Math 231br
Problem Set 2

Spring 2014

You should hand in solutions to at least four problems, but please feel free to work on as many problems as you like and to hand in all your solutions. For any questions, please send me an email to gquick@math.harvard.edu and/or come to my office hours on Wednesdays 1.30-2.30pm in SC 341. Solutions are due by Friday, March 7, at the beginning of class.

Problem 2.1. Let $f, g\colon A \to B$ be two maps between paracompact spaces and let ξ be a k-dimensional vector bundle over B. Show that if f and g are homotopic, then there is an isomorphism $f^*\xi \cong g^*\xi$ of bundles over A.

Problem 2.2. Show that the projection $q\colon V_k(\mathbb{R}^{n+k}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k})$ is a fiber bundle with fiber $O(k)$, the group of orthogonal $k \times k$-matrices.

Problem 2.3. Show that $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is a Hausdorff space by defining a suitable continuous map $\mathrm{Gr}_k(\mathbb{R}^{n+k}) \to \mathbb{R}$ that distinguishes points.

Problem 2.4. Show that the correspondence $X \overset{f}{\mapsto} \mathbb{R}^1 \oplus X$ defines an embedding of the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ into $\mathrm{Gr}_{k+1}(\mathbb{R}^1 \oplus \mathbb{R}^{n+k}) = \mathrm{Gr}_{k+1}(\mathbb{R}^{n+k+1})$, and that f is covered by a bundle map

$$\epsilon^1 \oplus \gamma^k(\mathbb{R}^{n+k}) \to \gamma^{k+1}(\mathbb{R}^{n+k+1}).$$

Problem 2.5. The notation $\Omega_{\underline{a}}$ does not make reference to the integer n in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$. Show that if $n \leq m$ and $\Omega_{\underline{a}}$ is a Schubert variety in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$, then the inclusion $\mathrm{Gr}_k(\mathbb{R}^{n+k}) \subset \mathrm{Gr}_k(\mathbb{R}^{m+k})$ restricts to a homeomorphism of the Schubert variety $\Omega_{\underline{a}} \subset \mathrm{Gr}_k(\mathbb{R}^{n+k})$ and the Schubert variety $\Omega_{\underline{a}} \subset \mathrm{Gr}_k(\mathbb{R}^{m+k})$.

Problem 2.6. Show that if $\underline{a} = (0, a_2, \ldots, a_k)$ is a Schubert symbol for a Schubert variety $\Omega_{\underline{a}}$ in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$, then the projection map

$$\mathbb{R}^{n+k} \to \mathbb{R}^{n+k-1}$$

with kernel ϵ_1 gives a homeomorphism of $\Omega_{\underline{a}}$ with the Schubert variety in $\mathrm{Gr}_{k-1}(\mathbb{R}^{n+k-1})$ with symbol $(a_2, \ldots, a_k)$.