Math 231b
Lecture 36

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36. Lecture 36: Sullivan's proof of the Adams conjecture

Today we will have a look at Sullivan's beautiful ideas on Galois symmetries in topology and his proof of the Adams conjecture in the complex case. We will omit a lot of details and just outline the ideas. We encourage everyone to read Sullivan's original paper and lecture notes.

36.1. **The Adams conjecture.** Let X be a connected finite cell complex. We defined $\mathcal{SF}(X)$ as the Grothendieck group of sphere bundles over X modulo fiber homotopy equivalence. Sending a vector bundle to its fiberwise one-point compactification defines the J-homomorphism

$$J \colon KO(X) \to \mathcal{SF}(X).$$

For $X = S^n$ a sphere we showed that there is a natural isomorphism

$$\mathcal{SF}(S^n) \cong \pi^s_{n-1}(S^0)$$

with the stable homotopy group of the sphere.

Our goal is to show the following result.

Theorem 36.1 (The Adams Conjecture)**.** *Let X be a finite cell complex, k an integer, and $y \in KO(X)$. Then there exists a non-negative integer $e = e(k,y)$ such that*

$$k^e(\psi^k - 1)y \in \operatorname{Ker} J.$$

Last time we defined the monoid $G(n) = \operatorname{Equiv}(S^n, S^n)$ of self-homotopy equivalences of S^n. Taking smash product with a circle defines a map $G(n) \to G(n+1)$. Moreover, since a linear self-transformation of $\mathbb{R}^k$ extends via one-point compactification to a self-homotopy equivalence of S^n, we have a canonical map $O(n) \to G(n)$. Since we study only the complex case today (though the real case follows from an analogous argument), we compose this map with $U(n) \to O(2n)$ and get a map

$$U(n) \to G(2n).$$

This map induces a map of corresponding classifying spaces

$$BU(n) \to BG(2n).$$

We denote the colimit of the $BG(n)$ over n by BG:

$$BG := \operatorname*{colim}_{n\to\infty} BG(n).$$

Overall, we obtain a commutative diagram

$$\begin{array}{ccc} BU(n) & \longrightarrow & BG(2n) \\ \downarrow & & \downarrow \\ BU & \longrightarrow & BG. \end{array} \tag{1}$$

The space BG is the classifying space of (stable) spherical fibration (sphere bundles up to fiber homotopy equivalence). Hence the set of spherical fibrations over X is in bijection to the set of homotopy classes of maps

$$[X, BG].$$

Now the (complex) J-homomorphism $K(X) \to \mathcal{SF}(X)$ corresponds to a map

$$[X, BU] \to [X, BG]$$

which is induced by the above map of classifying spaces which we also denote by

$$J\colon BU \to BG.$$

Furthermore, the kth Adams operation corresponds to a map of classifying spaces

$$\psi^k\colon BU \to BU.$$

Now given an n-dimensional complex vector bundle E over X, its associated sphere bundle corresponds to a map

$$X \xrightarrow{E} BU(n) \xrightarrow{i} BU \xrightarrow{J} BG$$

where i is the inclusion. If we apply the kth Adams operation we get a corresponding map

$$X \xrightarrow{E} BU(n) \xrightarrow{\psi^k} BU \xrightarrow{J} BG.$$

Hence to prove the Adams conjecture we need to show that up to multiplication by some power k^e the map

$$BU(n) \xrightarrow{\psi^k - i} BU \xrightarrow{J} BG \tag{2}$$

is null-homotopic, that is homotopic to a constant map.

Let us dream about a strategy for the proof for a moment. The homotopy class of the map

$$J \circ i\colon BU(n) \to BG$$

classifies a sphere bundle up to fiber homotopy. This bundle is the sphere bundle associated to the canonical bundle γ_n over $BU(n)$. Now it turns out that this bundle is fiber homotopy equivalent to the fibration

$$BU(n-1) \to BU(n).$$

Hence we can also think of $BU(n-1)$ as the total space of the spherical fibration $J(\gamma_n)$.

Then if we had a (homotopy) pullback diagram of the form

$$\begin{CD} BU(n-1) @>{\psi^k}>> BU(n-1) \\ @V{i}VV @VV{i}V \\ BU(n) @>{\psi^k}>> BU(n) \end{CD} \tag{3}$$

with self-homotopy equivalences ψ^k then we would be done. For, the diagram would show that

- the spherical fibration over $BU(n)$ classified by $J \circ \psi^k$ is the pullback of
$$i \colon BU(n-1) \to BU(n) \text{ along } \psi^k \colon BU(n) \to BU(n);$$
- and hence, since the maps ψ^k are equivalences, the sphere bundles corresponding to $J \circ i$ and $J \circ \psi^k$ are fiber homotopy equivalent.

Unfortunately, the Adams operations ψ^k are self-homotopy equivalences of BU and there is no way to produce them as operations on $BU(n)$ (at least not compatibly for all n and with all properties).

This is a bummer. But here comes Sullivan's great idea. Even though the ψ^k do not exist on the $BU(n)$, they exist on the profinite completion $\hat{BU}(n)$. Moreover, they fit into a beautiful picture of Galois symmetries in topology. Let us have a look at how this works.

36.2. **Galois symmetries.** The crucial observation is that the homotopy groups of BG are finite (remember they are isomorphic to the stable homotopy groups of the sphere spectrum). This implies that the map $J \colon BU \to BG$ factors through the profinite completion of BU

$$\begin{array}{ccccc} & & \hat{BU} & & \\ & \nearrow & & \searrow^{\hat{J}} & \\ BU & & \xrightarrow{\quad J \quad} & & BG. \end{array}$$

The space $\hat{BU}$ is the *profinite completion of* BU, i.e., it is a space endowed with a map $BU \to \hat{BU}$ which induces the profinite completion on homotopy groups

$$\pi_* BU \to \pi_* \hat{BU} = (\pi_* BU)^\wedge,$$

which, in even degrees, is just the completion of the integers $\mathbb{Z} \to \hat{\mathbb{Z}}$.

We call

$$\hat{K}(X) = [X, \hat{BU}]$$

the *profinite* K*-theory* of X.

Remark 36.2. Such a space $\hat{BU}$ exists and Sullivan establishes a lot of interesting results about profinite homotopy. We will skip to explain how you obtain $\hat{BU}$ and omit the technical subtleties, since there is more interesting theory to explore. Another source for profinite completion in homotopy theory is the work of Artin-Mazur.

Now Sullivan shows that the map from stable fiber homotopy types to profinite stable homotopy types is injective. Hence it suffices to show that, up to multiplication by some power k^e, the induced composite map

$$\hat{BU(n)} \xrightarrow{\psi^k - i} \hat{BU} \xrightarrow{\hat{J}} BG \tag{4}$$

is null-homotopic. In fact, since we are only interested in showing that the map is null-homotopic after localizing at p, $(p, k) = 1$, it suffices to consider pro-p-completions. So we consider $\hat{BU}$ as the p-completed space if necessary, even though we will omit the p in the notation. (The smarter way to handle this is to redefine the ψ^k on the profinite completion as the identity if p divides k.)

Next comes a really cool move of Sullivan's. Using algebraic geometry, in particular étale homotopy theory, he interprets the Adams operations on the profinite completion of BU as elements in the absolute Galois group of $\mathbb{Q}$ and shows that there are unstable operations ψ^k on each $\hat{BU(n)}$. This is all the more remarkable, since the ψ^k do not exist as operations $BU(n) \to BU(n)$ (if we require all the nice properties they have as self-maps of BU).

Here is the idea. We can write the complex Grassmannian $\mathrm{Gr}_n(\mathbb{C}^{n+k})$ as a quotient

$$\mathrm{Gr}_n(\mathbb{C}^{n+k}) \cong \mathrm{GL}(n+k, \mathbb{C})/(\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C})). \tag{5}$$

So we may consider the Grassmannian as an affine smooth complex algebraic variety (for the real Grassmannian replace $GL(-, \mathbb{C})$ with $O(-, \mathbb{C})$).

Now there is a purely algebraic way to assign to every algebraic variety V over any base field a homotopy type represented by a CW-complex. The machinery which does this is called *étale homotopy theory* and has been developed by Artin-Mazur and Friedlander. The idea is similar to computing cohomology via Čech coverings. If X is a nice topological space we can compute its cohomology by taking an open covering $U \to X$ and form the Čech nerve. If the covering is nice, i.e., if each intersection of open sets is contractible, then the cohomology of the Čech nerve is equal to the cohomology of X. Not every space admits nice coverings, but if we take the limit over all coverings, i.e., the colimit over all cohomology groups of the corresponding Čech nerves, then we still recover the cohomology of X.

Now we transport this idea to algebraic geometry. Unfortunately, there are not enough open coverings of a variety V in its intrinsic topology, the Zariski topology. But Grothendieck showed that we do not actually need a topology in the usual sense to compute cohomology, it suffices to consider maps $U \to X$ of a certain types (instead of taking open *subsets*). The correct generalization of an open subset in our case is the notion of an *étale map*. An étale map between (smooth) algebraic varieties is the analogue of a local diffeomorphism between manifolds. You should think about what that means or read about it. There is actually a criterion using Jacobian determinants which makes the analogy obvious.

So we can speak of an *étale open covering* by taking an étale surjective map $U \to V$. Now we can apply the Čech construction and form a simplicial variety $U.$ whose nth term is the $(n+1)$-fold fiber product

$$U \times_X U \times_X \cdots \times_X U$$

of U over X. Applying the connected component functor to $U.$ in each degree yields a simplicial set $\pi_0(U.)$. Taking its geometric realization gives us a CW-complex. If V is a finite-dimensional smooth variety, then this is actually a finite cell complex.

As in topology taking just one such covering is not enough to describe the homotopy type of V. But if we make the coverings finer and finer and consider the colimit over all of them (actually the cofiltering system of all such coverings), then we get the correct *profinite homotopy type*.

Remark 36.3. Using étale Čech coverings is actually sufficient for smooth quasi-projective varieties over a field. For more general schemes one has to consider hypercoverings. But that's a different story.

So let us focus on our case. What we learn from this story is that there is a purely algebraic construction of the profinite homotopy type of the Grassmannian manifold and we can write

$$\hat{\mathrm{Gr}}_n(\mathbb{C}^{n+k}) \simeq \lim_{\alpha} N_\alpha \tag{6}$$

where the N_α run through these algebraic étale coverings space (actually the associated finite cell complexes).

Now we come to the crucial point. The equations defining the Grassmannian in (5) actually have rational (in fact integer) coefficients. So we can consider the Grassmannian as a variety *defined over* $\mathbb{Q}$. Hence each automorphism σ of $\mathbb{C}$ fixing $\mathbb{Q}$ acts on the (complex) points of the Grassmannian. This is nice, though there is the problem: the action of σ is "highly discontinuous", at least in the sense that it does not induce an interesting automorphism on cohomology.

That's bad news. But here is the solution: Each variety N_α in (6) is defined over $\mathbb{Q}$ and the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on the system of the N_α's. After taking the union over all k, *this defines an action of* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ *on the* profinite classifying space $B\hat{U}(n)$ *(and on $\hat{BU}$).*

Now consider the natural surjective homomorphism

$$\chi\colon \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) \to \hat{\mathbb{Z}}_p^*$$

obtained by letting $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ act on the roots of unity. (This is also called the *cyclotomic character*.)

Example 36.4. One can check that $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on $B\hat{U}(1) = \hat{\mathbb{C}\mathrm{P}}^\infty = K(\hat{Z}_p, 2)$ via χ and the natural action of $\hat{Z}_p^*$ on $K(\hat{Z}_p, 2)$. (You should do this yourself after reading more about étale coverings, but you could also look it up in Sullivan's MIT notes §5.)

From this example it follows by naturality and the splitting principle that $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts through $\hat{\mathbb{Z}}_p^*$ and χ on $B\hat{U}(n)$. That means that σ acts on cohomology via

$$\sigma(c_i) = \chi(\sigma)^{-1} c_i$$

where c_i is the ith Chern class (which is a generator of the cohomology of $BU(n)$).

Proposition 36.5. *Given k in $\hat{Z}_p^*$, choose a $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ such that $\chi(\sigma) = k^{-1}$. Then*

$$\sigma\colon B\hat{U}(n) \to B\hat{U}(n)$$

is an unstable Adams operation *in the sense that the diagram*

$$\begin{array}{ccc} B\hat{U}(n) & \longrightarrow & \hat{BU} \\ \downarrow \sigma & & \downarrow \hat{\psi}^k \\ B\hat{U}(n) & \longrightarrow & \hat{BU} \end{array}$$

is commutative up to homotopy. Moreover, the operations σ are compatible if n varies.

Sketch of the proof. To show that the diagram is homotopy commutative amounts to show that the elements in profinitely completed K-theory corresponding to the horizontal maps agree. For this it suffices to show by the splitting principle that the diagram

$$\begin{array}{ccc} \widehat{BU(1)} & \longrightarrow & \widehat{BU} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \hat{\psi}^k} \\ \widehat{BU(1)} & \longrightarrow & \widehat{BU} \end{array}$$

is commutative up to homotopy. But we know from the above example that σ raises elements to the kth power and this is what ψ^k does on line bundles. □

Remark 36.6. The fact that we can define Adams operations on the profinite completion $\widehat{BU(n)}$ is very remarkable, since there are no unstable Adams operations on $BU(n)$ itself. *The key is the natural Galois action on the inverse system of étale coverings.*

So Sullivan concludes that we can reformulate the Adams conjecture in the following way.

Theorem 36.7. *The stable fiber homotopy type of elements in profinite K-theory is constant on the orbits of the Galois group.*

Proof. Proposition 36.5 shows that we have a homotopy pullback diagram

$$\begin{array}{ccc} \widehat{BU(n-1)} & \xrightarrow{\psi^k} & \widehat{BU(n-1)} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ \widehat{BU(n)} & \xrightarrow{\psi^k} & \widehat{BU(n)} \end{array} \tag{7}$$

where the ψ^k are given by the Galois symmetries σ and are homotopy equivalences. So for the profinite completions we can argue as we wanted that

- the completed spherical fibration over $\widehat{BU(n)}$ classified by $\hat{J} \circ \psi^k$ is the pullback of

  $$i\colon \widehat{BU(n-1)} \to \widehat{BU(n)} \text{ along } \psi^k = \sigma\colon \widehat{BU(n)} \to \widehat{BU(n)};$$

- and hence, since the maps $\psi^k = \sigma$ are equivalences, the completed sphere bundles corresponding to $\hat{J} \circ i$ and $\hat{J} \circ \psi^k$ are fiber homotopy equivalent.

This shows that the sphere bundles associated to $\hat{\gamma}_n$ and $\psi^k(\hat{\gamma}_n) = \hat{\gamma}_n^\sigma$ have the same unstable profinite homotopy types. But this implies that also the stable sphere bundles associated to $\hat{\gamma}$ and $\psi^k(\hat{\gamma}) = \hat{\gamma}^\sigma$ have the same stable profinite homotopy types. □

Remark 36.8. 1. This completes the proof of the Adams conjecture in the complex case. The argument for the real case is similar. We just have to take care of the extra information of the extension $\mathbb{C}/\mathbb{R}$.
2. The proof shows more than just the stable version in Theorem 36.7. It also proves an unstable (real and complex) profinite version of the Adams conjecture.
3. It is in fact not necessary to just complete at primes p with $(p, k) = 1$. If one redefines the Adams operations appropriately at the primes p dividing k one can take profinite completions with respect to all primes at once.