Math 231br
Problem Set 1

Spring 2014

You should hand in solutions to at least three problems, but please feel free to work on as many problems as you like and to hand in all your solutions. For any questions, please send me an email to gquick@math.harvard.edu and/or come to my office hours on Wednesdays 1.30-2.30pm in SC 341. Solutions are due by Friday, February 28, at the beginning of class.

Problem 1.1. Using a partition of unity, show that any real vector bundle over a paracompact base space can be given a Euclidean metric.

Recall that a smooth *tangent vector field* on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ is a smooth map

$$v\colon S^n \to \mathbb{R}^{n+1} \text{ such that } v(x) \cdot x = 0 \text{ for all } x \in S^n.$$

Problem 1.2. a) Show that the unit n-sphere S^n admits a tangent vector field which is nowhere zero providing that n is odd.
b) If S^n admits a tangent vector field which is nowhere zero, show that the identity map of S^n is homotopic to the antipodal map.

Recall that the degree of a self-map $f\colon S^n \to S^n$ is defined as follows. The map f induces a map $f_*\colon \pi_n(S^n) \to \pi_n(S^n)$ on homotopy groups. The group $\pi_n(S^n)$ is isomorphic to $\mathbb{Z}$ and f_* is given by multiplication by an integer. This integer is called the *degree of* f and is denoted by $d(f)$.
If you feel familiar enough with the notions just mentioned, please try to show the following application of Problem 1.2.

Problem 1.3. For $n \geq 2$ even, show that the antipodal map of S^n is homotopic to the reflection

$$r(x_1, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1})$$

and therefore has degree -1.
Follow from this fact and the result in Problem 1.2 that the tangent bundle of S^n is not trivial for n even (and $n \geq 2$).

Problem 1.4. Let R be the commutative ring $R := \mathbb{R}[x,y,z]/(x^2 + y^2 + z^2 = 1)$ and let A be the 3×1-matrix (x,y,z) over R. We define P to be the kernel of the homomorphism $R^3 \to R^1$ given by multiplication with A.
a) Show that every element in P defines a tangent vector field of the unit sphere S^2 in $\mathbb{R}^3$.
b) Using the result of Problem 1.3, show that P is not a free R-module.
(Even though $P \oplus R \cong R^3$ is free.)

Problem 1.5. Let M be a smooth manifold of dimension n. As in class we denote by $w = 1 + w_1 + \ldots + w_n$ the total Stiefel-Whitney class of M.
a) If M admits an immersion $f\colon M \hookrightarrow \mathbb{R}^{n+1}$, show that $(1 + w_1)w = 1$ in $H^*(M;\mathbb{Z}/2)$.
b) If M admits an immersion $f\colon M \hookrightarrow \mathbb{R}^{n+2}$, show that $(1 + w_1 + w_1^2 - w_2)w = 1$ in $H^*(M;\mathbb{Z}/2)$.

Orientations and Stiefel-Whitney classes

Recall that an *orientation* of a real vector space V of dimension $n > 0$ is an equivalence class of bases, where two ordered bases $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ are said to be equivalent if and only if the matrix (a_{ij}) defined by the equation

$$v'_i = \sum a_{ij} v_j$$

has positive determinant.

Let ξ be a real vector bundle given by the map $\pi\colon E \to B$. An *orientation* of ξ is a function assigning an orientation to each fiber in such a way that near each point of B there is a local trivialization $h\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$ carrying the canonical orientation of $\mathbb{R}^n$ in the fibers of $U \times \mathbb{R}^n$ to the orientations of the fibers in $\pi^{-1}(U)$. An oriented vector bundle ξ is a real vector bundle together with a choice of orientation.

We would like to draw a connection between orientations and the first Stiefel-Whitney class. Let B be a path-connected space and let $E \to B$ be a vector bundle. Let $\pi_1(B) \to \mathbb{Z}/2$ be the homomorphism that assigns 0 or 1 to each loop according to whether orientations of fibers are preserved or reversed as one goes around the loop. Since $\mathbb{Z}/2$ is abelian, this homomorphism factors through the abelianization of $\pi_1(B)$, i.e., it factors through the first homology group $H_1(B;\mathbb{Z})$ of B. A homomorphism $H_1(B;\mathbb{Z}) \to \mathbb{Z}/2$ corresponds uniquely to an element in $H^1(B;\mathbb{Z}/2)$. Hence we have defined a class in $H^1(B;\mathbb{Z}/2)$ which is zero exactly if E is orientable.

Problem 1.6. Show that the just defined cohomology class is $w_1(E)$.

(It is ok if you assume for the proof that B is a CW-complex, but you may also want to try to prove the general case.)