Math 231br
Paper suggestions

Spring 2014

Here is a list of possible topics, but feel free to choose your own. Please send me an email to gquick@math.harvard.edu if you have your own topic in mind, or if you have any other questions.

The general idea of writing the final paper is that you should learn something. So pick or think about a topic or some part of the suggested topics that really interests you. Then work on it as far as you get and make it into a coherent story. Indicate where you omit details or where you have not understood seomthing in full detail. Some of the topics are difficult (and too long), so it's ok if you cannot explain everything. Cite all sources that you used!

The final paper should be about 10 pages and is due at the end of reading period, Wednesday May 7.

Topic 1: Let β denote the Bott periodicity map in real K-theory. The main result of the paper
M. F. Atiyah, F. Hirzebruch, Bott periodicity and the parallelizability of spheres, Proc. Cambridge Philos. Soc. 57 (1961), 223-226.
is a formula expressing the total Stiefel-Whitney class of $\beta(\xi)$ in terms of the Stiefel-Whitney class of ξ, where ξ is a real vector bundle. As one consequence one gets another proof of the non-parallelizability of the n-sphere for $n \neq 1, 3, 7$. Your task is to explain the arguments and ideas of the paper.
This topic is a nice and very interesting combination of many ideas we have seen in class. The only drawback is that one has to work with real K-theory (just accept that it is 8-periodic).

Topic 2: Investigate other proofs of the Bott periodicity for complex K-theory. For example, you could look at the rather elementary proof using simplicial methods by Bruno Harris:
B. Harris, Bott periodicity via simplicial spaces, J. Algebra 62 (1980), 450-454.

Topic 3: Discuss power operations in K-theory. We have seen the close relationship between cohomology and K-theory in class. Atiyah extends the relationship by generalizing Adams operations on complex K-theory. To investigate these operations you could look at Atiyah's original paper. For this topic you will need some familiarity with representations of finite groups.
M. F. Atiyah, Power operations in K-theory, Quart. J. Math. 17 (1966), 165-193.

It has also been reprinted in Atiyah's lecture notes:
M. F. Atiyah, K-theory. Notes by D. W. Anderson. Addison-Wesley. 1967.

Topic 4: Discuss the problem of vector fields on spheres.
This is a rather difficult topic. The full solution of the problem of determining all vector fields on spheres requires techniques beyond our class. But if you choose this topic you see some beautiful mathematics. You can look at Adams's original paper and work out as many arguments as you can. It is totally ok if you use some things as a black box, e.g., duality. Just tell it as a consistent story.
J. F. Adams, Vector fields on spheres. Ann. of Math. (2) 75 (1962), 603-632.

Topic 5: Investigate transfers in K-theory and Becker-Gottlieb's proof of the Adams conjecture. You could look at the original paper:
J. C. Becker, D. H. Gottlieb, The transfer map and fiber bundles, Topology 14 (1975), 1-12.

Topic 6: Let $\mathcal{F}$ be the space of Fredholm operators on a separable complex Hilbert space. For a finite CW-complex X, the index defines a natural isomorphism

$$[X, \mathcal{F}] \to K(X).$$

Discuss this result following the appendix in Atiyah's lecture notes.
If you want to work on this topic you should be familiar with Fredholm operators (or you should be eager to learn a bit about them).

Topic 7: Discuss the Hirzebruch-Riemann-Roch formula. You could study the paper by Atiyah and Hirzebruch to understand the relationship of the Chern character and Gysin/transfer maps:
M. F. Atiyah, F. Hirzebruch, Riemann-Roch theorems for differentiable manifolds, Bull. Amer. Math. Soc. 65 (1959), 276-281.
F. Hirzebruch, Topological methods in algebraic geometry. Die Grundlehren der Mathematischen Wissenschaften, Band 131 Springer-Verlag New York, Inc., New York 1966.

Topic 8: Dive into Atiyah's world of K-theory and reality. Atiyah defines K-theory on spaces with an involution. This is useful for many applications. You could study the periodicity theorem for this theory and show how this leads to an elegant proof of Bott periodicity in real K-theory.
M. F. Atiyah, K-theory and reality, Quart. J. Math. 17 (1966), 367-386.
It has also been reprinted in Atiyah's lecture notes.