Math 231br
Problem Set 3

Spring 2014

You should hand in solutions to at least four problems, but please feel free to work on as many problems as you like and to hand in all your solutions. For any questions, please send me an email to gquick@math.harvard.edu and/or come to my office hours on Wednesdays 1.30-2.30pm in SC 341. Solutions are due by Friday, March 14, at the beginning of class.

Problem 3.1. Show that if the homology of $\Omega_{\underline{a}}$ satisfies Poincaré duality in the sense that

$$\dim H_i(\Omega_{\underline{a}}) = \dim H_{|\underline{a}|-i}(\Omega_{\underline{a}})$$

then $\Omega_{\underline{a}}$ is homeomorphic to $\mathrm{Gr}_\ell(\mathbb{R}^{m+\ell})$ for some (ℓ, m) and so in fact is a manifold. (Hint: Show that the Poincaré duality condition implies that the Schubert symbol a must have exactly one immediate predecessor).

Problem 3.2. Show that the two CW-complexes $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ and $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ are actually isomorphic by proving that the number of r-cells in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is equal the number of r-cells of $\mathrm{Gr}_n(\mathbb{R}^{n+k})$.

Problem 3.3. Let $E \to B$ be a fiber bundle with fiber F. Show that if B and F are manifolds, then so is E.

Problem 3.4. The correspondence $X \overset{f}{\mapsto} \mathbb{R}^1 \oplus X$ defines an embedding of the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ into $\mathrm{Gr}_{k+1}(\mathbb{R}^1 \oplus \mathbb{R}^{n+k}) = \mathrm{Gr}_{k+1}(\mathbb{R}^{n+k+1})$. Show that f carries the $|\underline{a}|$-cell of $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ which corresponds to a given Schubert symbol $\underline{a}$ onto the $|\underline{a}|$-cell of $\mathrm{Gr}_{k+1}(\mathbb{R}^{n+k+1})$ which corresponds to the same Schubert symbol $\underline{a}$.

Problem 3.5. Show that the cohomology algebra $H^*(\mathrm{Gr}_k(\mathbb{R}^{n+k}; \mathbb{Z}/2)$ is generated as an algebra over $\mathbb{Z}/2$ by the Stiefel-Whitney classes $w_1, \ldots, w_k$ of γ^k and the dual classes $\bar{w}_1, \ldots, \bar{w}_n$ subject to the $n + k$ defining relations

$$(1 + w_1 + \cdots + w_k)(1 + \bar{w}_1 + \cdots + \bar{w}_n) = 1.$$

Problem 3.6. Show that $w_i(\gamma^k) \in H^i(\mathrm{Gr}_k; \mathbb{Z}/2)$ corresponds to the cocycle which assigns 1 to the Schubert cell $\Omega^0_{\underline{a}}$ with Schubert symbol $\underline{a} = (1, \ldots, 1)$ of length i and zero to all other cells