Algebraic cycles in generalized cohomology theories

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Lefschetz's theorem: X projective complex surface

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2-dim. topological cycle

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Lefschetz:

class in homology

If (N) holds for Γ, then $[\Gamma]$ is "algebraic".

there is an algebraic curve $C \sim \Gamma$

Higher (co-)dimensions:

X smooth projective complex algebraic variety

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(smooth) subvariety of dim. n

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free abelian group on alg. subvarieties of codim. p modulo rat. equiv.

$CH^p(X)$

$Z \subset X$

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dual of fund. class (of desingularization)

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- Hodge's question: Is this map surjective?

- Question: What is the kernel of this map?

How to do homotopy on Man?

category of complex manifolds

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category of complex manifolds

-

presheaves of sets, i.e., functors: $\mathrm{Man}^{op} \longrightarrow \mathrm{Set}$

$$\mathrm{Man} \dashrightarrow \mathrm{Pre}$$

How to do homotopy on Man?

category of complex manifolds

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$F_M: X \longmapsto \mathrm{Hom}_{\mathrm{Man}}(X,M)$

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Presheaves "remember"

$$\mathrm{Hom}_{\mathrm{Pre}}(F_M,F_{M'}) = \mathrm{Hom}_{\mathrm{Man}}(M,M')$$

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- presheaves of Sets — switch to — presheaves of Sets_Δ

"rigid" — "allow homotopy"

How to do homotopy on Man?

-

presheaves of simplicial sets
functors: $\mathbf{Man}^{op} \longrightarrow \mathbf{Set}_\Delta$

$$\mathbf{Man} \dashrightarrow \mathbf{Pre}_\Delta$$
$$M \longmapsto F_M$$

F_M: $X \longmapsto$ discrete
simplicial set $F_M(X)$

How to do homotopy on Man?

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- Given $n \geq 0$, the n-dimensional stalk of $F_\bullet$

$$F_\bullet^{(n)} = \operatorname*{colim}_{r \to 0} F_\bullet(B^n(r)) \quad \text{in } \mathrm{Set}_\Delta$$

ball of radius r in n-dim.
complex affine space

How to do homotopy on Man?

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- A map $F_\bullet \to G_\bullet$ is a weak equivalence in Pre_Δ

  if $F_\bullet^{(n)} \to G_\bullet^{(n)}$ is a weak equivalence in Set_Δ for all $n \geq 0$.

Homotopy category of Man:

- $\text{Man} \dashrightarrow \text{Pre}_{\Delta}$

Homotopy category of Man:

homotopy category of simplicial presheaves on Man

- $$\mathrm{Man} \dashrightarrow \mathrm{hoPre}_\Delta = \mathrm{Pre}_\Delta[\mathrm{w.e.}^{-1}]$$

Homotopy category of Man: homotopy category of simplicial presheaves on Man

- Man $\dashrightarrow$ $hoPre_\Delta = Pre_\Delta[w.e.^{-1}]$

- Given M with an open cover $\{U_\alpha\}$:

  $F_{U_\bullet} \to F_M$ is a weak equivalence.

  $\coprod U_\alpha \leftleftarrows \coprod U_\alpha \times_X U_\beta \Lleftarrow \ldots$

Homotopy category of Man: homotopy category of simplicial presheaves on Man

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sequence of spaces $\dots, E_n, E_{n+1}, \dots$ with maps $S^1 \wedge E_n \to E_{n+1}$

- Can replace Set_Δ with Spectra and get a stable homotopy category $\mathrm{hoPre}_{\mathrm{Spectra}}$ of Man.
  - $S^1 \wedge -$ with S^1 viewed as a simplicial (constant) presheaf is made invertible.

Homotopy category of $Sm_{\mathbb{C}}$:

smooth complex varieties

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smooth complex varieties

simplicial presheaves on $Sm_{\mathbb{C}}$

- $Sm_{\mathbb{C}} \dashrightarrow Pre_{\Delta}$

Morel
Voevodsky
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- Localize with respect to

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Homotopy category of Sm_C:

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simplicial presheaves on Sm_C

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  $F_{U.} \to F_X$ for any X and any (hyper)cover $U \to X$

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Homotopy category of Sm_C:

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Homotopy category of Sm_C: motivic homotopy category of simplicial presheaves on Sm_C

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affine line over C

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affine line over C

- stable motivic homotopy category of Sm_C
  - $P^1 \wedge -$ the projective line
  - $S^1 \wedge -$ the "simplicial circle" and $(A^1-0) \wedge -$ the "Tate circle"

Topological realization:

$$\begin{array}{ccc} \mathrm{Sm}_{\mathbb{C}} & \xrightarrow{\rho} & \mathrm{Man}_{\mathbb{C}} \\ X & \longmapsto & X(\mathbb{C}) \end{array}$$

e.g. $\mathbb{P}^1 \longmapsto \mathbb{P}^1(\mathbb{C}) = \mathbb{CP}^1 \simeq S^2$

"complex manifold of solutions in $\mathbb{C}$"

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motivic spectrum

induced map

top. real.

$$E^{a,b}_{mot}(X) \xrightarrow{\rho_E} E^{a}_{top}(X(\mathbb{C}))$$

"algebraic" "topological"

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- in $E^{*}_{top}(X(\mathbb{C}))$ are algebraic,

  i.e., are in the image of ρ_E?

Atiyah-Hirzebruch, Totaro, Levine-Morel:

$$CH^*(X)=H^{2*,*}_{mót}(X;Z) \xrightarrow{\rho_H = cl_H} Alg^{2*}_H(X) \subseteq H^{2*}(X;Z)$$

Atiyah-Hirzebruch, Totaro, Levine-Morel:

universal oriented theories

$$
\begin{tikzcd}
MGL^{2*,*}(X) \arrow[r, "\rho_{MGL}"] \arrow[d, bend right] & MU^{2*}(X) \arrow[d, bend left] \\
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coeff. ring = Lazard ring

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MGL^{2*,*}(X)\otimes_{L^*}\mathbb{Z} \arrow[r] \arrow[d, "\text{Levine + Levine-Morel } \approx"'] & MU^{2*}(X)\otimes_{L^*}\mathbb{Z} \arrow[d, dotted] \arrow[dr, bend left] & \\
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coeff. ring = Lazard ring

Levine + Levine-Morel $\approx$

Totaro

$\circlearrowright$

- Atiyah-Hirzebruch: cl_H is not surjective onto integral Hodge classes.

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$\circlearrowright$

- Atiyah-Hirzebruch: cl_H is not surjective onto integral Hodge classes.
- Totaro: new classes in kernel of cl_H.

I. Kernel: X smooth projective

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Recall Deligne's diagram

$$0 \to J^{2p-1}(X) \to H_D^{2p}(X;\mathbb{Z}(p)) \to Hdg^{2p}(X) \to 0$$

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Griffiths' Jacobian

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\begin{tikzcd}
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Abel-Jacobi map

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Generalized Hodge filtered cohomology theories (joint work with Mike Hopkins):

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Griffiths' Jacobian

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Generalized Hodge filtered cohomology theories (joint work with Mike Hopkins):

$$0 \to J^{2p-1}_{MU}(X) \to MU^{2p}_D(X;Z(p)) \to Hdg^{2p}_{MU}(X) \to 0$$

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Deligne
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$$\begin{array}{ccccccccc} & & \text{Kernel of } cl_H & \subset & CH^p(X) & & & & \\ & & \mu_H \downarrow & & \downarrow cl_{HD} & \searrow^{cl_H} & & & \\ 0 & \to & J^{2p-1}(X) & \to & H^{2p}_D(X;Z(p)) & \to & Hdg^{2p}(X) & \to & 0 \end{array}$$

Griffiths'
Jacobian

Hodge
classes

Generalized Hodge filtered cohomology theories (joint work with Mike Hopkins):

MU-Hodge
classes

$$0 \to J^{2p-1}_{MU}(X) \to MU^{2p}_D(X;Z(p)) \to Hdg^{2p}_{MU}(X) \to 0$$

I. Kernel: X smooth projective

Recall Deligne's diagram

Abel-Jacobi map

Deligne cohomology

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Examples:

The new Abel-Jacobi map is able to detect interesting algebraic cobordism classes:

$$\begin{tikzcd}
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"f_* of universal genus of curvature form" of normal bundle of Y if Y is a smooth projective variety

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homotopy fibre represents
"Arakelov algebraic cobordism"

Question: What is the arithmetic-geometric information encoded in the Chern classes in Arakelov algebraic cobordism?

II. Image:

Recall: $\mathrm{Sm} \xrightarrow{\rho} \mathrm{Man}$

$$X \longmapsto X(\mathbb{C})$$

manifold of solutions in $\mathbb{C}$

motivic spectrum

induced map

$$E^{a,b}_{mot}(X) \xrightarrow{\rho_E} E^{a}_{top}(X(\mathbb{C}))$$

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Question: • How can we detect whether classes in $E^{*}_{top}(X(\mathbb{C}))$ are algebraic, i.e., are in the image of ρ_E?

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- How can we detect whether classes in $E^{*}_{top}(X(\mathbb{C}))$ are not algebraic, i.e., are not in the image of ρ_E?

- How can we construct such classes?

A different perspective:

Fix a prime p.

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Brown-Peterson, Quillen

$|v_i|=2(p^i-1)$

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For every n: $BP \xrightarrow{\text{quotient map}} BP/(v_{n+1}, \ldots) =: BP\langle n\rangle$

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The Brown-Peterson tower (Wilson):

$$BP \longrightarrow \cdots \longrightarrow BP\langle n\rangle \longrightarrow \cdots \longrightarrow BP\langle 1\rangle \longrightarrow BP\langle 0\rangle \longrightarrow BP\langle -1\rangle$$

p=2: 2-local connective K-theory

$HZ_{(p)} \longrightarrow HF_p$

Milnor operations:

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For every n:

stable cofibre sequence

$$\Sigma^{|v_n|} BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \longrightarrow BP\langle n-1\rangle \longrightarrow \Sigma^{|v_n|+1} BP\langle n\rangle$$

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with an induced exact sequence (for any space X)

$$\begin{tikzcd} BP\langle n\rangle^{*+|v_n|}(X) \arrow[r] & BP\langle n\rangle^{*}(X) \arrow[dl] \\ BP\langle n-1\rangle^{*}(X) \arrow[r, "q_n"] & BP\langle n\rangle^{*+|v_n|+1}(X) \end{tikzcd}$$

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H^{*}(X;F_p) \arrow[r, "Q_n"'] & H^{*+|v_n|+1}(X;F_p)
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nth Milnor operation:

Q_0=Bockstein

$Q_n=P^{p^{n-1}}Q_{n-1}-Q_{n-1}P^{p^{n-1}}$

The LMT obstruction in action:

$$\begin{tikzcd}
 & BP^{2*}(X) \arrow[dl, bend right] \arrow[d] & \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
 & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p)
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α

Question: Is α algebraic?

The LMT obstruction in action:

$$
\begin{tikzcd}
 & BP^{2*}(X) \arrow[dl, bend right] \arrow[d] \arrow[dl, phantom, shift left=6, "\circlearrowleft"] & \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
CH^*(X) \arrow[r] & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p) \\
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CH^*(X) \arrow[r] \arrow[uur, bend left, dotted, "LMT"] & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p)
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\begin{tikzcd}
 & BP^{2*}(X) \arrow[d] \arrow[dl, bend right] \arrow[d, phantom, shift right=8ex, "\circlearrowleft"] & \\
BP\langle n\rangle^{2*}(X) \arrow[r] & \overset{\alpha_{n-1}}{BP\langle n-1\rangle^{2*}(X)} \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
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α_{n-1}

α

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$$
\begin{tikzcd}
 & BP^{2*}(X) \arrow[dl, bend right] \arrow[d] & \\
 & \alpha_{n-1} \arrow[r, mapsto] & q_n\alpha_{n-1} \\
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\end{tikzcd}
$$

if

Question: Is α algebraic?

The LMT obstruction in action:

then $\circlearrowleft$ if

$$\begin{tikzcd}
 & BP^{2*}(X) \arrow[dd] \arrow[ddl, bend right] & \\
\alpha_n & \alpha_{n-1} \arrow[l, dotted, bend right] \arrow[u, dotted, bend right] \arrow[r, mapsto] & q_n\alpha_{n-1} = 0 \\
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BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
CH^*(X) \arrow[r] \arrow[uur, dotted, bend left, "\mathrm{LMT}"] & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p)
\end{tikzcd}
$$

LMT

then $\circlearrowleft$ $\quad \boldsymbol{\alpha}_n \leftarrow \boldsymbol{\alpha}_{n-1}$

if $\quad \boldsymbol{\alpha}_{n-1} \mapsto q_n\boldsymbol{\alpha}_{n-1} = 0$

$\boldsymbol{\alpha} \mapsto Q_n\boldsymbol{\alpha}$

Question: Is $\boldsymbol{\alpha}$ algebraic?

The LMT obstruction in action:

LMT

then $\circlearrowleft$

if

$$\alpha_n \qquad \alpha_{n-1} \longmapsto q_n\alpha_{n-1} = 0$$

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$$\alpha \longmapsto Q_n\alpha$$

The LMT obstruction in action:

$$
\begin{tikzcd}
 & BP^{2*}(X) \arrow[dl, bend right] \arrow[d] & \\
\alpha_n & \alpha_{n-1} \arrow[r, mapsto] \arrow[l, dotted, bend right, "\text{then } \circlearrowleft"'] \arrow[u, dotted, bend right] & q_n\alpha_{n-1} = 0 \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
CH^*(X) \arrow[r] \arrow[uuur, dotted, bend left, "LMT"] & H^{2*}(X;F_p) \arrow[r, "Q_n"'] \arrow[u, dotted, bend right] & H^{2*+|v_n|+1}(X;F_p) \\
 & \alpha \arrow[r, mapsto] & Q_n\alpha
\end{tikzcd}
$$

if

Levine-Morel-Totaro obstruction:

The LMT obstruction in action:

$$
\begin{tikzcd}
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BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
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LMT

then $\circlearrowleft$

$\boldsymbol{\alpha}_n \quad \boldsymbol{\alpha}_{n-1} \longmapsto q_n\boldsymbol{\alpha}_{n-1} = 0$

if

$\boldsymbol{\alpha} \longmapsto Q_n\boldsymbol{\alpha} \neq 0$

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If $Q_n\boldsymbol{\alpha} \neq 0$,

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Levine-Morel-Totaro obstruction:

If $Q_n\boldsymbol{\alpha} \neq 0$,

The LMT obstruction in action:

then ↺

$\alpha_n \qquad \alpha_{n-1} \longmapsto q_n\alpha_{n-1} \neq 0$

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$\alpha \longmapsto Q_n\alpha \neq 0$

if

Levine-Morel-Totaro obstruction:

If $Q_n\alpha \neq 0$,

The LMT obstruction in action:

LMT

then

$$
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 & BP^{2*}(X) \arrow[ld, bend right] \arrow[d] & \\
\alpha_n & \alpha_{n-1} \arrow[r, mapsto] & q_n\alpha_{n-1} \neq 0 \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[rd, phantom, "\circlearrowleft"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
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 & \alpha \arrow[r, mapsto] & Q_n\alpha \neq 0
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$$

if

Levine-Morel-Totaro obstruction:

If $Q_n\alpha \neq 0$, then α is not algebraic.

Voevodsky’s motivic Milnor operations:

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There are motivic operations

$$Q_n^{mot} \in \mathscr{A}^{2p^n-1,p^n-1}$$

mod p-motivic
Steenrod algebra

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For a smooth complex variety X:

$$H^{i,j}_{mot}(X;F_p) \xrightarrow{Q_n^{mot}} H^{i+2p^n-1,j+p^n-1}_{mot}(X;F_p)$$

mod p-motivic
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Voevodsky's motivic Milnor operations:

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mod p-motivic cohomology

Recall: $H^{2i,i}_{mot}(X;F_p) = CH^i(X;Z/p)$ and

$H^{i,j}_{mot}(X;F_p) = 0$ if $i>2j$.

Obstructions revisited: X smooth complex variety

$$
\begin{tikzcd}
H^{2i,i}_{mot}(X;F_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;F_p) \arrow[d] \\
H^{2i}(X;F_p) \arrow[r, "Q_n"'] & H^{2i+2p^n-1}(X;F_p)
\end{tikzcd}
$$

topological realization

Obstructions revisited: X smooth complex variety

$$\begin{tikzcd} H^{2i,i}_{mot}(X;\mathbb{F}_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;\mathbb{F}_p) = 0 \arrow[d] \\ H^{2i}(X;\mathbb{F}_p) \arrow[r, "Q_n"'] & H^{2i+2p^n-1}(X;\mathbb{F}_p) \end{tikzcd}$$

topological realization

Obstructions revisited: X smooth complex variety

$$
\begin{tikzcd}
H^{2i,i}_{mot}(X;F_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[rr, bend left, "Q_n^{mot}"] \arrow[dr, phantom, "\circlearrowleft"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;F_p) \arrow[d] & = 0 \\
H^{2i}(X;F_p) \arrow[r, "Q_n"'] & H^{2i+2p^n-1}(X;F_p) &
\end{tikzcd}
$$

topological realization

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$$\begin{tikzcd}[column sep=large]
H^{2i,i}_{mot}(X;F_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[rr, bend left, "Q_n^{mot}"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;F_p) \arrow[d] & = 0 \\
H^{2i}(X;F_p) \arrow[r, "Q_n"'] \arrow[ur, phantom, "\circlearrowleft"] & H^{2i+2p^n-1}(X;F_p) &
\end{tikzcd}$$

topological realization

α

Obstructions revisited: X smooth complex variety

$$
\begin{tikzcd}
H^{2i,i}_{mot}(X;F_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[rr, bend left, "Q_n^{mot}"] \arrow[dr, phantom, "\circlearrowleft"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;F_p) \arrow[d] & = 0 \\
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\end{tikzcd}
$$

topological realization

$$\alpha \longmapsto Q_n\alpha \neq 0$$

if

Obstructions revisited: X smooth complex variety

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\begin{tikzcd}
H^{2i,i}_{mot}(X;\mathbb{F}_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[rr, bend left, "Q_n^{mot}"] \arrow[dr, phantom, "\circlearrowleft"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;\mathbb{F}_p) \arrow[d] \arrow[r, phantom, "="] & 0 \\
H^{2i}(X;\mathbb{F}_p) \arrow[r, "Q_n"'] & H^{2i+2p^n-1}(X;\mathbb{F}_p) &
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Obstructions revisited: X smooth complex variety

$$
\begin{array}{ccccc}
H^{2i,i}_{mot}(X;F_p) & \xrightarrow{Q_n^{mot}} & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;F_p) & = & 0 \\
\downarrow & \circlearrowleft & \downarrow & & \\
H^{2i}(X;F_p) & \xrightarrow[Q_n]{} & H^{2i+2p^n-1}(X;F_p) & & \\
\alpha & \longmapsto & Q_n\alpha \neq 0 & &
\end{array}
$$

topological realization

if

Observation: The LMT-obstruction is particular to smooth varieties and bidegrees (2i,i).

Obstructions revisited: X smooth complex variety

$$
\begin{tikzcd}
H^{2i,i}_{mot}(X;F_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[rr, bend left, "Q_n^{mot}"] \arrow[dr, phantom, "\circlearrowleft"] & H^{2i+2p^n-1,\,i+p^n-1}_{mot}(X;F_p) \arrow[d] & = 0 \\
H^{2i}(X;F_p) \arrow[r, "Q_n"'] \arrow[u, dotted, bend left, "\times"] & H^{2i+2p^n-1}(X;F_p) &
\end{tikzcd}
$$

topological realization

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if

Observation: The LMT-obstruction is particular to smooth varieties and bidegrees $(2i,i)$.

Example: $Q_n\iota\neq 0$ for ι the fundamental class of a suitable Eilenberg-MacLane space, though ι is algebraic.

Back to our task:

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Study $Alg_E^{2*}(X)$ and its complement in $E_{top}^{2*}(X)$.

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Recall: BP and $BP\langle n\rangle$ exist in the motivic world (e.g. Vezzosi, Hopkins, Hu-Kriz, Ormsby, Hoyois, Ormsby-Østvær).

Back to our task:

Study $\mathrm{Alg}^{2*}_E(X)$ and its complement in $E^{2*}_{top}(X)$.

For example: $E=BP\langle n\rangle$?

Recall: BP and $BP\langle n\rangle$ exist in the motivic world (e.g. Vezzosi, Hopkins, Hu-Kriz, Ormsby, Hoyois, Ormsby-Østvær).

Question: How can we produce non-algebraic elements in $BP\langle n\rangle^{2*}_{top}(X)$?

will drop the "top" again

Back to the cofibre sequence:

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Recall:

stable cofibre sequence

$$\Sigma^{|v_n|}BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \longrightarrow BP\langle n-1\rangle \longrightarrow \Sigma^{|v_n|+1}BP\langle n\rangle$$

and the induced map

$$BP\langle n-1\rangle^{*}(X) \xrightarrow{q_n} BP\langle n\rangle^{*+|v_n|+1}(X).$$

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For example:

Bockstein homomorphism

$$n=0: \quad H^*(X;F_p) \xrightarrow{q_0} H^{*+1}(X;Z_{(p)}),$$

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For example:

Bockstein homomorphism

$$n=0: \quad H^*(X;F_p) \xrightarrow{q_0} H^{*+1}(X;Z_{(p)}),$$

$$n=1: \quad H^*(X;Z_{(p)}) \xrightarrow{q_1} BP\langle 1\rangle^{*+2p-1}(X),$$

$$\vdots$$

A diagram chase:

$H^k(X;F_p)$

A diagram chase:

$$\begin{array}{c} H^{k+1}(X;\mathbb{Z}_{(p)}) \\ {\scriptstyle q_0}\uparrow \\ H^k(X;\mathbb{F}_p) \end{array}$$

A diagram chase:

$$\begin{tikzcd} BP\langle 1\rangle^{k+1+2p-1}(X) \\ H^{k+1}(X;Z_{(p)}) \arrow[u, "q_1"] \\ H^k(X;F_p) \arrow[u, "q_0"] \end{tikzcd}$$

A diagram chase:

$$\begin{tikzcd}
BP\langle n\rangle^{k+|v_0|+\ldots+|v_n|}(X) \\
\vdots \arrow[u, "q_n"] \\
BP\langle 1\rangle^{k+1+2p-1}(X) \arrow[u, "q_2"] \\
H^{k+1}(X;Z_{(p)}) \arrow[u, "q_1"] \\
H^k(X;F_p) \arrow[u, "q_0"]
\end{tikzcd}$$

A diagram chase:

$$
\begin{tikzcd}
BP\langle n\rangle^{k+|v_0|+\dots+|v_n|}(X) \arrow[r, "q_{n+1}"] & BP\langle n+1\rangle^{k+|v_0|+\dots+|v_{n+1}|}(X) \\
\vdots \arrow[u, "q_n"] & \\
BP\langle 1\rangle^{k+1+2p-1}(X) \arrow[u, "q_2"] & \\
H^{k+1}(X;Z_{(p)}) \arrow[u, "q_1"] & \\
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\end{tikzcd}
$$

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\vdots \arrow[u, "q_n"] & \\
BP\langle 1\rangle^{k+1+2p-1}(X) \arrow[u, "q_2"] & \\
H^{k+1}(X;Z_{(p)}) \arrow[u, "q_1"] & \\
H^k(X;F_p) \arrow[u, "q_0"] & H^{k+|v_0|+\ldots+|v_{n+1}|}(X;F_p)
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$$
\begin{tikzcd}
BP\langle n+1\rangle \arrow[d, "\text{Thom map}"] \\
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A diagram chase:

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\begin{tikzcd}
BP\langle n\rangle^{k+|v_0|+\dots+|v_n|}(X) \arrow[r, "q_{n+1}"] & BP\langle n+1\rangle^{k+|v_0|+\dots+|v_{n+1}|}(X) \arrow[dddd] \\
\vdots \arrow[u, "q_n"] & \\
BP\langle 1\rangle^{k+1+2p-1}(X) \arrow[u, "q_2"] & \\
H^{k+1}(X;Z_{(p)}) \arrow[u, "q_1"] & \\
H^k(X;F_p) \arrow[u, "q_0"] \arrow[r, "Q_{n+1}Q_n\dots Q_0"'] & H^{k+|v_0|+\dots+|v_{n+1}|}(X;F_p)
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$$

Lifting classes:

We get

Lifting classes:

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- a map φ

$$\begin{array}{ccc} H^k(X;F_p) & \xrightarrow{\varphi:=q_n\dots q_0} & BP\langle n\rangle^{k+|v_0|+\dots+|v_n|}(X) \\ \boldsymbol{\alpha} & \longmapsto & \varphi(\boldsymbol{\alpha}) \end{array}$$

Lifting classes:

We get

- a map φ
- an obstruction

$$
\begin{tikzcd}
 & BP^{k+|v_0|+\dots+|v_n|}(X) \arrow[d] \\
H^k(X;F_p) \arrow[r, "\varphi:=q_n\dots q_0"] & BP\langle n\rangle^{k+|v_0|+\dots+|v_n|}(X) \arrow[u, dotted, bend right] \\
\alpha \arrow[r, mapsto] \arrow[d, "Q_{n+1}Q_n\dots Q_0"'] & \varphi(\alpha) \arrow[d, "q_{n+1}"] \\
H^{k+|v_0|+\dots+|v_{n+1}|}(X;F_p) & BP\langle n+1\rangle^{k+|v_0|+\dots+|v_{n+1}|}(X) \arrow[l]
\end{tikzcd}
$$

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H^{k+|v_0|+\dots+|v_{n+1}|}(X;F_p) & BP\langle n+1\rangle^{k+|v_0|+\dots+|v_{n+1}|}(X) \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\alpha \arrow[r, mapsto] \arrow[d, mapsto] & \varphi(\alpha) \arrow[d, mapsto] \\
\neq 0 & \neq 0 \arrow[l, dotted]
\end{tikzcd}
$$

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$$
\begin{tikzcd}
 & & BP^{k+|v_0|+\dots+|v_n|}(X) \arrow[d] \\
H^k(X;F_p) \arrow[rr, "\varphi:=q_n\dots q_0"] \arrow[d, "Q_{n+1}Q_n\dots Q_0"'] & & BP\langle n\rangle^{k+|v_0|+\dots+|v_n|}(X) \arrow[d, "q_{n+1}"] \arrow[u, dotted, bend right] \\
H^{k+|v_0|+\dots+|v_{n+1}|}(X;F_p) & & BP\langle n+1\rangle^{k+|v_0|+\dots+|v_{n+1}|}(X) \arrow[ll]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\alpha \arrow[r, mapsto] \arrow[d, mapsto] & \varphi(\alpha) \arrow[d, mapsto] \\
\neq 0 & \neq 0 \arrow[l, dotted]
\end{tikzcd}
$$

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$$
\begin{tikzcd}
 & & & BP^{k+|v_0|+\ldots+|v_n|}(X) \arrow[d] \\
H^k(X;F_p) \arrow[d, "Q_{n+1}Q_n\ldots Q_0"'] & \alpha \arrow[r, mapsto] \arrow[d, mapsto] & \varphi(\alpha) \arrow[d, mapsto] & BP\langle n\rangle^{k+|v_0|+\ldots+|v_n|}(X) \arrow[d, "q_{n+1}"] \arrow[u, dotted, "\times"'] \\
H^{k+|v_0|+\ldots+|v_{n+1}|}(X;F_p) & \neq 0 & \neq 0 \arrow[l, dotted] & BP\langle n+1\rangle^{k+|v_0|+\ldots+|v_{n+1}|}(X) \arrow[lll]
\end{tikzcd}
$$

$$H^k(X;F_p) \xrightarrow{\varphi:=q_n\ldots q_0} BP\langle n\rangle^{k+|v_0|+\ldots+|v_n|}(X)$$

- If $Q_{n+1}..Q_0(\boldsymbol{\alpha})\neq 0$, then $\varphi(\boldsymbol{\alpha})$ is not algebraic.

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$$
\begin{tikzcd}
 & BP^{k+|v_0|+\dots+|v_n|}(X) \arrow[d] \\
H^k(X;F_p) \arrow[r, "\varphi:=q_n\dots q_0"] \arrow[d, "Q_{n+1}Q_n\dots Q_0"'] & BP\langle n\rangle^{k+|v_0|+\dots+|v_n|}(X) \arrow[d, "q_{n+1}"] \arrow[u, dotted, bend right, "\times"'] \\
H^{k+|v_0|+\dots+|v_{n+1}|}(X;F_p) & BP\langle n+1\rangle^{k+|v_0|+\dots+|v_{n+1}|}(X) \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\alpha \arrow[r, mapsto] \arrow[d, mapsto] & \varphi(\alpha) \arrow[d, mapsto] \\
\neq 0 & \neq 0 \arrow[l, dotted]
\end{tikzcd}
$$

the degree increases

- If $Q_{n+1}..Q_0(\alpha)\neq 0$, then $\varphi(\alpha)$ is not algebraic.

But we also pay a price...

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Finally, set X = Godeaux-Serre variety associated to the group G_{n+3} and pullback x via

$$X \longrightarrow BG_{n+3}\times CP^\infty.$$

a $2(p^{n+1}+\dots+1)+1$-connected map

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There is a smooth proj. variety X of dimension 15 over $\mathbb{C}$ with a non-algebraic class in $BP\langle 1\rangle^8(X)$.

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Thank you!