Math 231br
Problem Set 4

Spring 2014

You should hand in solutions to at least four problems, but please feel free to work on as many problems as you like and to hand in all your solutions. For any questions, please send me an email to gquick@math.harvard.edu and/or come to my office hours on Wednesdays 1.30-2.30pm in SC 341. Solutions are due by Friday, March 28, at the beginning of class.

Problem 4.1. a) Show that there is a bijection

$$\varphi\colon [S^0, \mathrm{GL}_n(\mathbb{C})] \to \mathrm{Vect}^n_{\mathbb{C}}(S^1).$$

Conclude that every finite dimensional complex vector bundle over S^1 is trivial and that $\pi_1(BU)$ is the trivial group.
(Hint: Let $D^1_+ = \{(x,y)\mathbb{R}^2 | x^2 + y^2 = 1,\ y \geq 0\}$ and $D^1_- = \{(x,y)\mathbb{R}^2 | x^2 + y^2 = 1,\ y \leq 0\}$. Given a function a map $f\colon S^0 \to \mathrm{GL}_n(\mathbb{C})$ we can define a complex vector bundle E_f over S^1 by taking the disjoint union $D^1_+ \times \mathbb{C}^n \amalg D^1_- \times \mathbb{C}^n$ and identifying (x, v) on $D^1_- \times \mathbb{C}^n$ with $(x, f(x)(v))$ on $D^1_+ \times \mathbb{C}^n$. This gives us a map φ. For a map $\psi\colon \mathrm{Vect}^n_{\mathbb{C}}(S^1) \to [S^0, \mathrm{GL}_n(\mathbb{C})]$, let E be an n-dimensional complex vector bundle over S^1. Its restrictions to E_+ and E_- to D^1_+ and D^1_- are trivial since D^1_+ and D^1_- are contractible. Choose trivializations $h_\pm\colon E_\pm \to D^1_\pm \times \mathbb{C}^n$. This defines a map $S^0 \to \mathrm{GL}_n(\mathbb{C})$. Now you only need to show that φ and ψ are well-defined and inverse to each other.)
b) Generalize this to a bijection

$$\varphi\colon [S^{k-1}, \mathrm{GL}_n(\mathbb{C})] \to \mathrm{Vect}^n_{\mathbb{C}}(S^k).$$

A function $f\colon S^{k-1} \to \mathrm{GL}_n(\mathbb{C})$ is called a *clutching function*.

Problem 4.2. Show that for any pointed compact space X there is a group isomorphism

$$\operatorname*{colim}_{n\to\infty} [X, \mathrm{GL}_n(\mathbb{C})] \to \tilde{K}(S^1 \wedge X),$$

where the group structure on the left is induced from that of $GL_n(\mathbb{C})$.

Problem 4.3. Let $\gamma = \gamma^1_{\mathbb{C}}$ be the canonical line bundle over $\mathbb{CP}^1 = S^2$. Show that the bundles $(\gamma \otimes \gamma) \oplus \epsilon^1$ and $\gamma \oplus \gamma$ are given by the clutching functions (see Problem 4.2) $S^1 \to \mathrm{GL}_2(\mathbb{C})$

$$z \mapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

respectively. Show that there is an isomorphism

$$(\gamma \otimes \gamma) \oplus \epsilon^1 \cong \gamma \oplus \gamma.$$

In particular, we have the relation $[H]^2 = 2[H] - 1$ in $K(S^2)$.

Problem 4.4. Use complex K-theory to prove Brouwer's fixed point theorem: Let D^n be the unit disk in Euclidean n-space. Then for every continuous function $f\colon D^n \to D^n$ there is a point $x \in D^n$ with $f(x) = x$.

Problem 4.5. Compute the complex K-theory groups of the complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^{n+k})$.
(Hint: One could use the cell decomposition of $\mathrm{Gr}_k(\mathbb{C}^{n+k})$.)

Problem 4.6. Let (X,Y) be a pair of compact space. Show that if Y is a retract of X, then for all $n \geq 0$, the sequence

$$K^{-n}(X,Y) \to K^{-n}(X) \to K^{-n}(Y)$$

is a split short exact sequence, and

$$K^{-n}(X) \cong K^{-n}(X,Y) \oplus K^{-n}(Y).$$