Math 231br
Problem Set 6

Spring 2014

You should hand in solutions to at least three problems. For any questions, please send me an email to gquick@math.harvard.edu and/or come to my office hours on Wednesdays 1.30-2.30pm in SC 341. Solutions are due by Friday, April 18, at the beginning of class.

The definition of the Hopf invariant via cohomology is the following. For $n \geq 2$, let S^n be an oriented n-sphere. Assume we are given a pointed map $f\colon S^{2n-1} \to S^n$. Considering S^{2n-1} as the boundary of an oriented $2n$-cell, we can form the cell complex $X = S^n \cup_f e^{2n}$, the *cofiber of* f. It is the complex formed from the disjoint union of S^n and e^{2n} by identifying each point in $S^{2n-1} = \mathring{e}^{2n}$ with its image under f. The cell complex X has a single vertex, a single n-cell i and a single $2n$-cell j. Hence the integral cohomology $H^*(X;\mathbb{Z})$ is zero except in dimensions 0, n, and $2n$ and is $\mathbb{Z}$ in those three dimensions. Denote the generators determined by the given orientations of the cohomology groups in dimension n and $2n$ by $\sigma = [i]$ and $\tau = [j]$, respectively. Then the cup-product square σ^2 is some integral multiple of τ.

The *Hopf invariant* of f is the integer $h(f)$ such that

$$\sigma^2 = h(f) \cdot \tau.$$

Problem 6.1. For $n \geq 2$ even, show that this definition of $h(f)$ agrees (at least up to sign) with the definition given in Lecture 26.

Problem 6.2. Prove the following properties of the Hopf invariant:

(a) If n is odd, then $h(f) = 0$ for all f.
(b) If $g\colon S^{2n-1} \to S^{2n-1}$ has degree d, then $h(f \circ g) = d \cdot h(f)$.
(c) If $e\colon S^n \to S^n$ has degree d, then $h(e \circ f) = d^2 \cdot h(f)$.

Problem 6.3. Let $n \geq 2$ be an even integer. Consider the product space $S^n \times S^n$ as the cell complex formed by attaching an $2n$-cell to the wedge of two spheres $S^n \vee S^n$ using an attaching map $g\colon S^{2n-1} \to S^n \vee S^n$. Let $F = \mathrm{id} \vee \mathrm{id}\colon S^n \vee S^n \to S^n$ be the "folding map", and let

$$f = F \circ g\colon S^{2n-1} \to S^n \vee S^n \to S^n.$$

Show that f has Hopf invariant two.

Problem 6.4. Use the cohomological definition above to show that the Hopf fibration

$$f\colon S^3 \to S^2$$

has Hopf invariant one.

Problem 6.5. Let H be the dual of the canonical line bundle, and write

$$x = [H] - 1 \in K(\mathbb{C}\mathrm{P}^n).$$

Use the Chern character (and what you about K-theory and cohomology) to reprove that $K(\mathbb{C}\mathrm{P}^n)$ is a free abelian group with generators $1, x, x^2, \ldots, x^{n-1}$, and that $x^n = 0$.

Problem 6.6. Let E be a vector bundle on $\mathbb{C}\mathrm{P}^4$ whose Chern classes satisfy $c_i(E) = 0$ for $i = 1$, 3, 4. We can write $c_2(E) = ax^2$ for some integer a, where $x \in H^2(\mathbb{C}\mathrm{P}^4; \mathbb{Z})$ is a generator. Show that a^2 must be divisible by 12.