.

Theorem 35.5 (The Adams Conjecture)**.** *Let X be a finite cell complex, k an integer, and $y \in KO(X)$. Then there exists a non-negative integer $e = e(k, y)$ such that*

$$k^e(\psi^k - 1)y \in \operatorname{Ker} J.$$

Moreover, these elements (for all k) generate the kernel of J.

Remark 35.6. We could reformulate the assertion of the theorem as follows. For every prime p not dividing k the kernel of the map

$$KO(X)_{(p)} \to \mathcal{SF}(X)_{(p)}$$

is generated by elements of the form $(\psi^k - 1)y$.

Before we go on, let us see how the following result of Adams', used in the previous lecture for $X = S^{4n}$, follows from the first part of Theorem 35.5. (We use the notation of the previous lecture.)

Proposition 35.7. *The group $J''(X)$ is an upper bound for the image of J in $\mathcal{SF}(X)$.*

Proof. Let $T(X)$ be the kernel of J and $Y = KO(X)$. By Theorem 35.5 there is a function $e(k, y)$ such that $Y_e \subseteq T(X)$, where Y_e is the subgroup of Y generated by all elements of the form $k^e(\psi^k - 1)y$. This shows that the intersection $\cap_e Y_e$ is contained in $T(X)$. But $J''(X)$ is by definition the quotient

$$J''(X) = Y/\cap_e Y_e.$$

So we have a surjective map $KO(X)/\cap_e Y_e \to KO(X)/T(X)$. In particular, every element in the image of J is also in the image of the induced map $J''(X) \to SF(X)$. □

35.3. **Line bundles and the mod k Dold theorem.** We will sketch a proof of Adams' conjecture in the next lecture. Today we study some special cases. We begin with an easy observation.

Remark 35.8. If the first assertion of Theorem 35.5 holds for all vector bundles of even rank, then it holds for all vector bundles. For, if ξ is a bundle of odd rank, then by assumption there is an N such that

$$k^N(\psi^k - 1)(\xi \oplus \epsilon^1) \in \operatorname{Ker} J,$$

and hence

$$k^N(\psi^k - 1)\xi = k^N(\psi^k(\xi) - \xi) + k^N(\epsilon^1 - \epsilon^1) = k^N(\psi^k(\xi \oplus \epsilon^1) - (\xi \oplus \epsilon^1)) \in \mathrm{Ker}\, J.$$

Proposition 35.9. *Let $y \in KO(X)$ be a linear combination of real line bundles over the finite cell complex X. Then there exists an $e \in \mathbb{N}$ (depending only on the dimension of X) such that*

$$k^e(\psi^k - 1)y = 0.$$

Proof. Since $k^e(\psi^k - 1)y$ is linear in y, it suffices to consider the case in which y is a real line bundle. In this case, since X is a finite cell complex, there exists a map $f\colon X \to \mathbb{R}\mathrm{P}^n$ for some n such that $y = f^*\gamma$, where γ is the canonical real line bundle over $\mathbb{R}\mathrm{P}^n$. Hence it suffices to prove the assertion for $y = \gamma$.

The $KO(\mathbb{R}\mathrm{P}^n)$ is a finite 2-group generated by $1-\gamma$. (If you know about spectral sequences, you can deduce this easily from the Atiyah-Hirzebruch spectral sequence and the fact that the cohomology of $\mathbb{R}\mathrm{P}^n$ is a finite 2-group.) Hence there is an $e \in \mathbb{N}$ such that

$$2^e(\psi^k - 1)y = 0.$$

If k is even, this implies $k^e(\psi^k - 1)y = 0$. If k is odd, then we have the relation $y^2 = 1$ in $KO(\mathbb{R}\mathrm{P}^n)$. This implies $\psi^k(y) = y^k = y$ and hence $(\psi^k - 1)y = 0$. To see that we have $y^2 = 1$ there are many different ways. For example, one could use the fact that real line bundles are characterized by their first Stiefel-Whitney class. Or one notices that the structure group of a real line bundle is $O(1) = \{+1, -1\}$ from which one sees $\gamma \otimes \gamma = 1$. □

The proof of Theorem 35.5 uses the following generalization of Dold's results.

Theorem 35.10 (mod k Dold theorem)**.** *Let X be a finite cell complex. Suppose there a map of sphere bundles $\xi_1 \to \xi_2$ of the same dimension such that the map on fibers $S^n \xrightarrow{k} S^n$ is of degree k. Then there exists a non-negative integer e such that $k^e\xi_1$ and $k^e\xi_2$ are fibre homotopy equivalent and hence $k^e\xi_1 = k^e\xi_2 \in \mathcal{SF}(X)$.*

Example 35.11. Let L be a complex line bundle, or equivalently an oriented 2-dimensional real vector bundle. Then the map

$$X \to \mathbb{C}\mathrm{P}^\infty \xrightarrow{k} \mathbb{C}\mathrm{P}^\infty$$

classifies $L^{\otimes k}$. The map $\mathbb{C}\mathrm{P}^\infty \xrightarrow{k} \mathbb{C}\mathrm{P}^\infty$ is covered by a map of universal bundles which is fiberwise the degree k map. For sending L to $L^{\otimes k}$ corresponds in each fiber to the map $z \mapsto z^k$. Then the mod k Dold theorem implies that there is an e such that $k^e\psi^k(L) = k^e L^{\otimes k}$ and $k^e L$ are fiber homotopy equivalent. Alternatively, we could say that $\psi^k(L) - L = 0 \in \mathcal{SF}(X)[k^{-1}]$.

35.4. **Sketch of Adams' proof for $X = S^{4n}$, $4n \equiv 4$ mod 8.** Let $X = S^{2n}$ such that the map

$$r\colon K(S^{2n}) \to KO(S^{2n})$$

is surjective. So given $y \in KO(S^{2n})$ there is a $z \in K(S^{2n})$ such that $y = r(z)$. Now consider the map

$$q\colon W = S^2 \times \cdots \times S^2 \to S^2 \wedge \cdots \wedge S^2 \to S^{2n}$$

Over W every vector bundle is a linear combination of complex line bundles (think of S^2 as $\mathbb{C}\mathrm{P}^1$). In particular, q^*z is such a linear combination. Therefore

$$q^*y = r(q^*z)$$

is a linear combination of oriented 2-dimensional real vector bundles. By Example 35.11 we know that there is an e such that

$$k^e(\psi^k - 1)q^*y = q^*(k^e(\psi^k - 1)y)$$

maps to zero in $\mathcal{SF}(W)$. Finally, Adams shows that the map

$$q^*\colon \mathcal{SF}(S^{2n}) \to \mathcal{SF}(W)$$

is a monomorphism. (This requires only some knowledge about the classifying space $BG(k)$ and mapping cones.)

Adams also proved the case that $y \in KO(X)$ is a linear combination of $O(1)$- and $O(2)$-bunldes. The general case was later proved independently and by very different methods by Quillen-Friedlander, Quillen, Sullivan, and Becker Gottlieb. We will sketch a proof in the next lecture.