

Algebraic cycles in generalized cohomology theories

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Lefschetz's theorem: X projective complex surface

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2-dim. topological
cycle



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homologous

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Lefschetz:

class in homology

If (N) holds for Γ , then $[\Gamma]$ is "algebraic".

there is an
algebraic
curve $C \sim \Gamma$

Higher (co-)dimensions:

X smooth projective complex
algebraic variety

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(smooth)
subvariety
of dim. n

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Then $\int_Z \iota^* \alpha = 0$ unless $\alpha \in A^{n,n}(X)$.

forms of
type (n,n)

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$l: Z \subset X$ smooth projective complex
algebraic variety

Then $\int_Z l^* \alpha = 0$ unless $\alpha \in A^{n,n}(X)$.

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free abelian
group on alg.
subvarieties
of codim. p
modulo
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$CH^p(X)$

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$CH^p(X)$ $\xrightarrow{cl_H}$

$Z \subset X$

\mapsto

$[Z_{sm}]$

$H^{2p}(X; Z)$

dual of fund. class
(of desingularization)

Higher (co-)dimensions:

(smooth)
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$$\text{CH}^p(X) \xrightarrow{\text{cl}_H} \text{Hdg}^{2p}(X) \subset H^{2p}(X; \mathbb{Z})$$

integral Hodge classes
of type (p,p)

$$\mathbb{Z} \subset X \mapsto [Z_{\text{sm}}] \quad \text{dual of fund. class (of desingularization)}$$

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integral Hodge classes
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$$\mathbb{Z} \subset X \longmapsto [Z_{\text{sm}}] \quad \text{dual of fund. class (of desingularization)}$$

- Hodge's question: Is this map **surjective**?

Higher (co-)dimensions:

(smooth) subvariety of dim. n $\xrightarrow{\quad}$ $\iota: Z \subset X$ smooth projective complex algebraic variety

Then $\int_Z \iota^* \alpha = 0$ unless $\alpha \in A^{n,n}(X)$.

forms of type (n,n)

free abelian group on alg. subvarieties of codim. p modulo rat. equiv.

$$\text{CH}^p(X) \xrightarrow{\text{cl}_H} \text{Hdg}^{2p}(X) \subset H^{2p}(X; \mathbb{Z})$$

integral Hodge classes of type (p,p)

$$\mathbb{Z} \subset X \mapsto [\mathbb{Z}_{\text{sm}}] \quad \text{dual of fund. class (of desingularization)}$$

- Hodge's question: Is this map **surjective**?
- Question: What is the **kernel** of this map?

How to do homotopy on **Man**?

category of complex
manifolds



How to do homotopy on **Man**?

category of complex manifolds



Man \dashrightarrow **Pre**

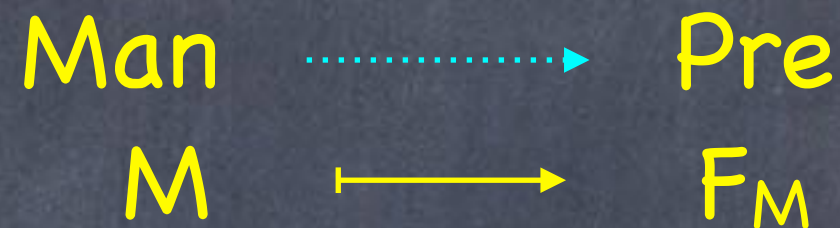
presheaves of sets, i.e.,
functors: $\text{Man}^{\text{op}} \longrightarrow \text{Set}$

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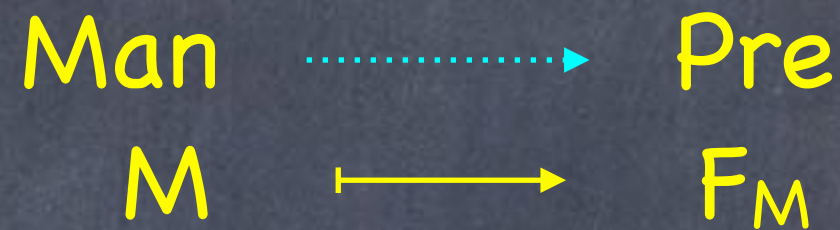
$$F_M: X \mapsto \text{Hom}_{\text{Man}}(X, M)$$

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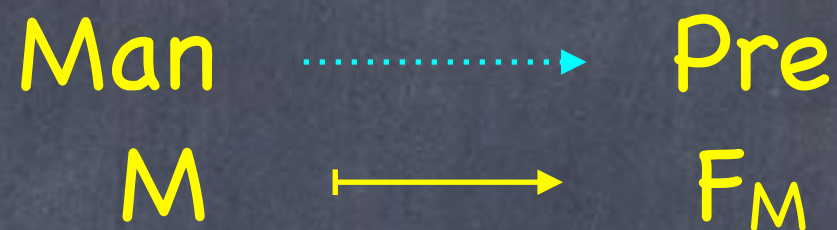
Presheaves "remember"

$$\text{Hom}_{\text{Pre}}(F_M, F_{M'}) = \text{Hom}_{\text{Man}}(M, M')$$

How to do homotopy on **Man**?

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presheaves
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Sets

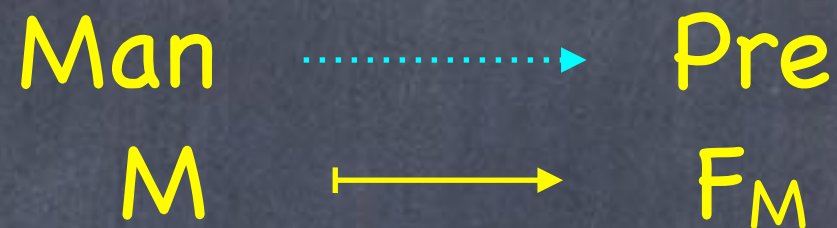
"rigid"

How to do homotopy on **Man**?

category of complex manifolds

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presheaves
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Sets

"rigid"

switch to

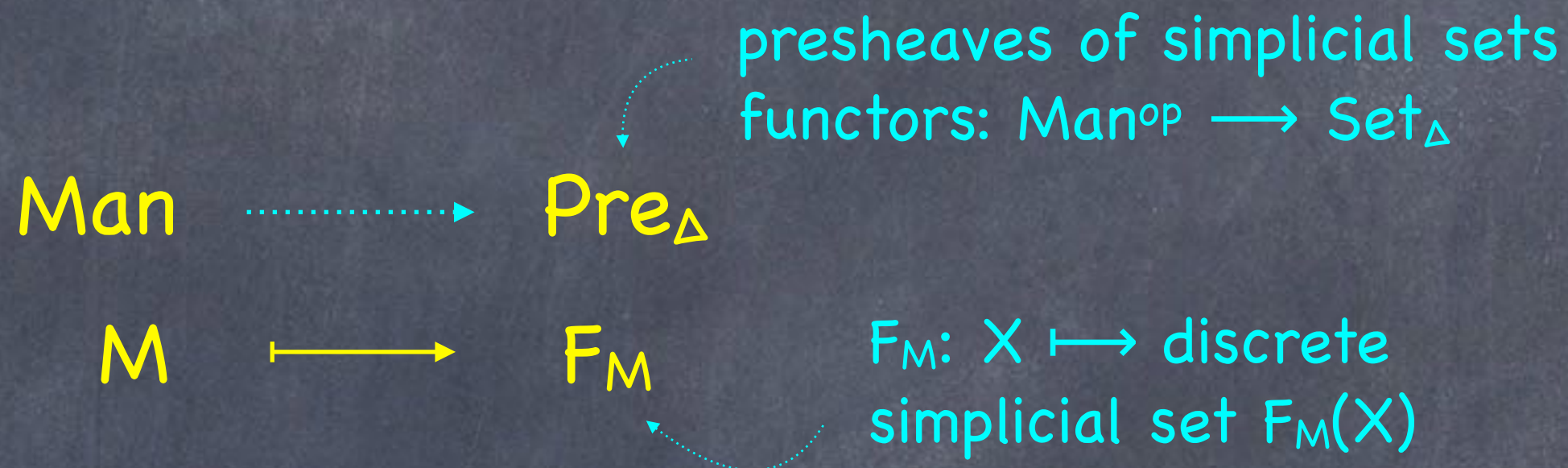
presheaves
of

Sets_Δ

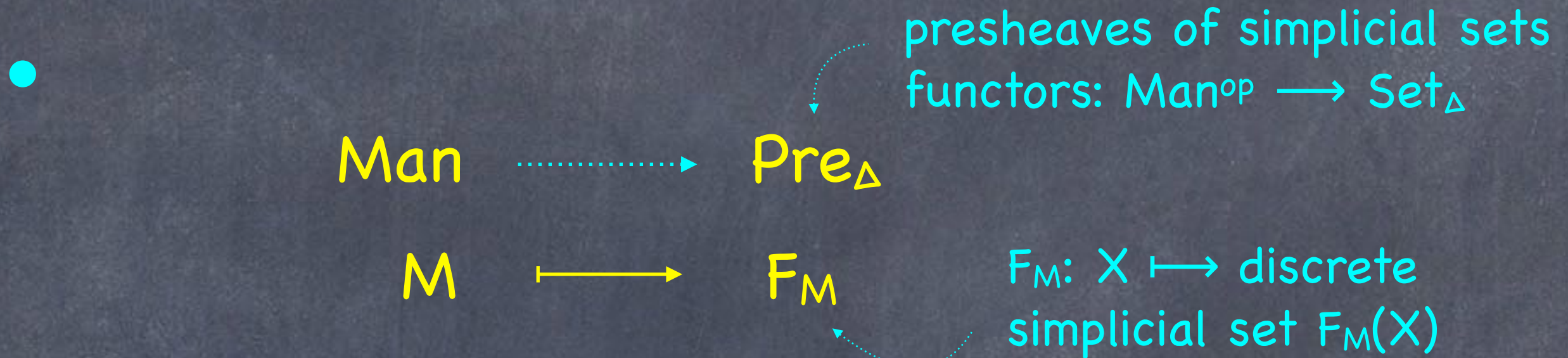
"allow homotopy"

How to do homotopy on \mathbf{Man} ?

-



How to do homotopy on \mathbf{Man} ?

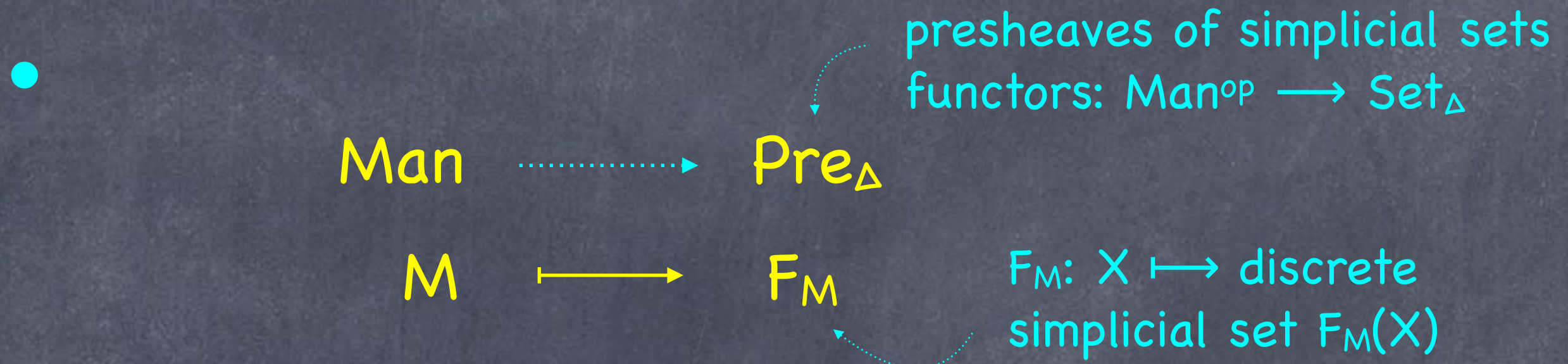


- Given $n \geq 0$, the n -dimensional stalk of F_\bullet

$$F_\bullet^{(n)} = \operatorname{colim}_{r \rightarrow 0} F_\bullet(B^n(r)) \text{ in } \mathbf{Set}_\Delta$$

ball of radius r in n -dim.
complex affine space

How to do homotopy on \mathbf{Man} ?



- Given $n \geq 0$, the n -dimensional stalk of F_\bullet .

$$F_\bullet^{(n)} = \operatorname{colim}_{r \rightarrow 0} F_\bullet(B^n(r)) \quad \text{in } \mathbf{Set}_\Delta$$

ball of radius r in n -dim. complex affine space

- A map $F_\bullet \rightarrow G_\bullet$ is a weak equivalence in \mathbf{Pre}_Δ if $F_\bullet^{(n)} \rightarrow G_\bullet^{(n)}$ is a weak equivalence in \mathbf{Set}_Δ for all $n \geq 0$.

Homotopy category of \mathbf{Man} :

- $\mathbf{Man} \xrightarrow{\dots\dots\dots} \mathbf{Pre}_\Delta$

Homotopy category of **Man**: homotopy category of simplicial presheaves on Man

• $\text{Man} \dashrightarrow \text{hoPre}_\Delta = \text{Pre}_\Delta[\text{w.e.}^{-1}]$

Homotopy category of **Man**: homotopy category of simplicial presheaves on Man

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- Given M with an open cover $\{U_\alpha\}$:

$\mathcal{F}_{U_\bullet} \rightarrow \mathcal{F}_M$ is a weak equivalence.

$$\coprod U_\alpha \rightrightarrows \coprod U_\alpha \times U_\beta \rightrightarrows \dots$$

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sequence of spaces $\dots, E_n, E_{n+1}, \dots$
with maps $S^1 \wedge E_n \rightarrow E_{n+1}$

- Can replace **Set**_Δ with **Spectra** and get a **stable** homotopy category **hoPre**_{Spectra} of **Man**.

- $S^1 \wedge -$ with S^1 viewed as a simplicial (constant) presheaf is made invertible.

Homotopy category of $S\mathbf{m}_C$:

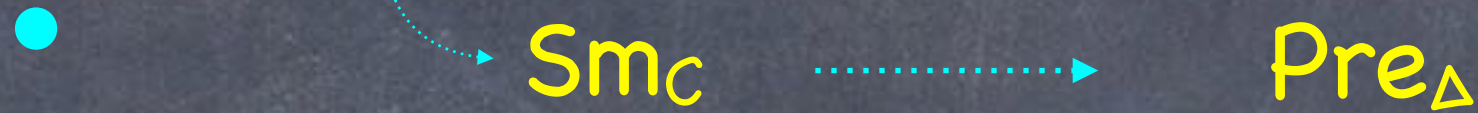
smooth complex varieties



 $S\mathbf{m}_C$

Homotopy category of \mathbf{Sm}_C :

smooth complex varieties



simplicial presheaves on \mathbf{Sm}_C

- Morel
- Voevodsky
- Jardine
- Joyal
- Isaksen
- Dugger
- ...

Homotopy category of \mathbf{Sm}_C :

- smooth complex varieties $\xrightarrow{\quad}$ \mathbf{Sm}_C $\xrightarrow{\quad}$ \mathbf{Pre}_Δ $\xleftarrow{\quad}$ simplicial presheaves on \mathbf{Sm}_C
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Homotopy category of \mathbf{Sm}_C :

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Homotopy category of \mathbf{Sm}_C : motivic homotopy category of simplicial presheaves on \mathbf{Sm}_C

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 - $\mathcal{F}_U \rightarrow \mathcal{F}_X$ for any X and any (hyper)cover $U \rightarrow X$
 - $A^1_C X \rightarrow X$ for any X
affine line over C

Homotopy category of \mathbf{Sm}_C : motivic homotopy category of simplicial presheaves on \mathbf{Sm}_C

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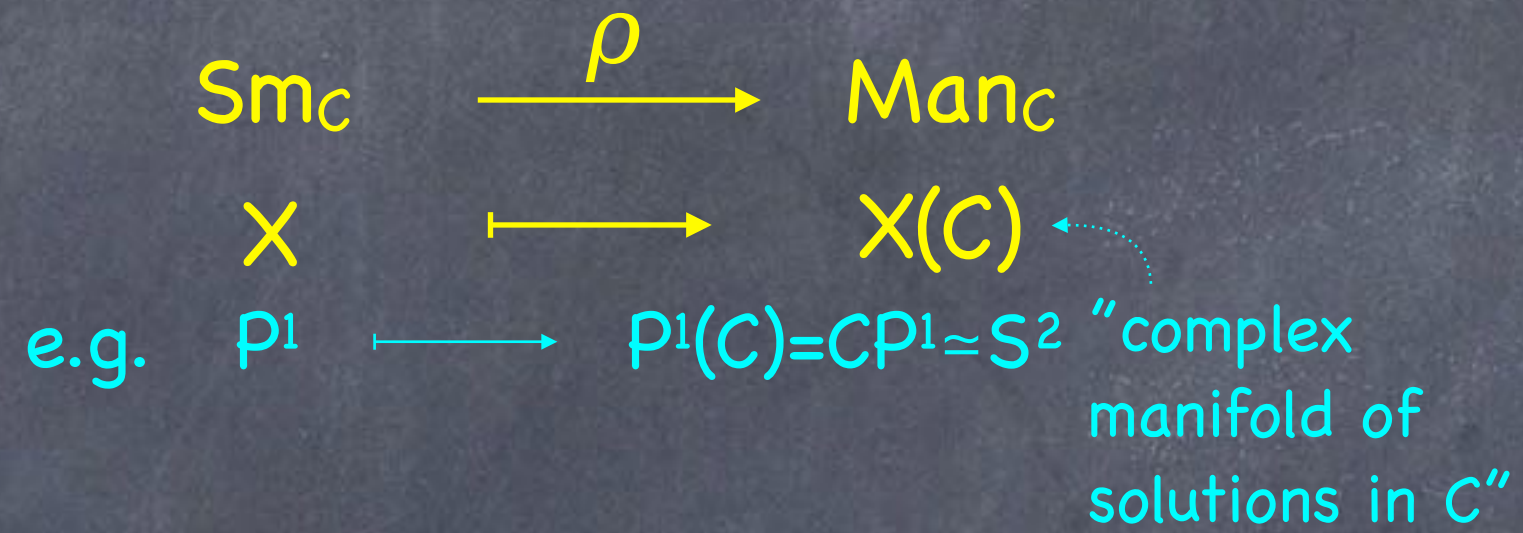
A^1_C : affine line over C

• stable motivic homotopy category of \mathbf{Sm}_C

• \mathbb{P}^1_C – the projective line

• S^1_C – the “simplicial circle” and $(A^1 - 0)_C$ – the “Tate circle”

Topological realization:



Topological realization:

$$\mathbf{Sm}_C \xrightarrow{\rho} \mathbf{Man}_C$$

$$X \longmapsto X(C)$$

e.g. $p^1 \longmapsto p^1(C) = CP^1 \simeq S^2$ "complex manifold of solutions in C "

motivic spectrum

$$E_{\text{mot}}^{a,b}(X)$$

"algebraic"

induced map

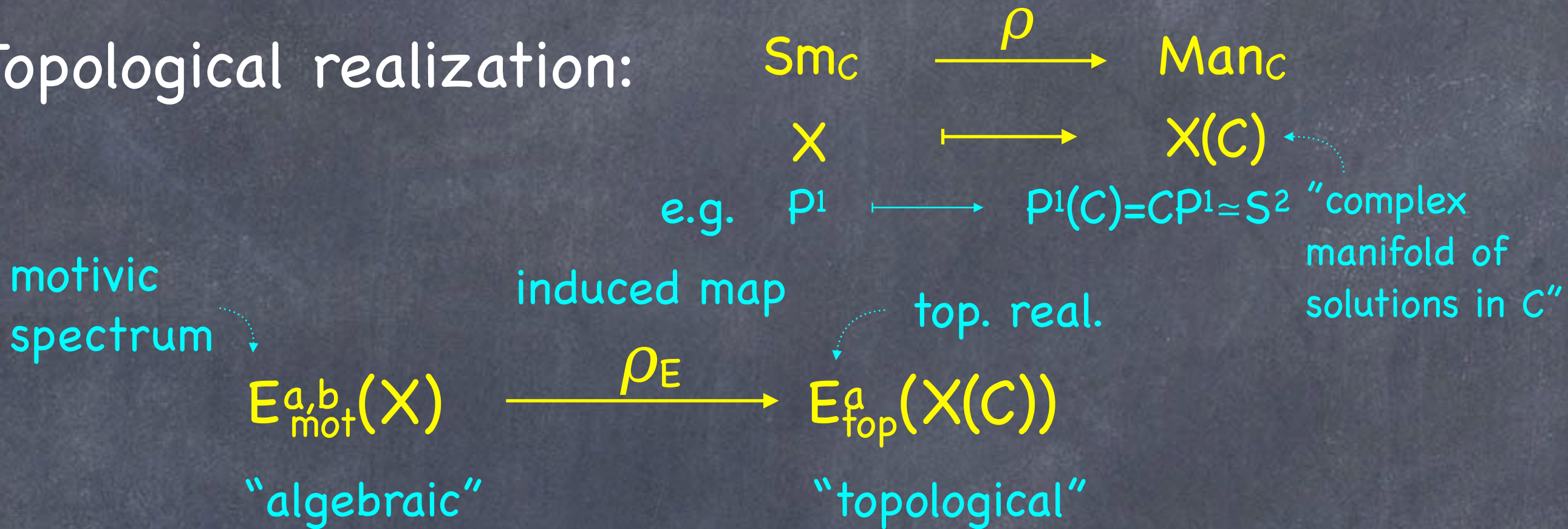
$$\xrightarrow{\rho_E}$$

$$E_{\text{top}}^a(X(C))$$

"topological"

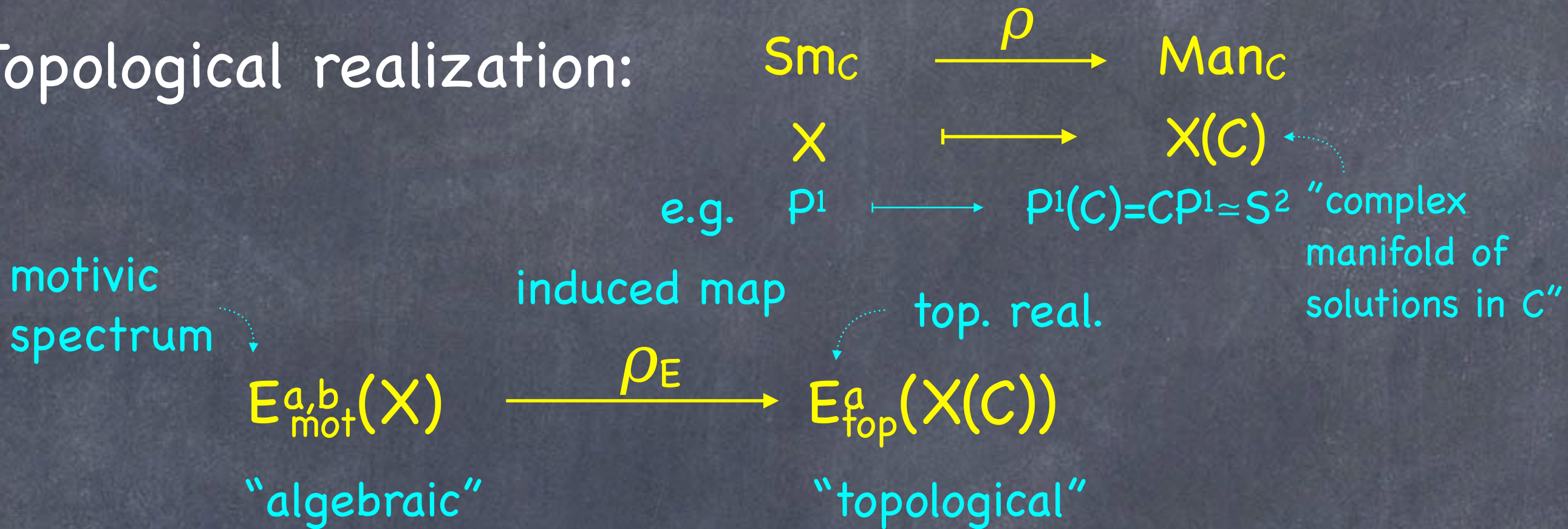
top. real.

Topological realization:



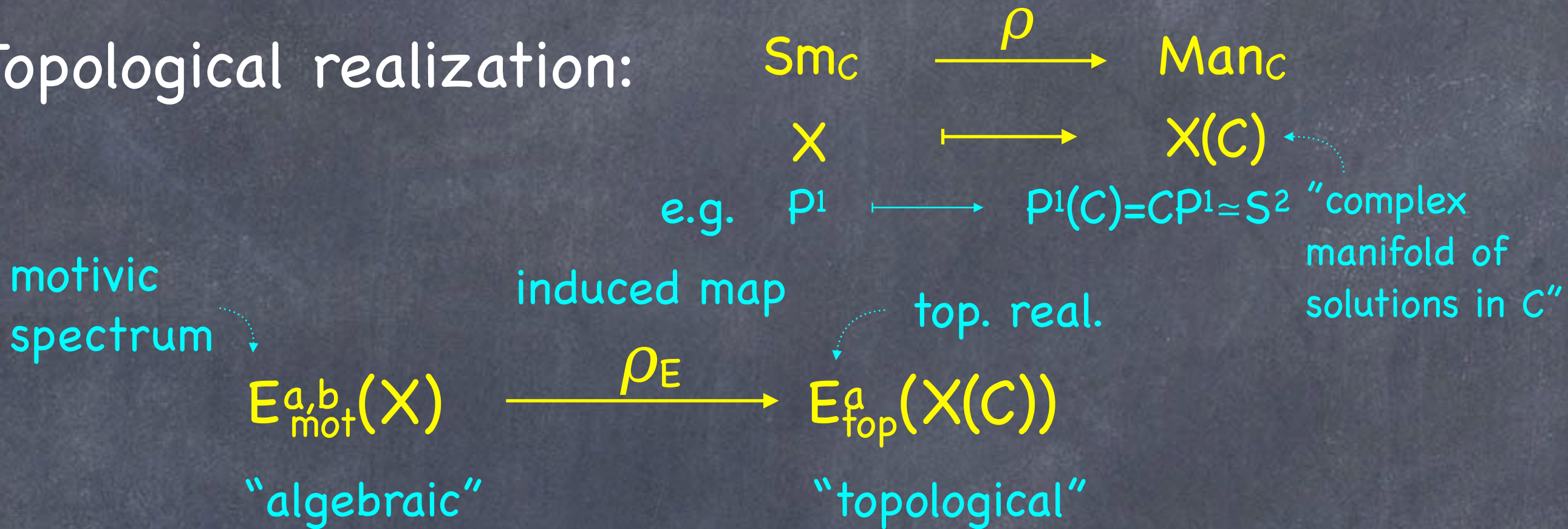
Questions:

Topological realization:



Questions: How can we detect whether classes

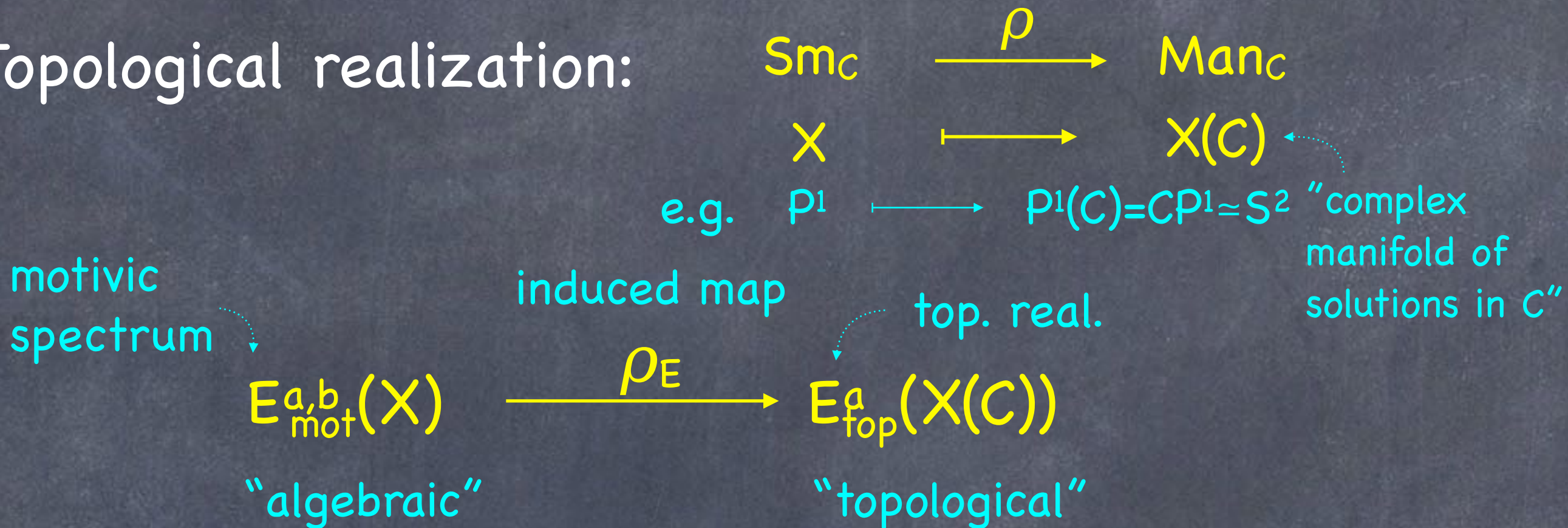
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Questions: How can we detect whether classes

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Topological realization:



Questions: How can we detect whether classes

- in $E_{\text{mot}}^{*,*}(X)$ are topologically trivial, i.e., become 0 in $E_{\text{top}}^*(X(C))$?
- in $E_{\text{top}}^*(X(C))$ are algebraic, i.e., are in the image of ρ_E ?

Atiyah-Hirzebruch, Totaro, Levine-Morel:

$$\mathrm{CH}^*(X) = \mathrm{H}_{\mathrm{mot}}^{2*}(X; \mathbb{Z}) \xrightarrow{\rho_H = \mathrm{cl}_H} \mathrm{Alg}_H^{2*}(X) \subseteq \mathrm{H}^{2*}(X; \mathbb{Z})$$

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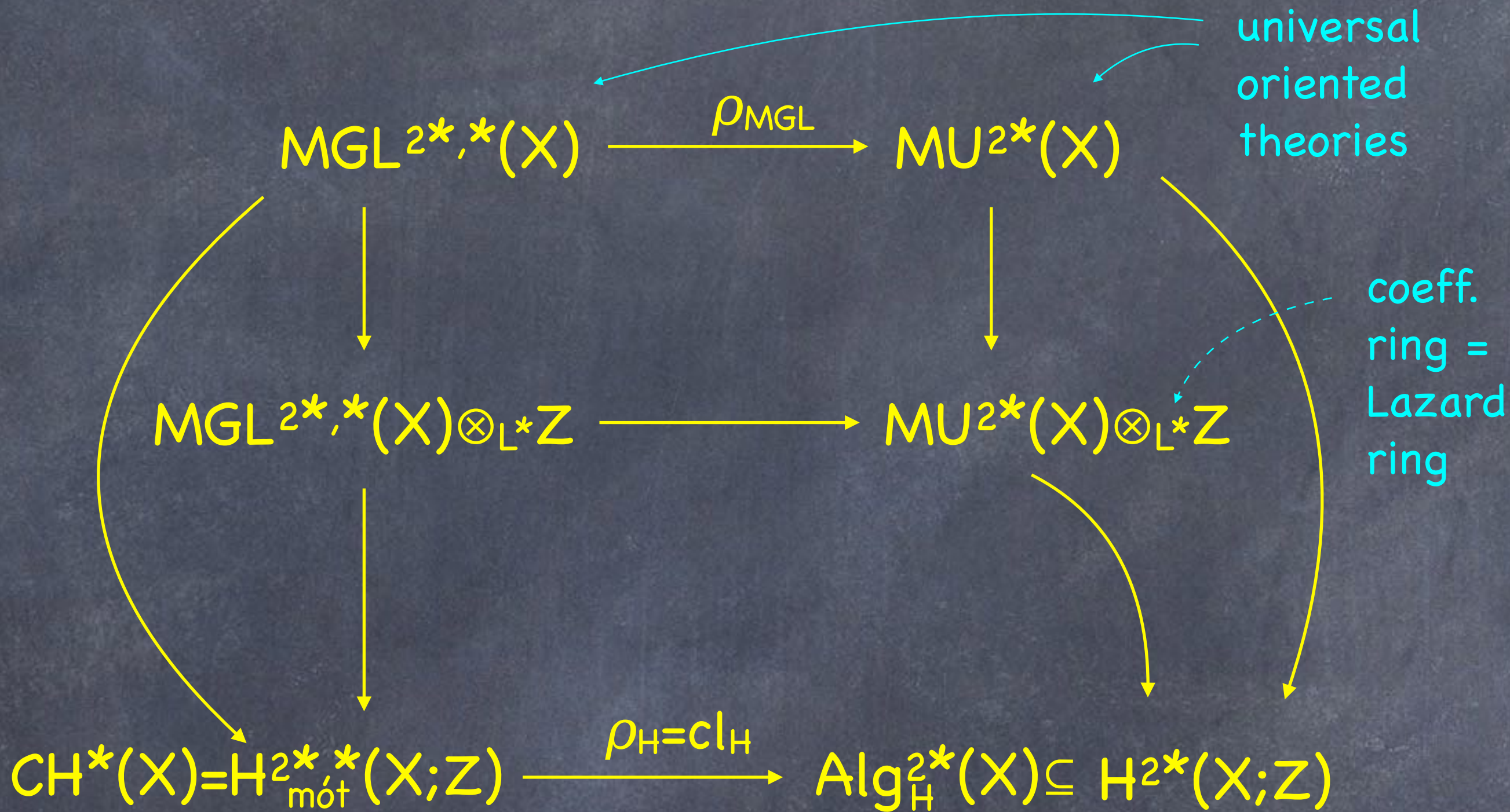
$$\text{MGL}^{2*,*}(X) \xrightarrow{\rho_{\text{MGL}}} \text{MU}^{2*}(X)$$

universal
oriented
theories

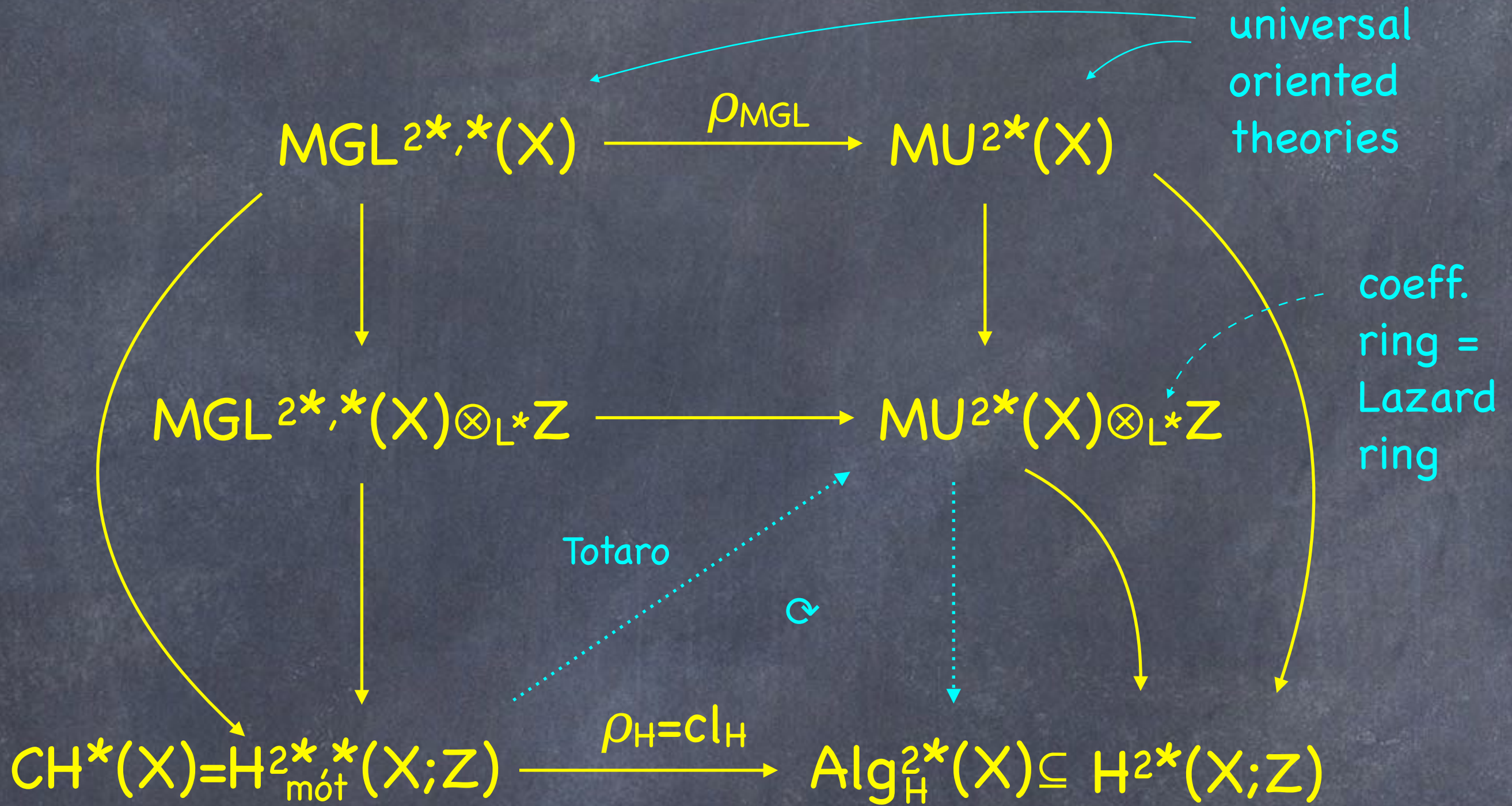
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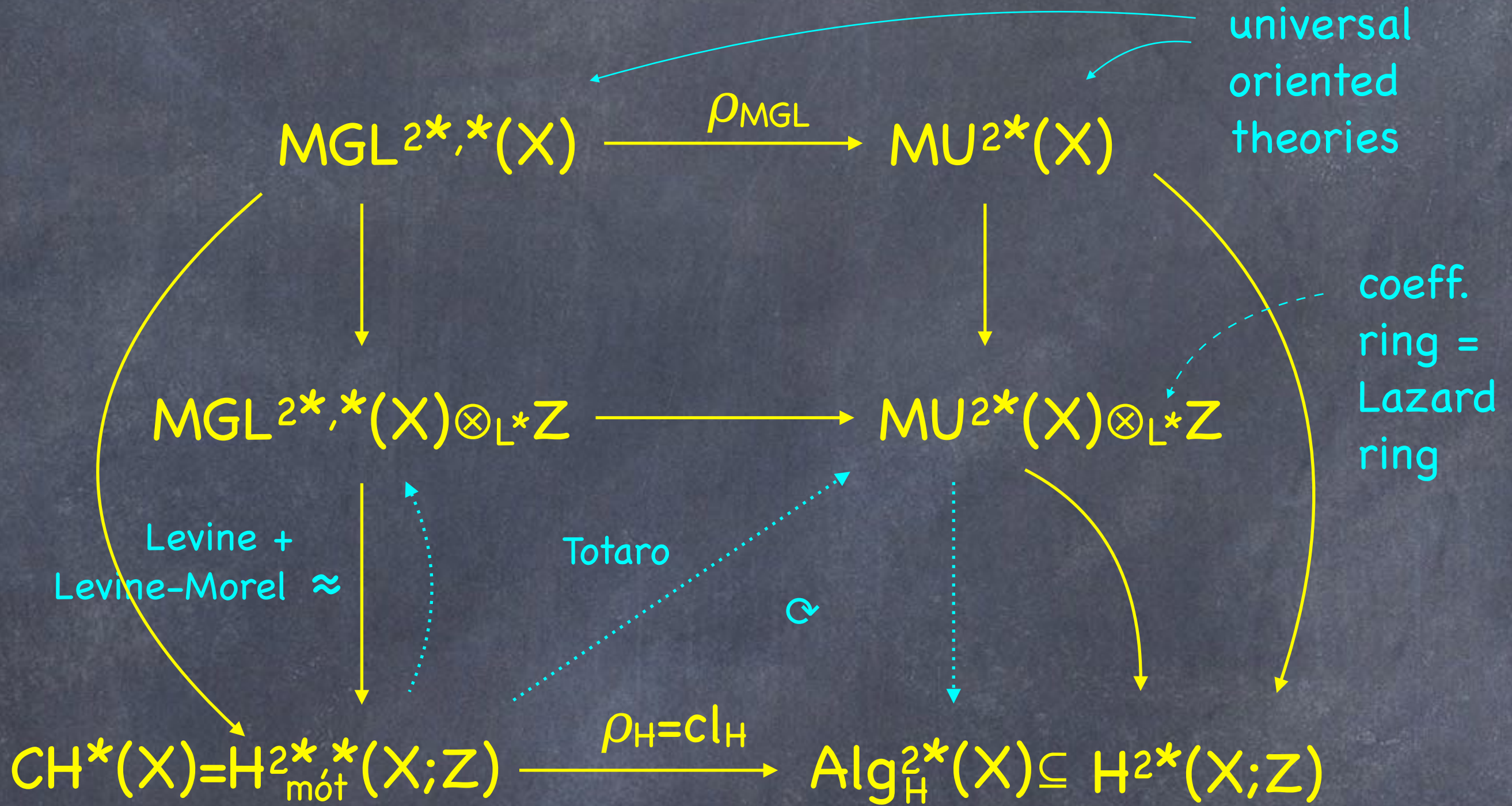
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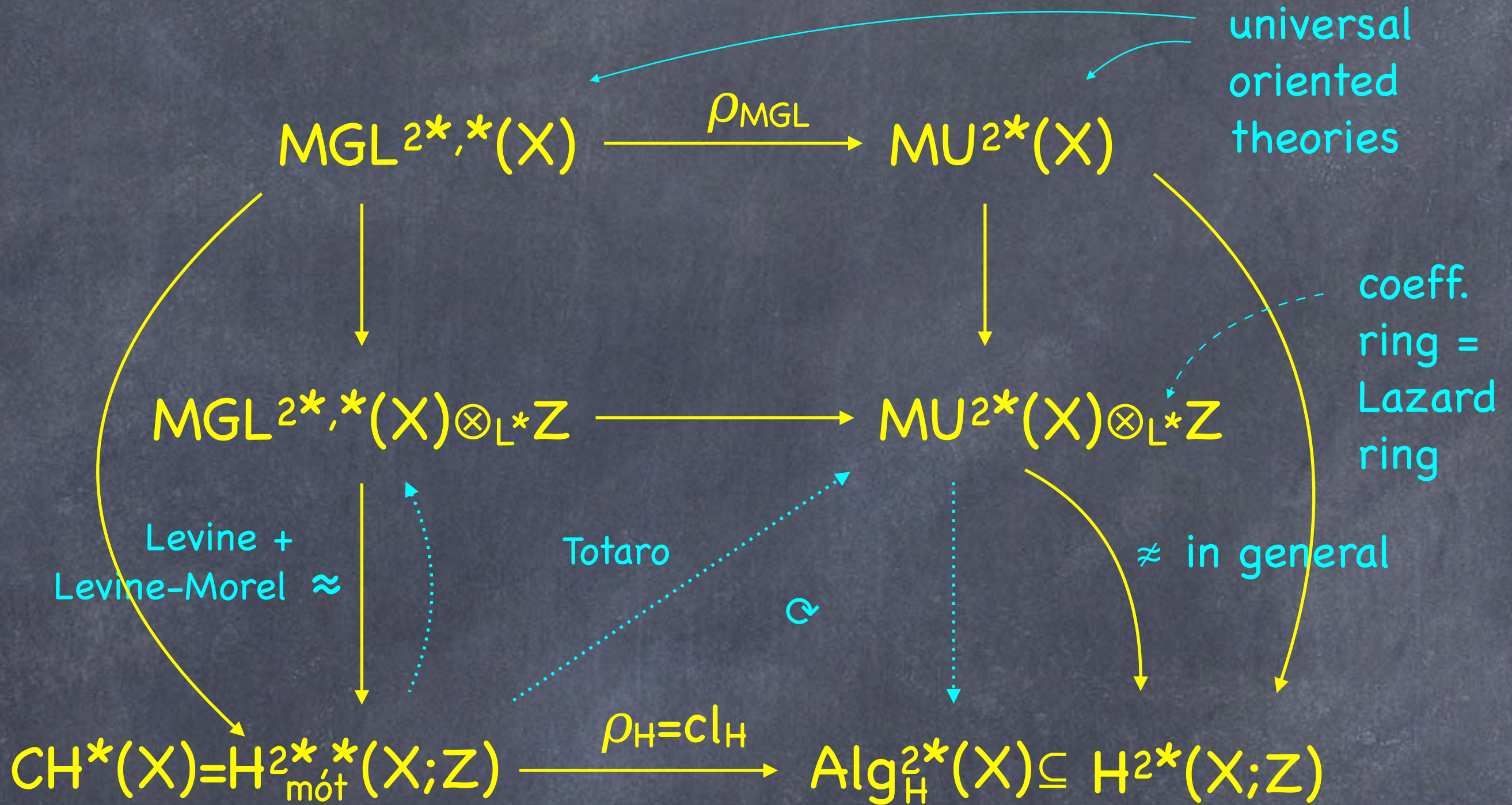
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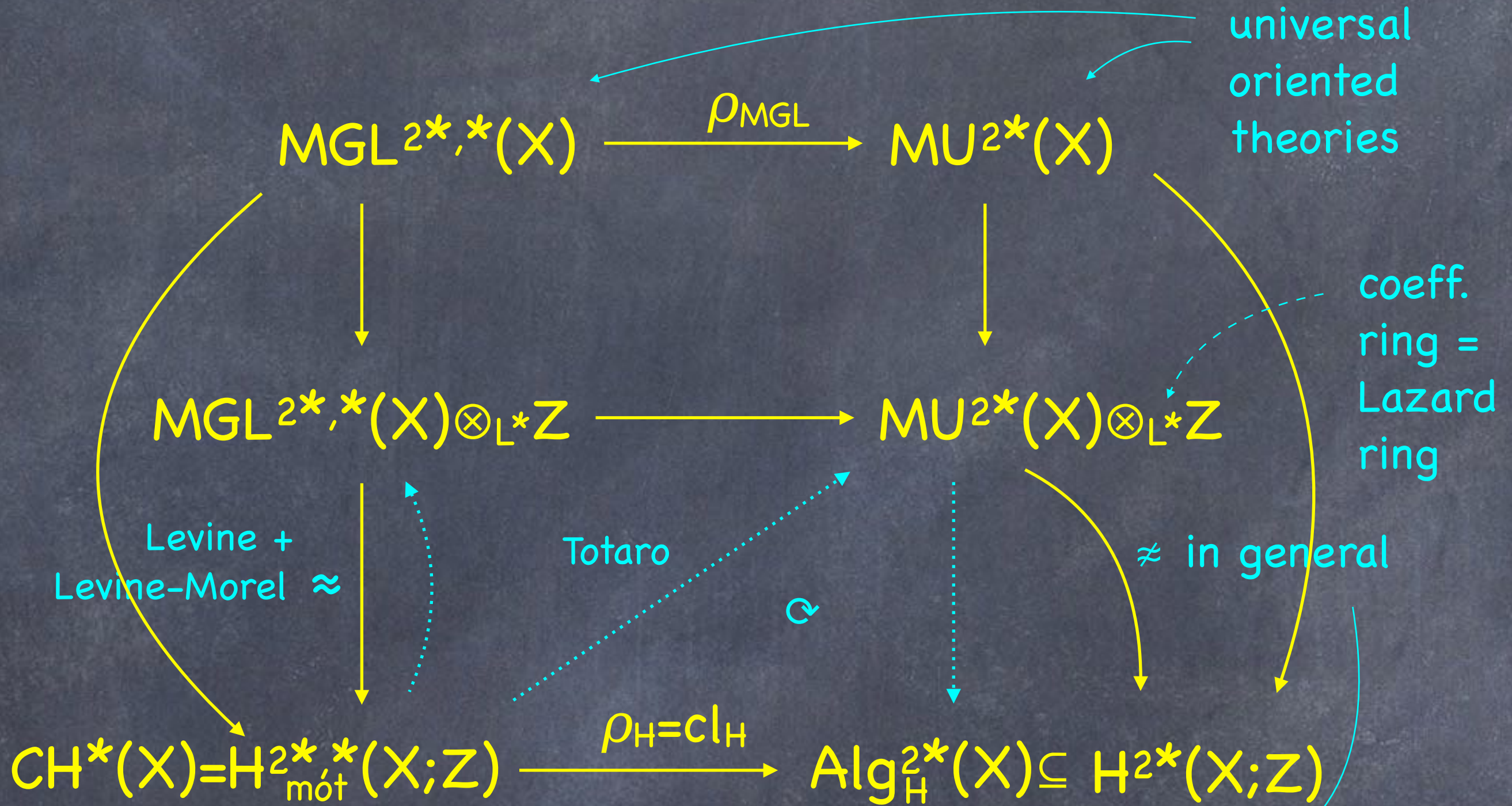
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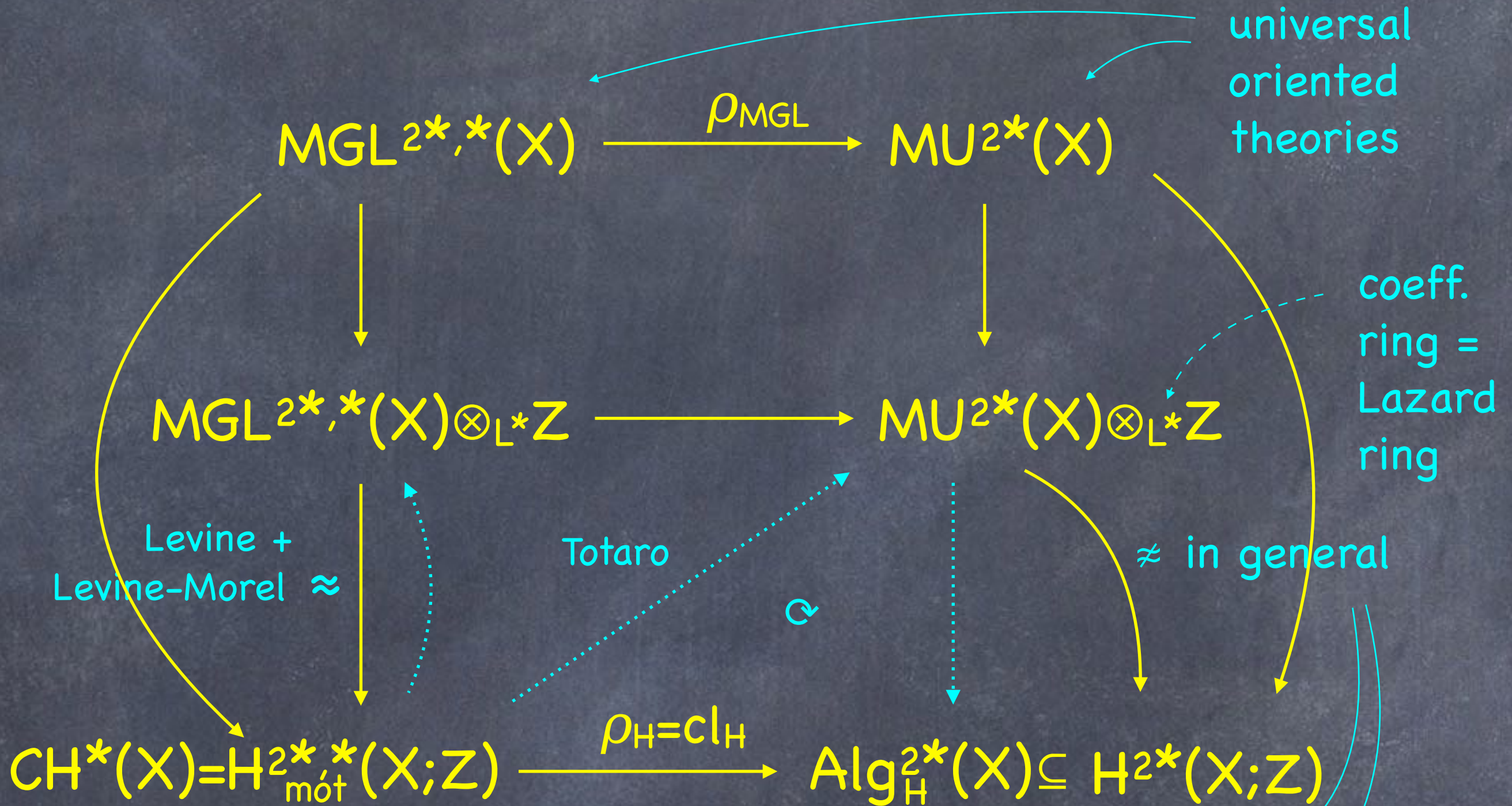


Atiyah-Hirzebruch, Totaro, Levine-Morel:



- Atiyah-Hirzebruch: cl_H is **not** surjective onto integral Hodge classes.

Atiyah-Hirzebruch, Totaro, Levine-Morel:



- Atiyah-Hirzebruch: cl_H is **not** surjective onto integral Hodge classes.
- Totaro: new classes in **kernel** of cl_H .

I. Kernel: X smooth projective

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Recall Deligne's diagram

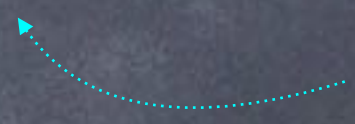
$$0 \rightarrow \mathcal{J}^{2p-1}(X) \rightarrow H_{\mathbb{D}}^{2p}(X; \mathbb{Z}(p)) \rightarrow \text{Hdg}^{2p}(X) \rightarrow 0$$

I. Kernel: X smooth projective

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Hodge
classes




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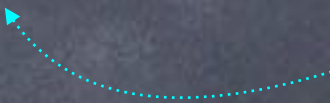
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Deligne
cohomology



Hodge
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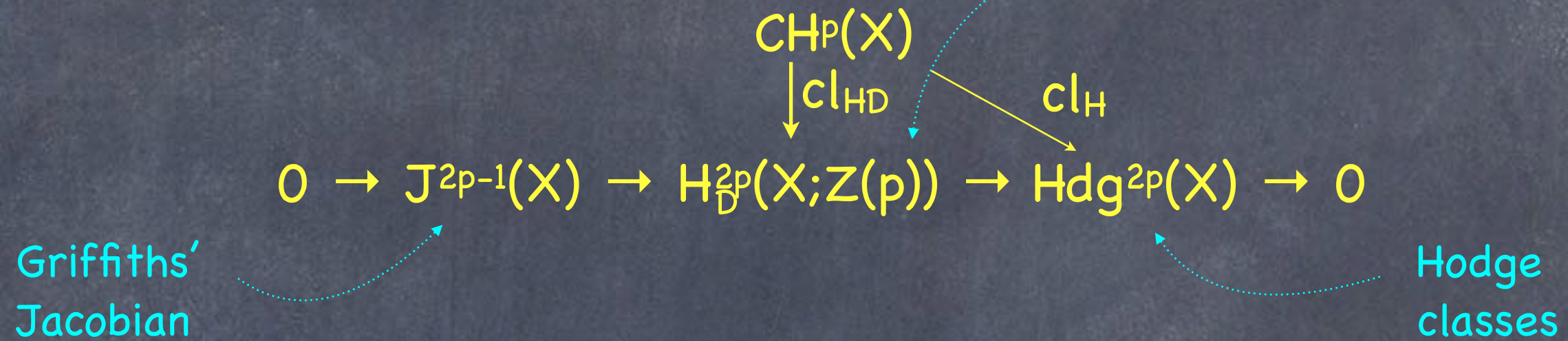
Griffiths'
Jacobian

Deligne
cohomology

Hodge
classes

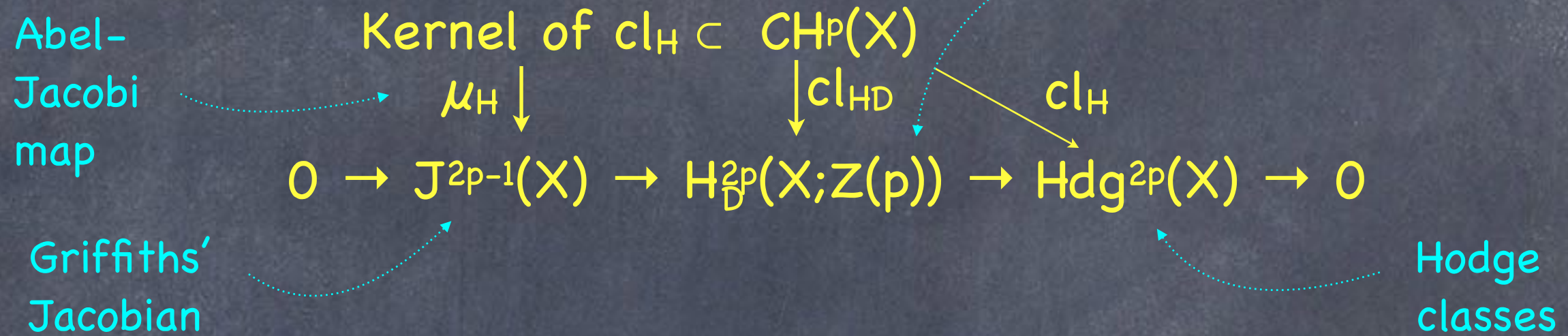
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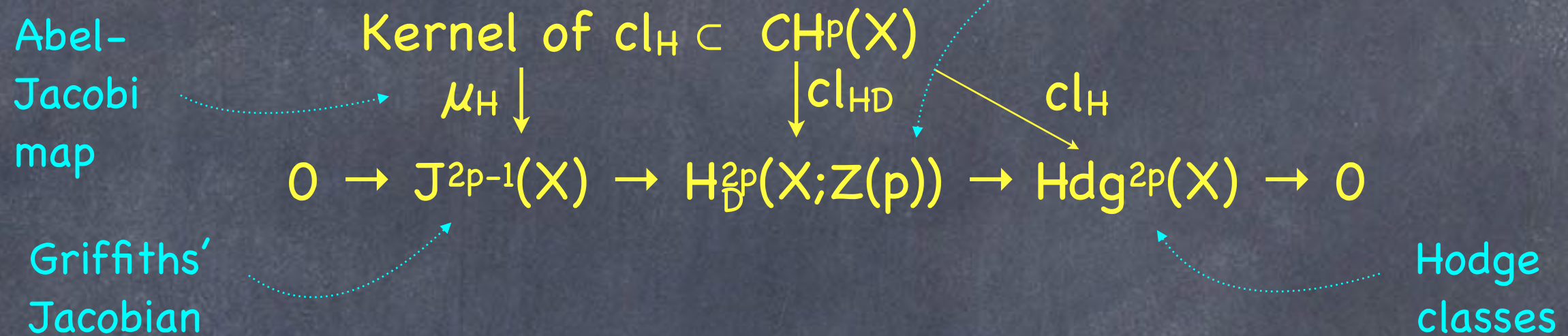
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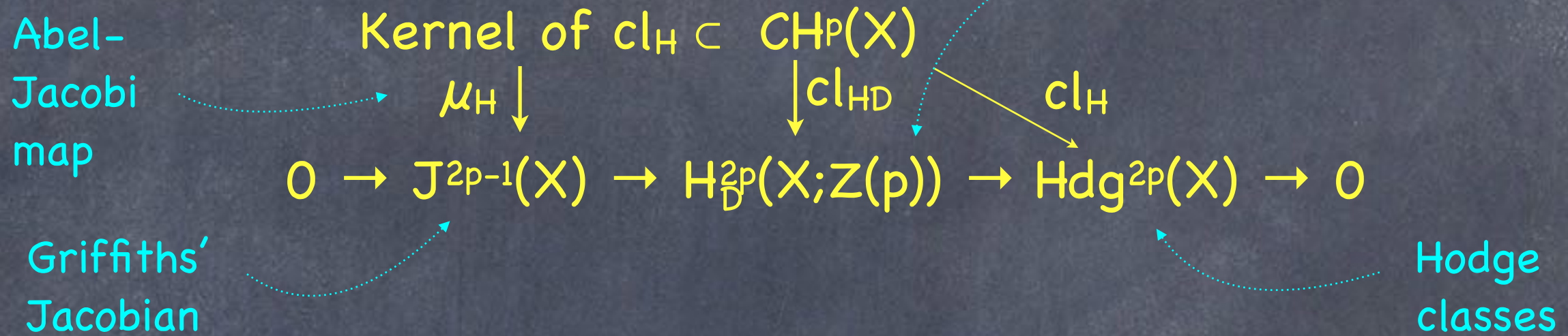
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Generalized Hodge filtered cohomology theories
(joint work with [Mike Hopkins](#)):

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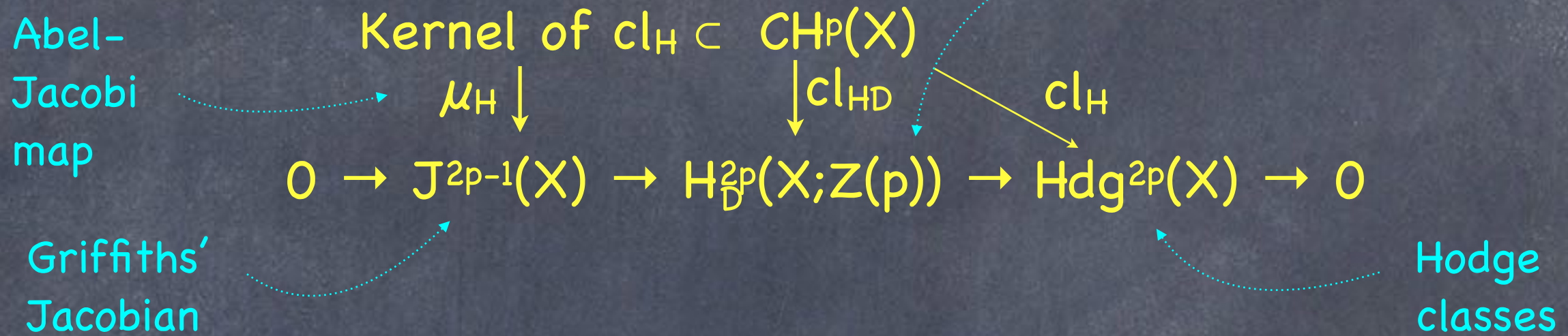


Generalized Hodge filtered cohomology theories
(joint work with [Mike Hopkins](#)):

$$0 \rightarrow J_{MU}^{2p-1}(X) \rightarrow MU_D^{2p}(X; Z(p)) \rightarrow Hdg_{MU}^{2p}(X) \rightarrow 0$$

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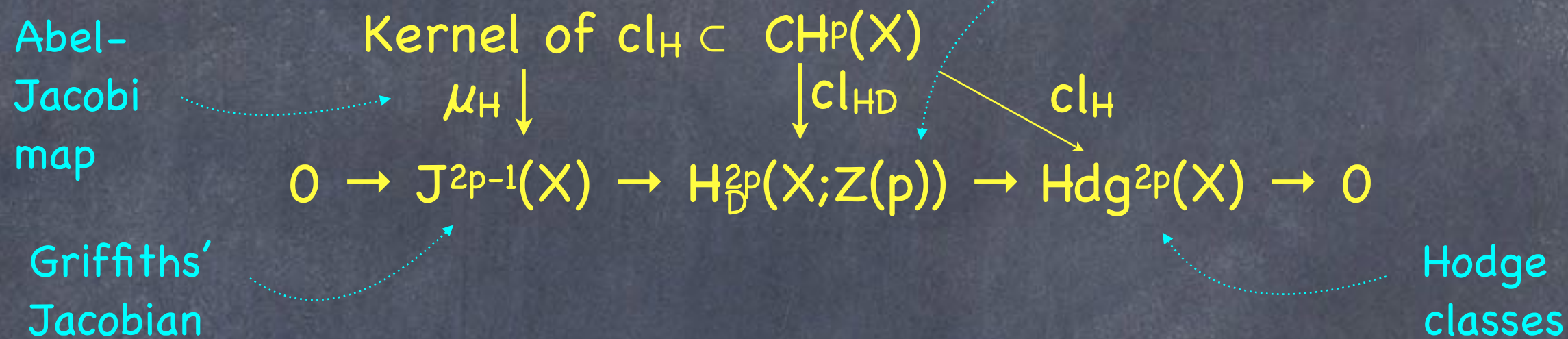
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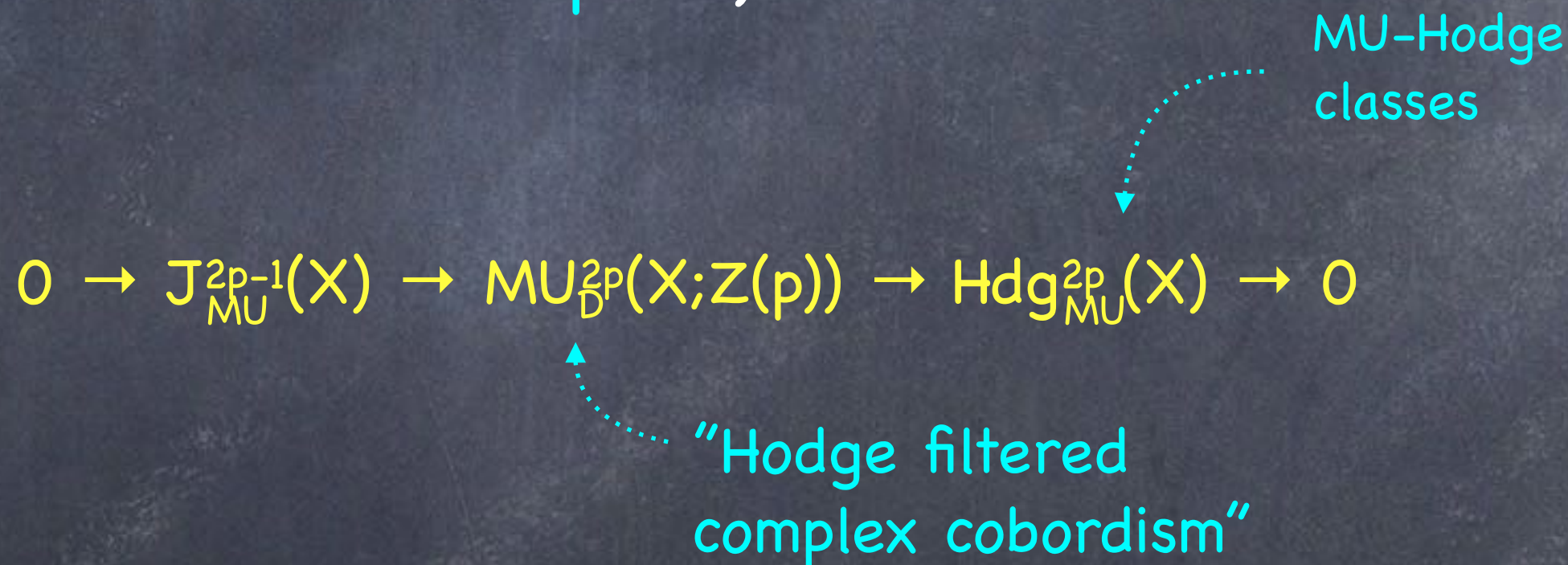
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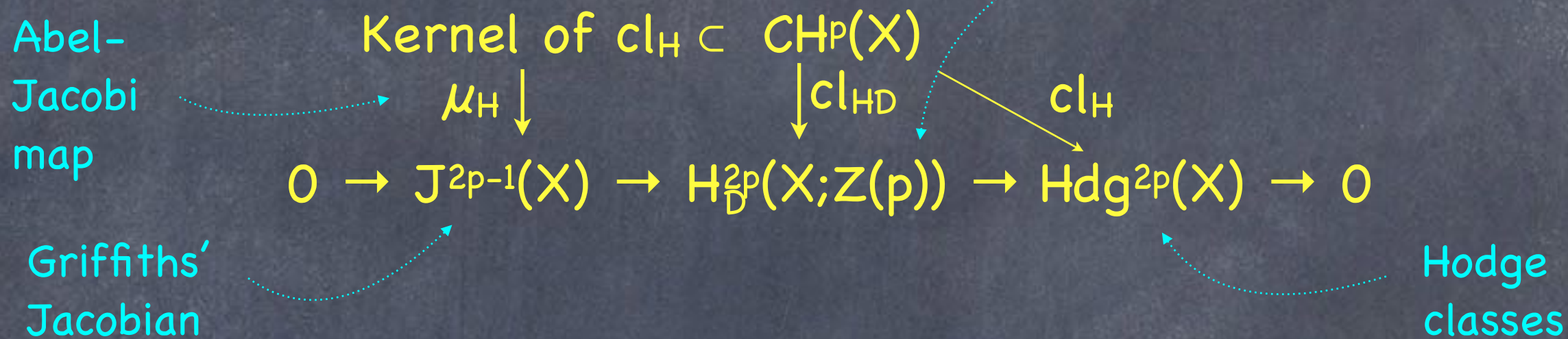


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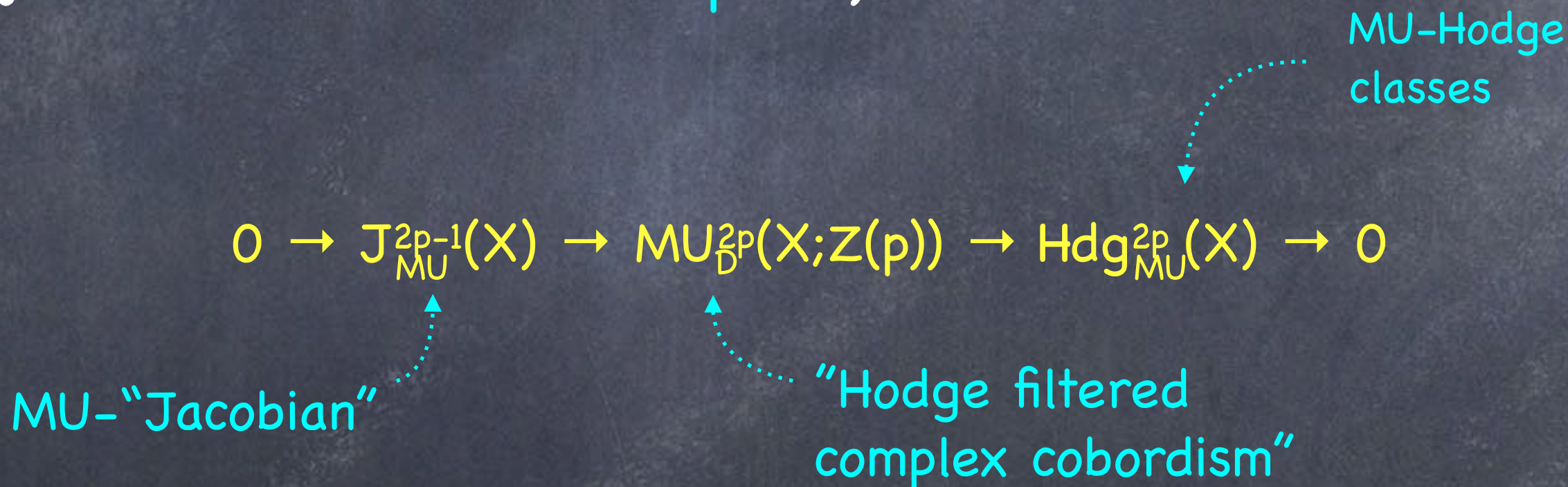


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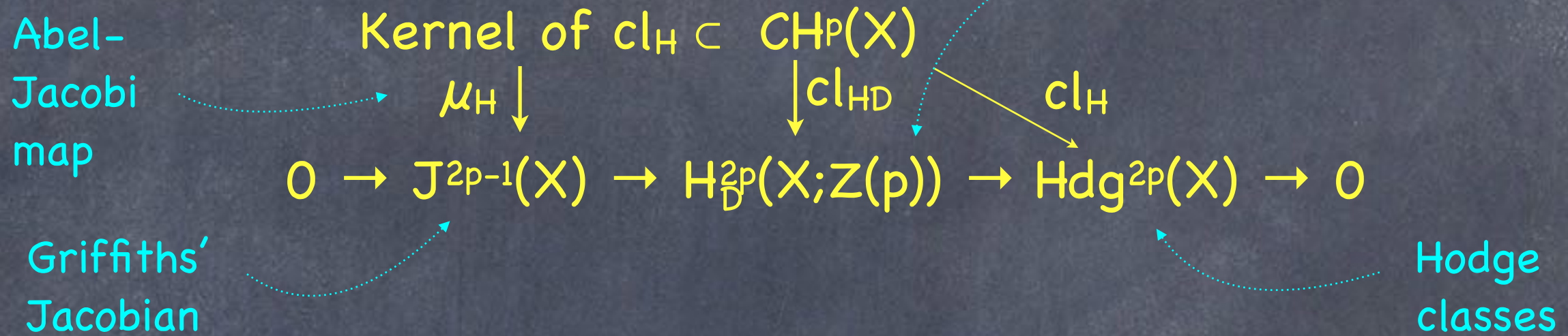


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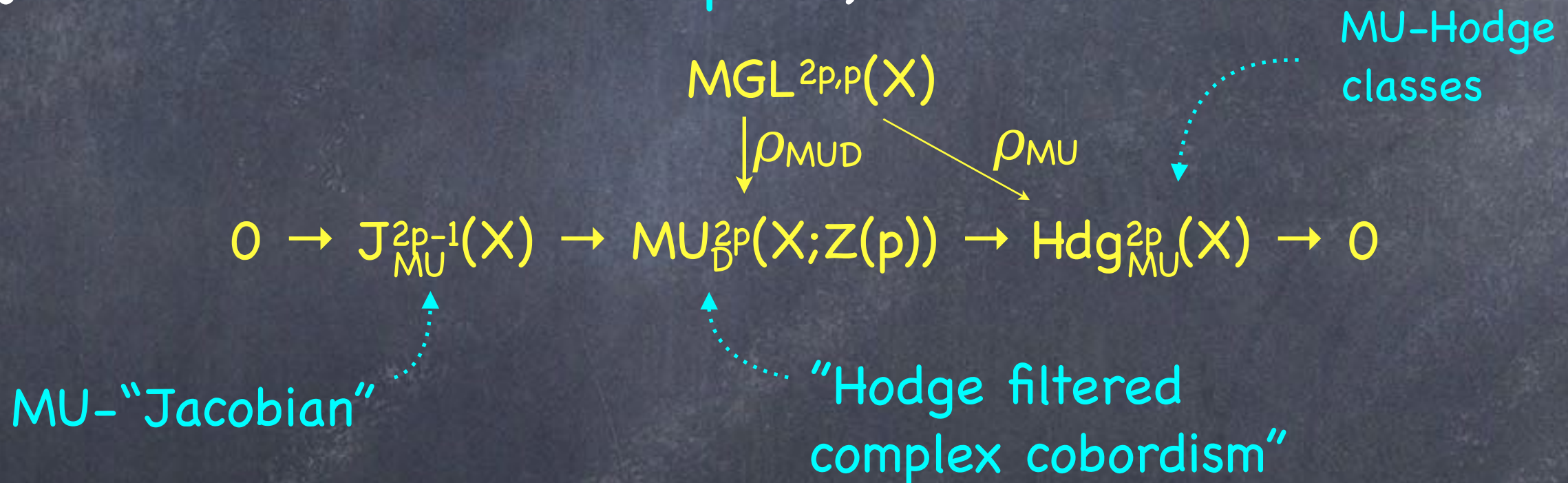


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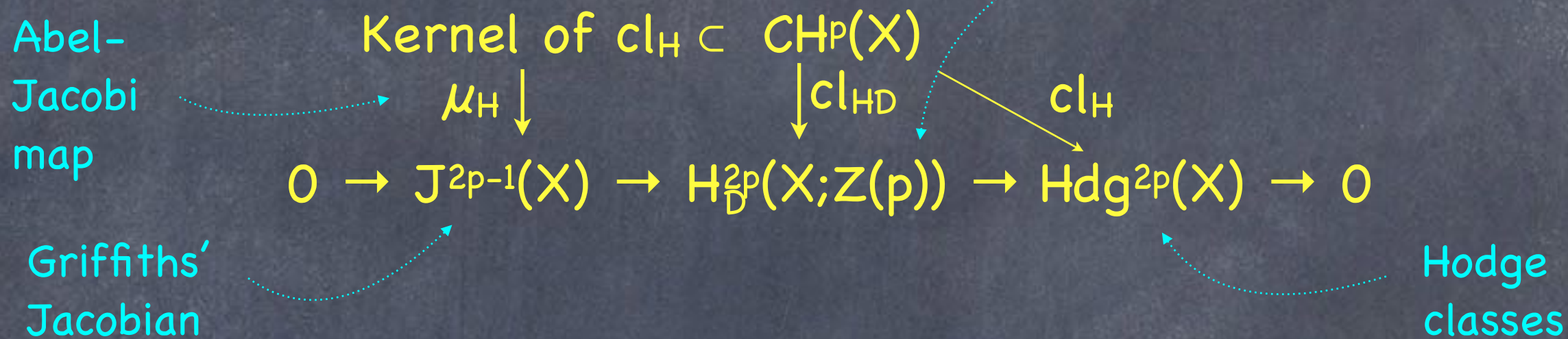


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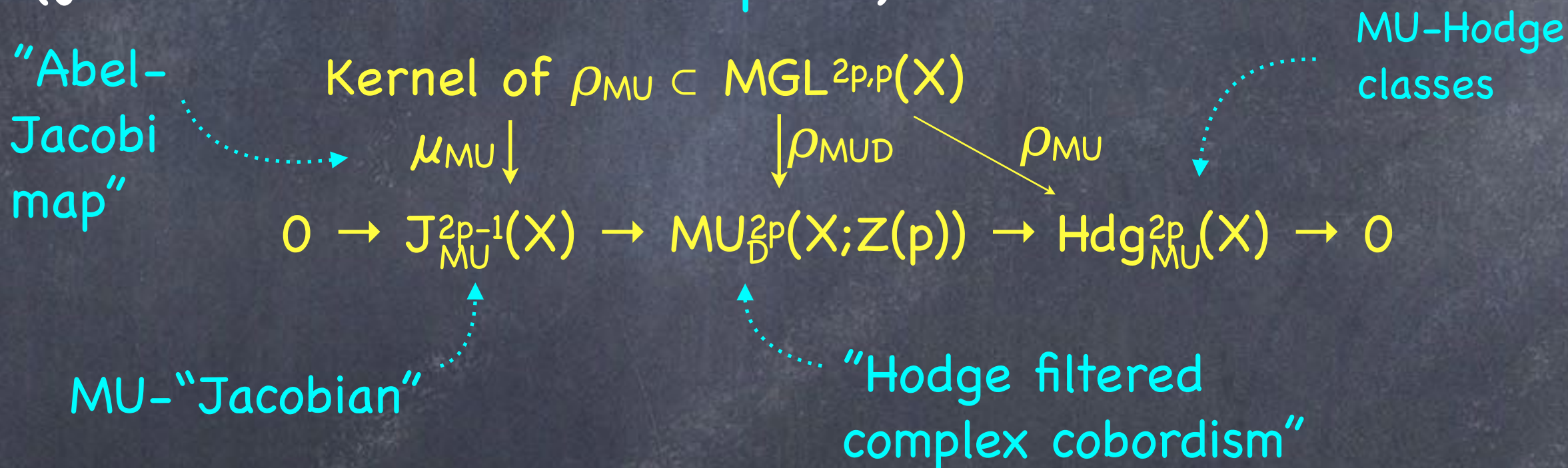


I. Kernel: X smooth projective

Recall Deligne's diagram

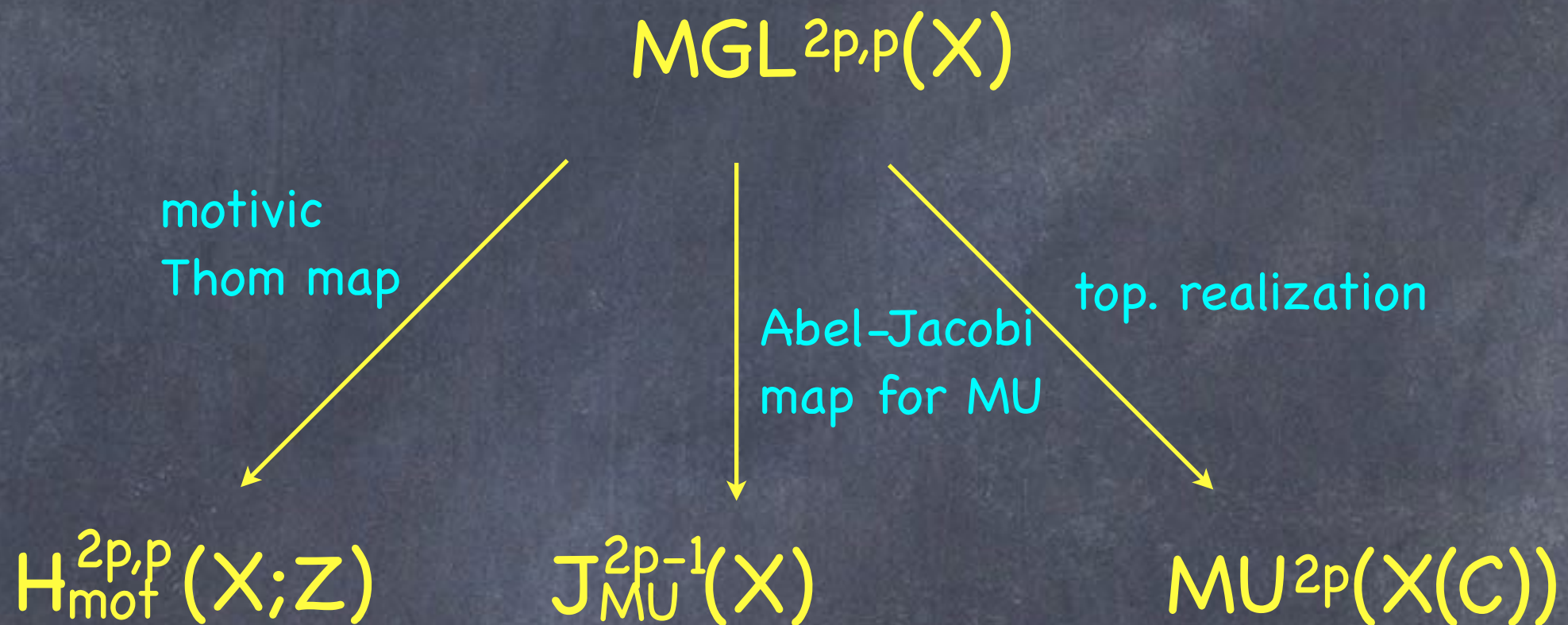


Generalized Hodge filtered cohomology theories
(joint work with Mike Hopkins):



Examples:

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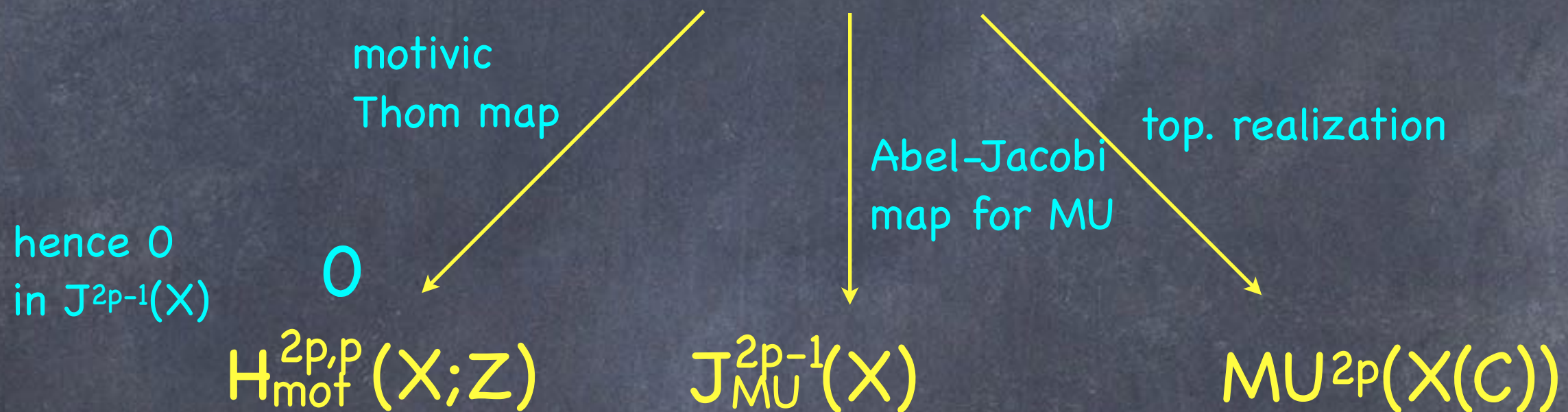
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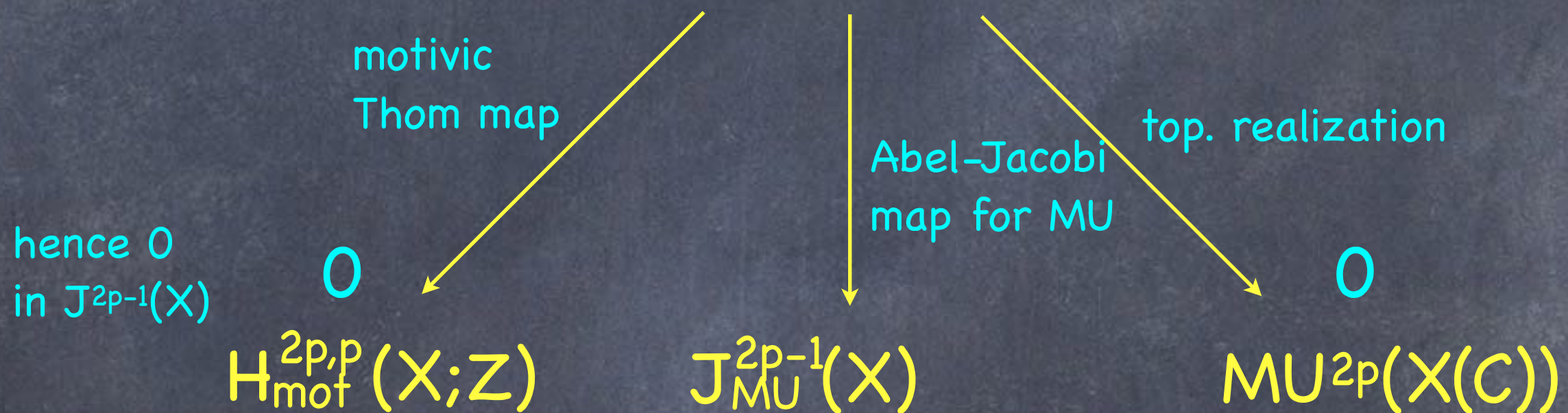
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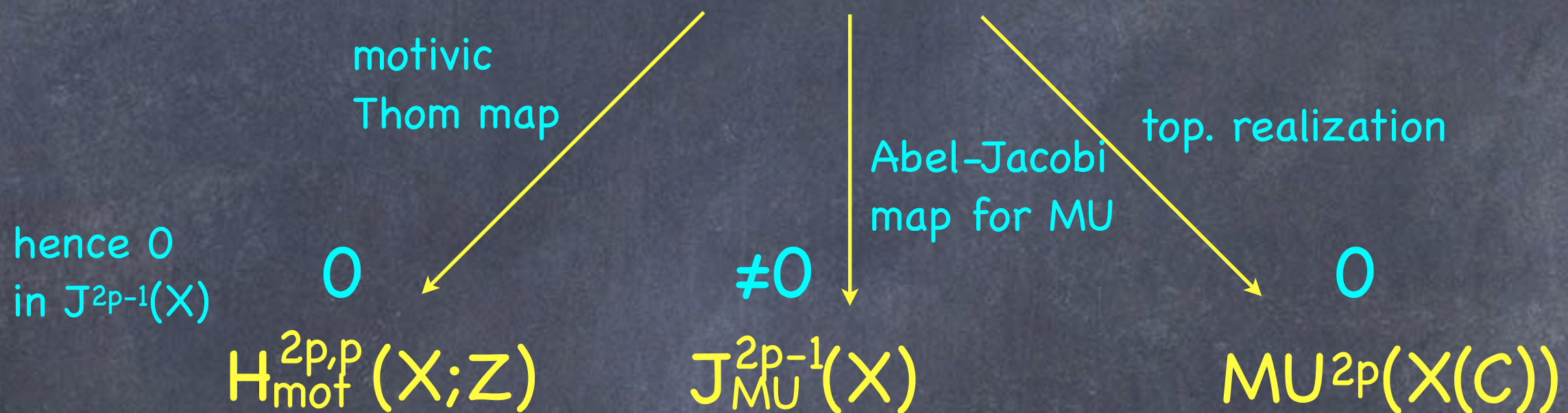
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" f_* of universal genus of curvature form" of normal bundle of Y if Y is a smooth projective variety

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Question: What is the arithmetic-geometric information encoded in the Chern classes in Arakelov algebraic cobordism?

II. Image:

Recall:

$$\begin{array}{ccc} \mathbf{Sm} & \xrightarrow{\rho} & \mathbf{Man} \\ X & \longmapsto & X(C) \end{array}$$

manifold of solutions in C

motivic spectrum



$$E_{\text{mot}}^{a,b}(X)$$

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- Question:
- How can we detect whether classes in $E_{\text{top}}^*(X(C))$ are **not algebraic**, i.e., are **not** in the image of ρ_E ?
 - How can we **construct** such classes?

A different perspective:


Fix a prime p .

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Brown-Peterson,
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The Brown-Peterson tower (Wilson):

$BP \longrightarrow \dots \longrightarrow BP\langle n \rangle \longrightarrow \dots \longrightarrow BP\langle 1 \rangle \longrightarrow BP\langle 0 \rangle \longrightarrow BP\langle -1 \rangle$

$p=2$: 2-local
connective K-theory

$HZ_{(p)} \longrightarrow HF_p$

Milnor operations:

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For every n :

stable cofibre sequence

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Thom
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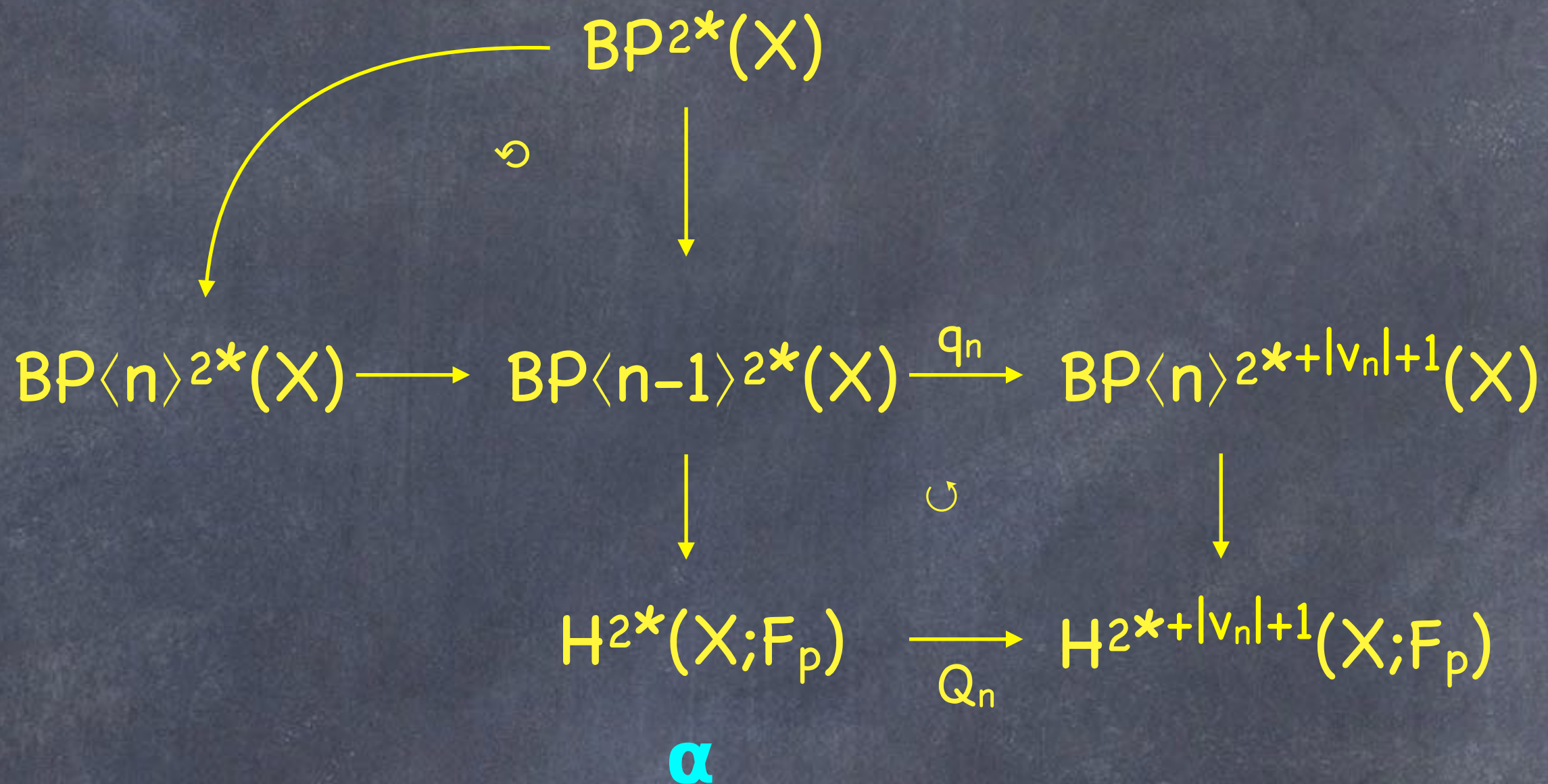
$Q_0 = \text{Bockstein}$

$Q_n = P^{p^{n-1}} Q_{n-1} - Q_{n-1} P^{p^{n-1}}$

The LMT obstruction in action:

$$\begin{array}{ccccc}
 & & \text{BP}^{2^*}(X) & & \\
 & \searrow & \downarrow & & \\
 & & & & \\
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α

Question: Is α algebraic?

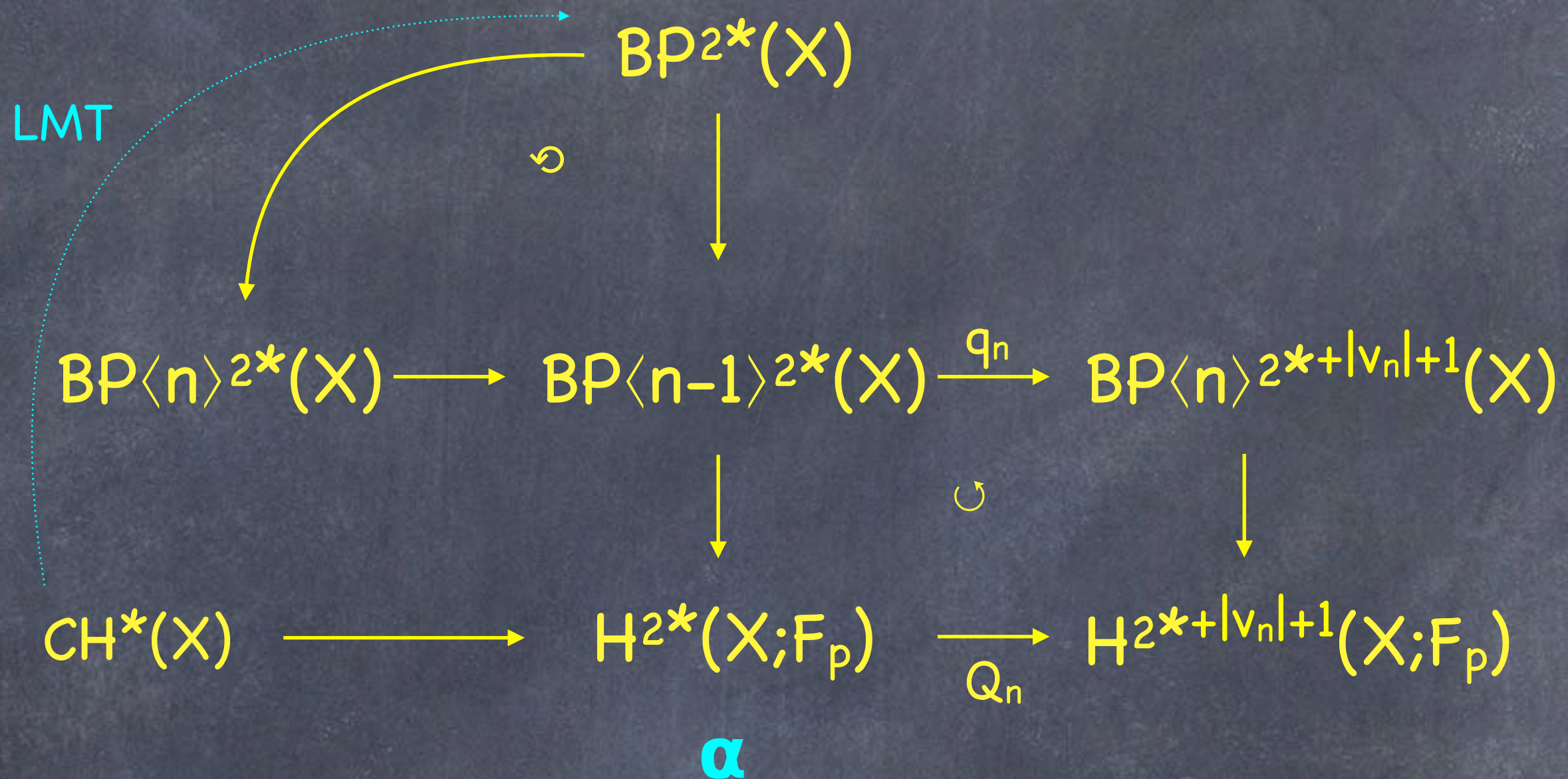
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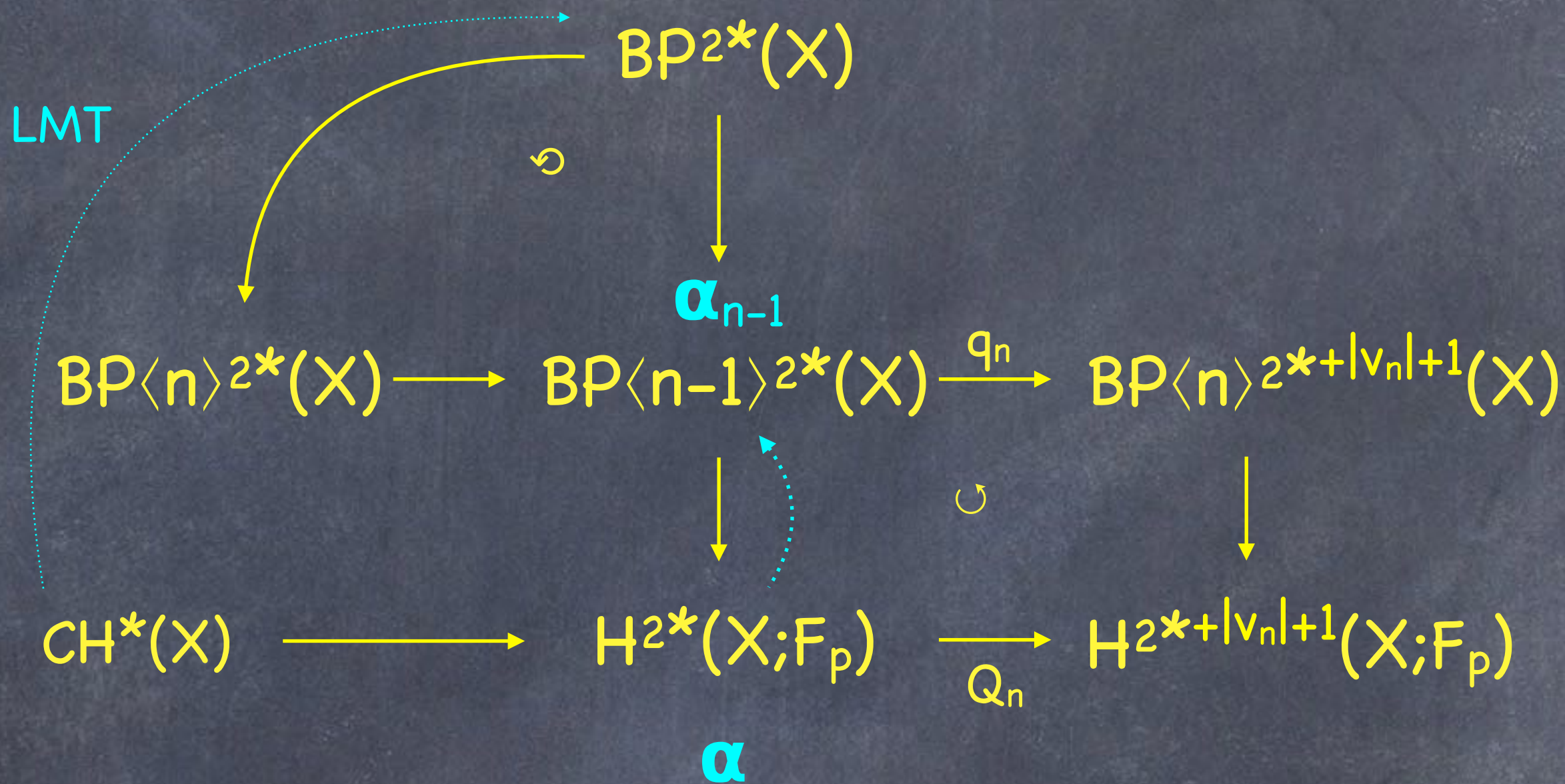
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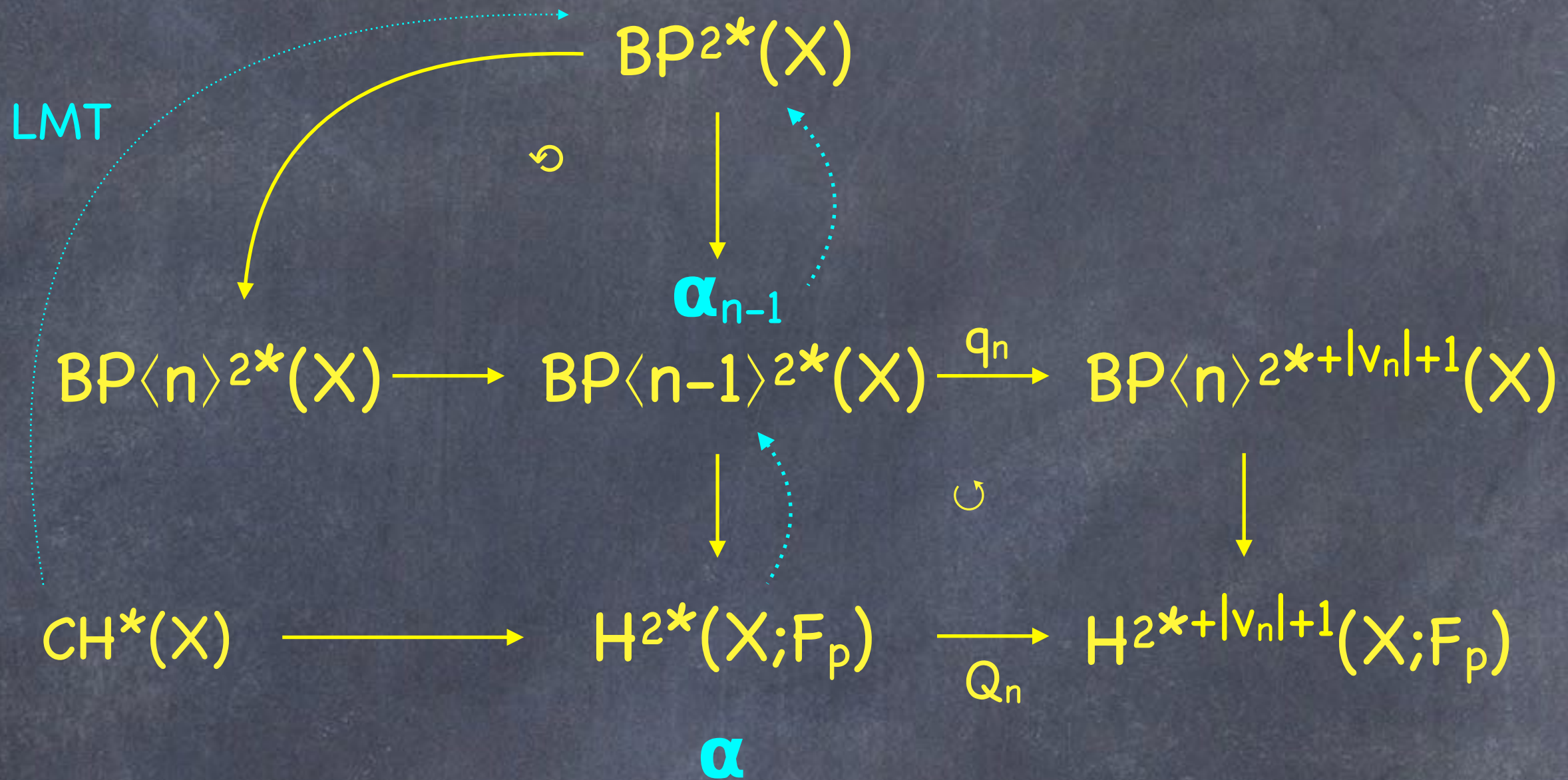
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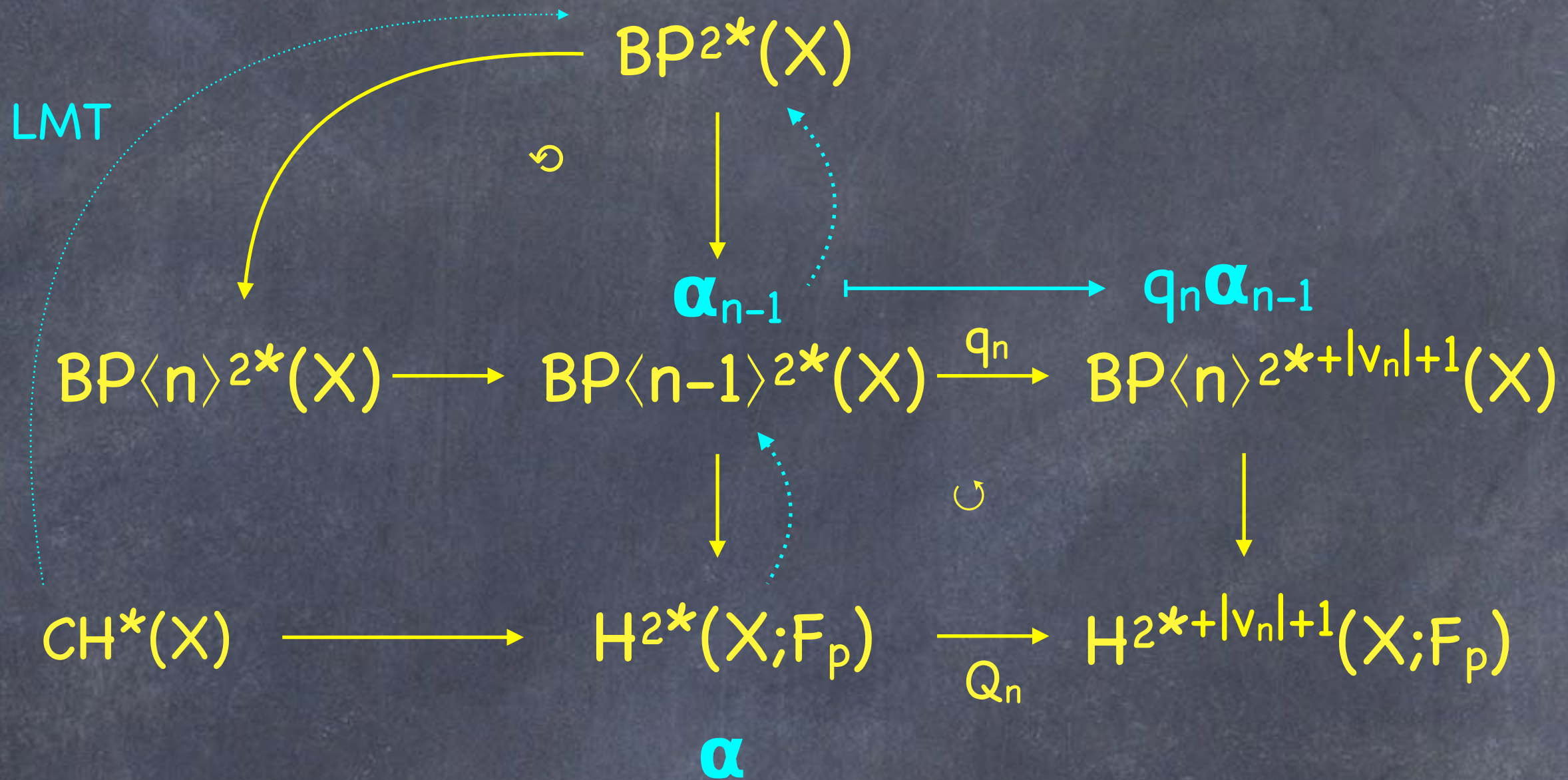
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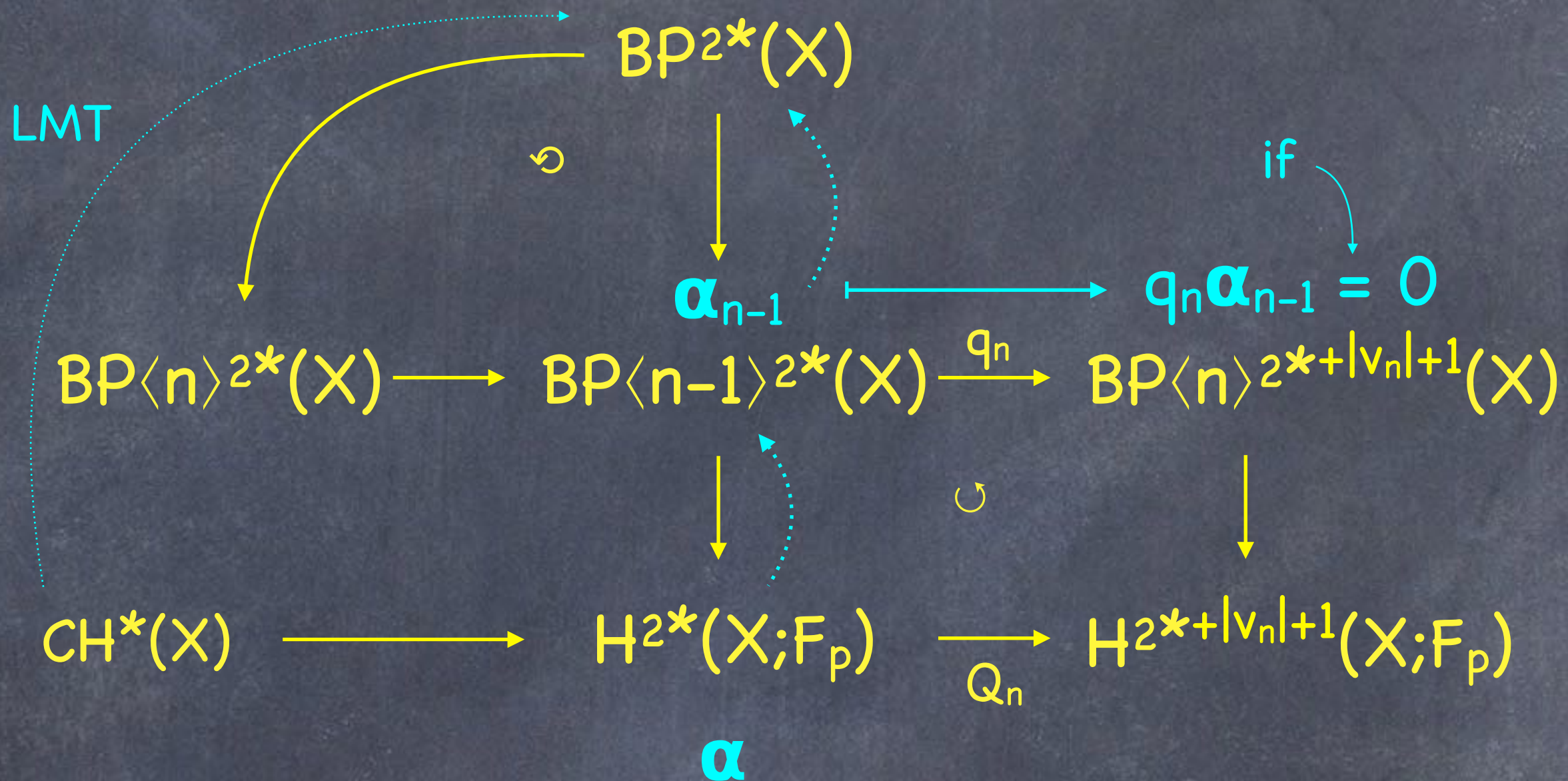
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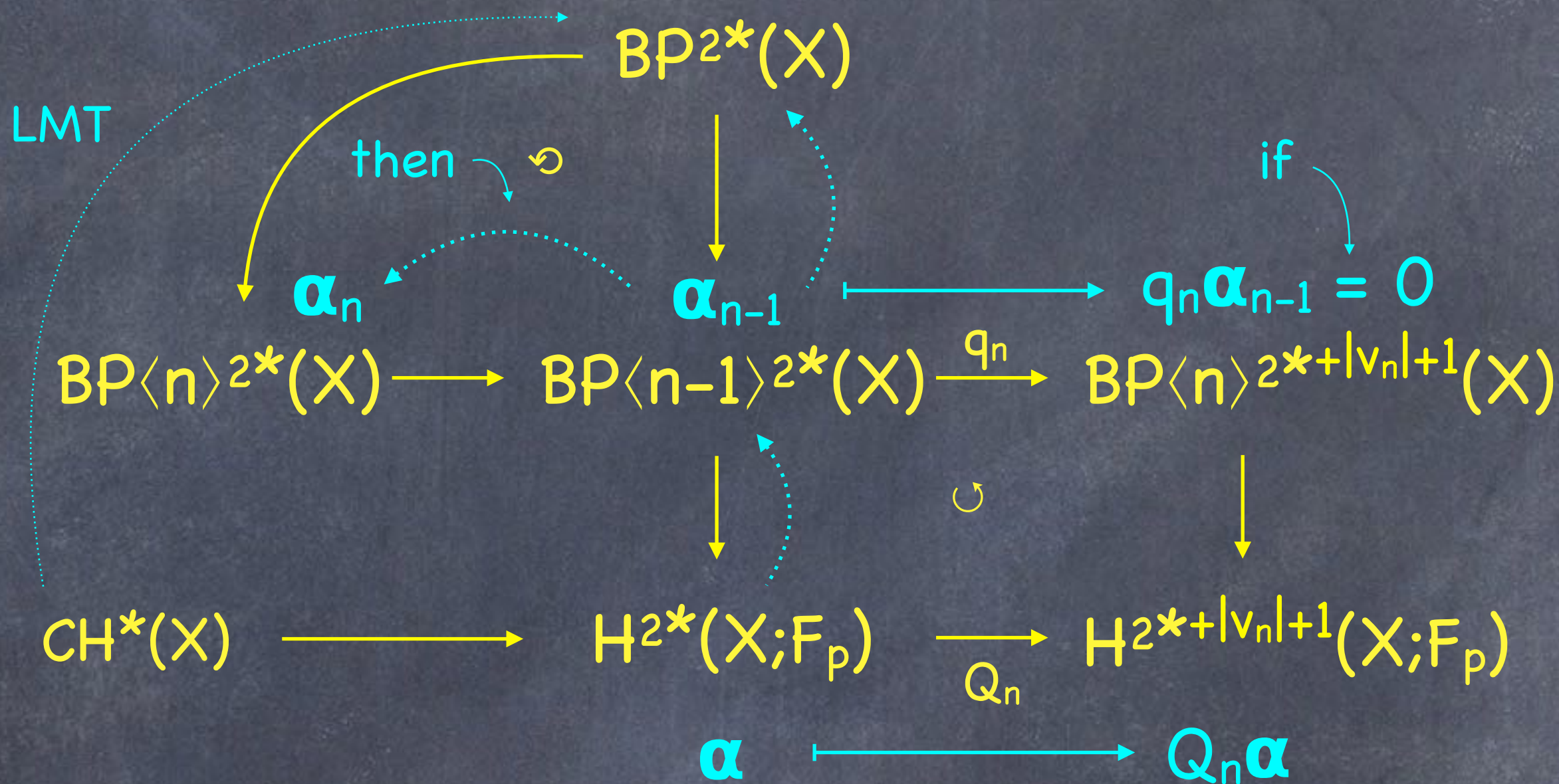
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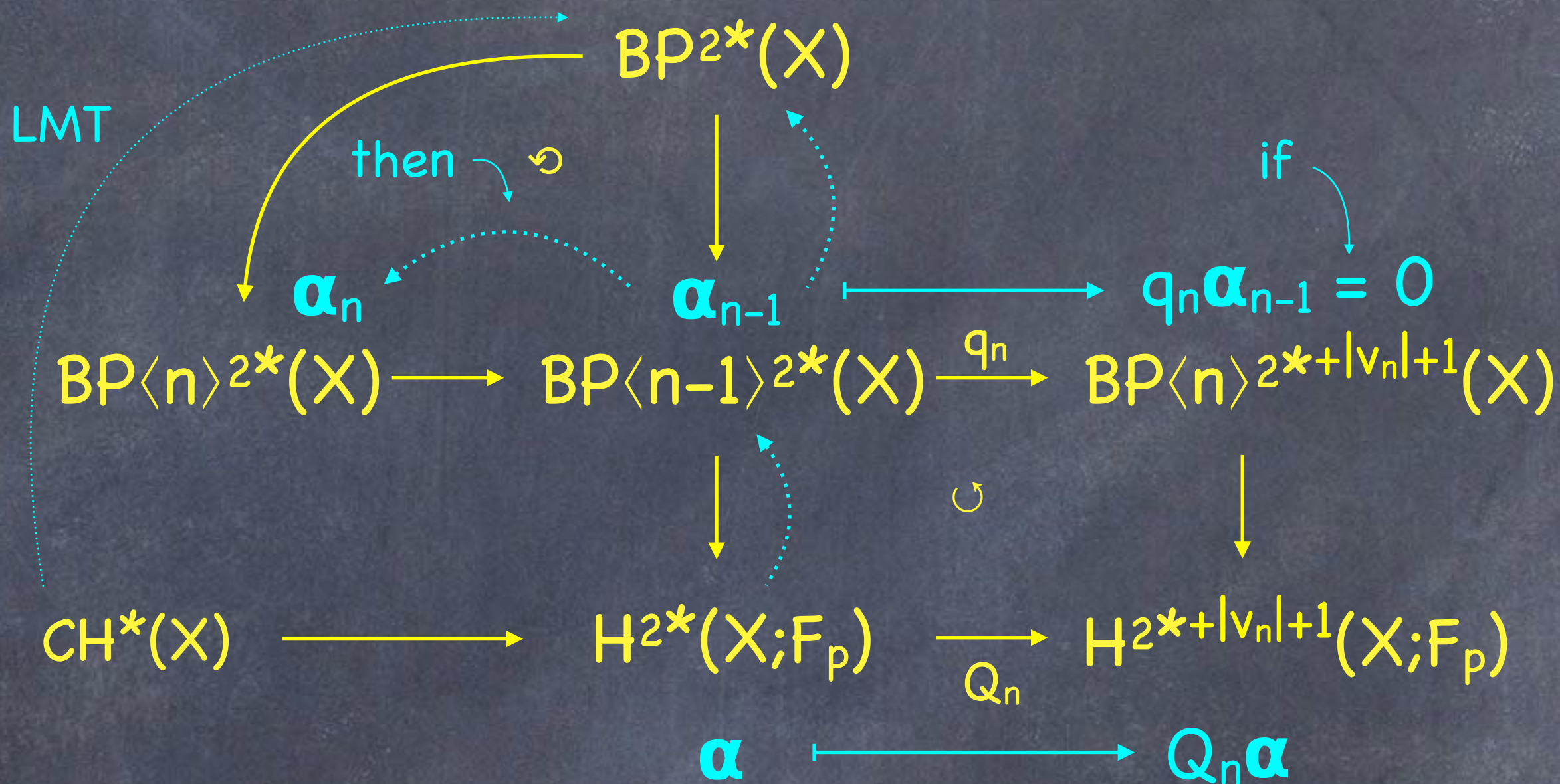
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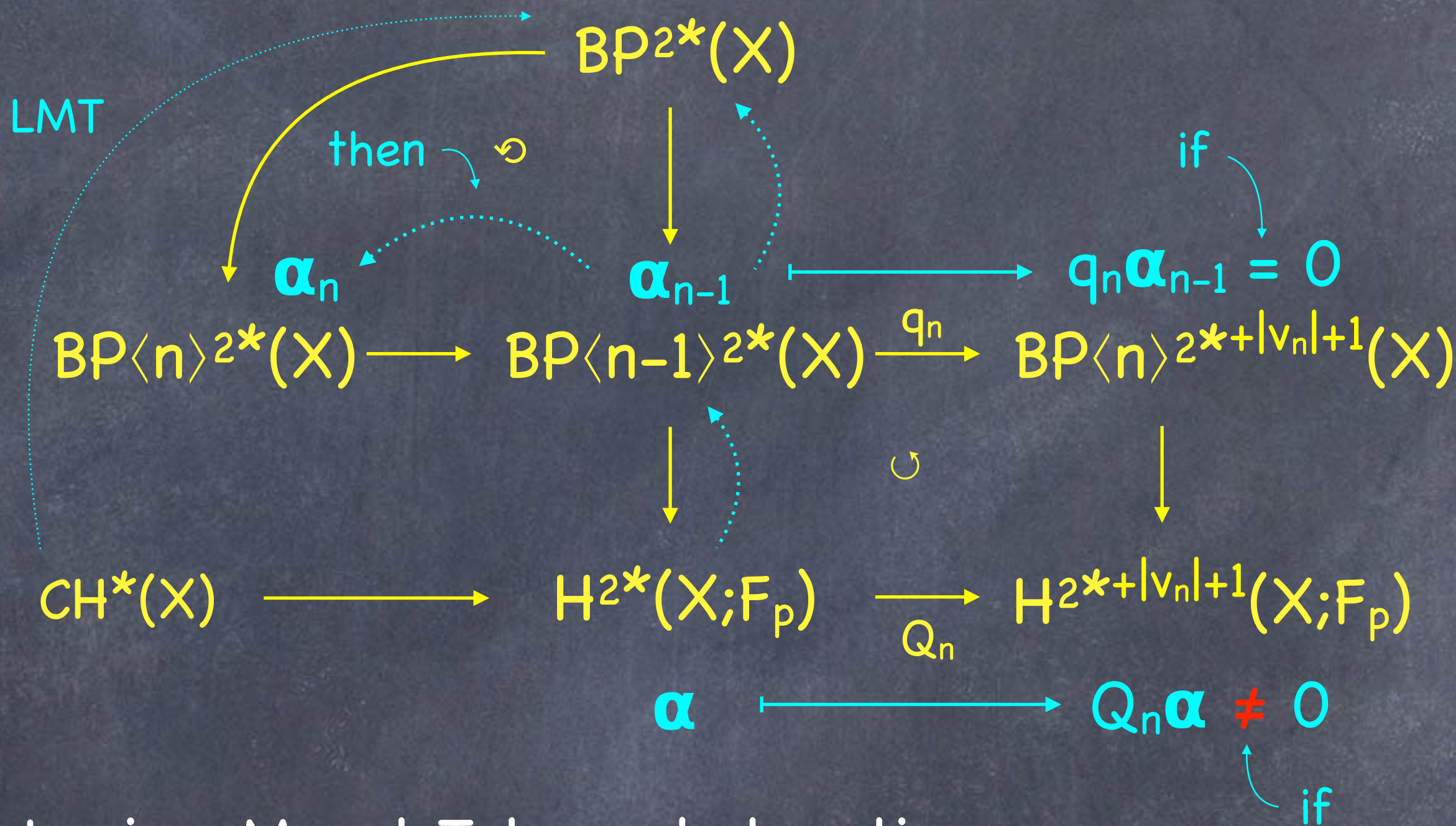


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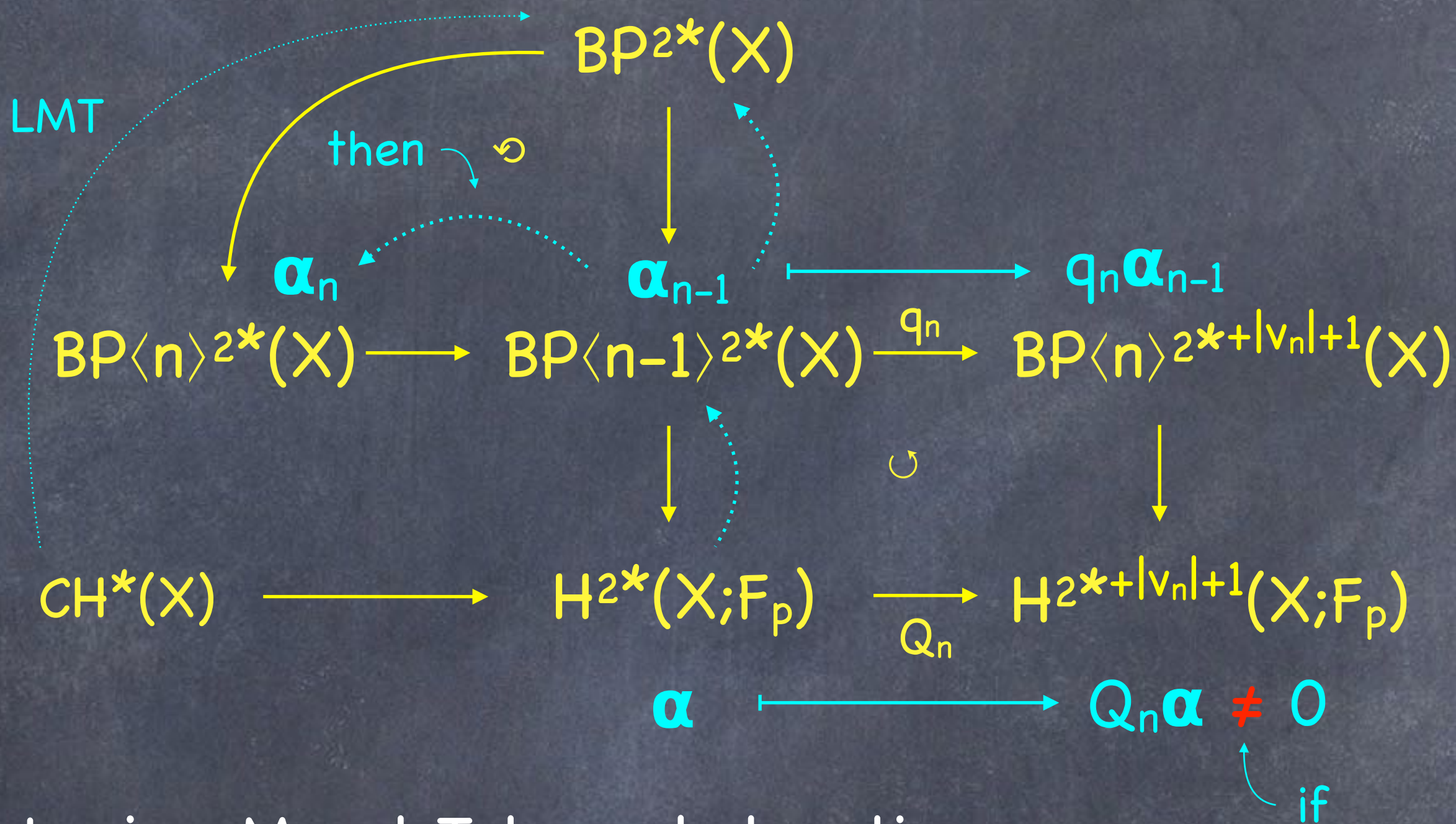
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Levine–Morel–Totaro obstruction:

If $Q_n \alpha \neq 0$,

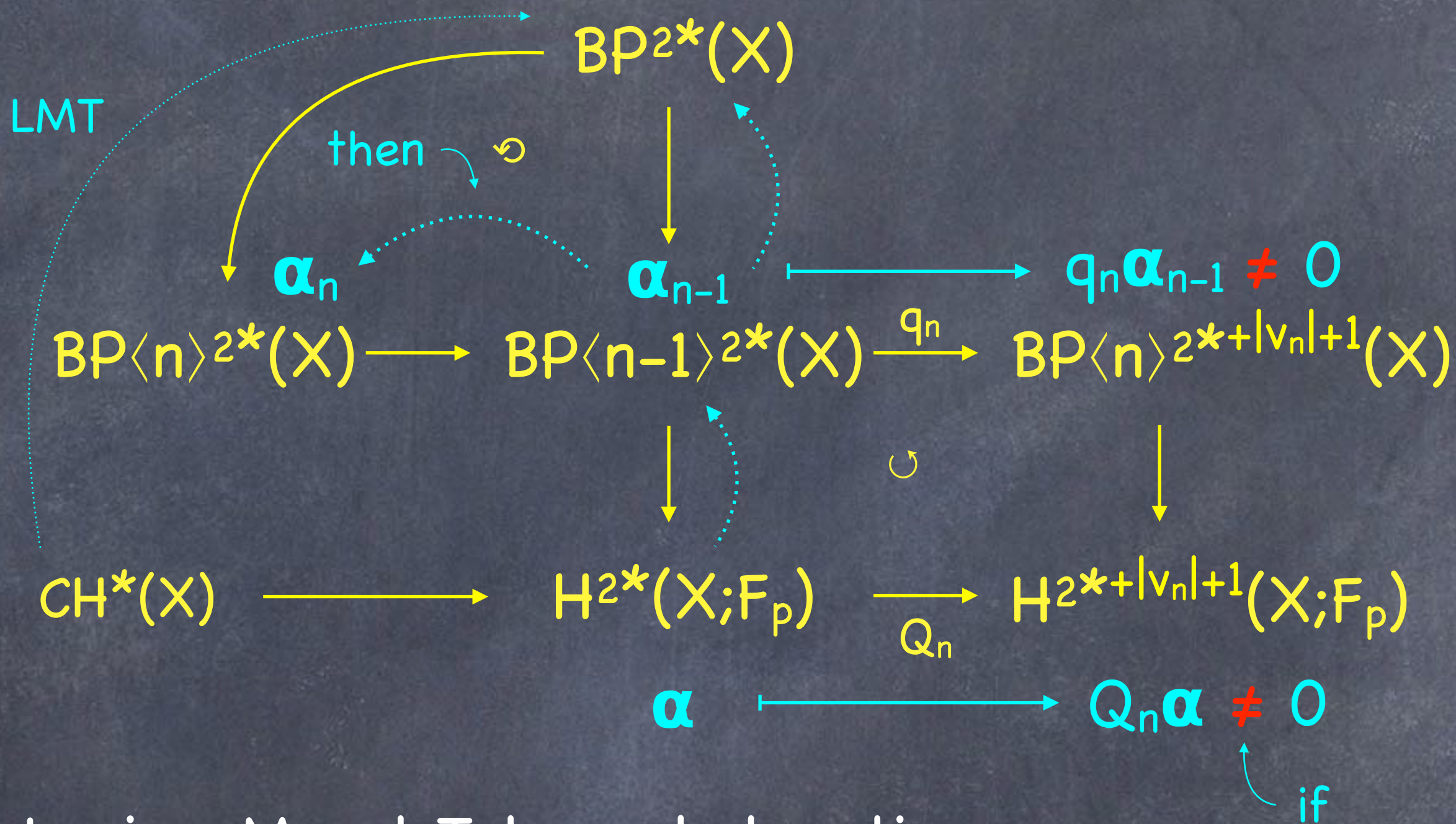
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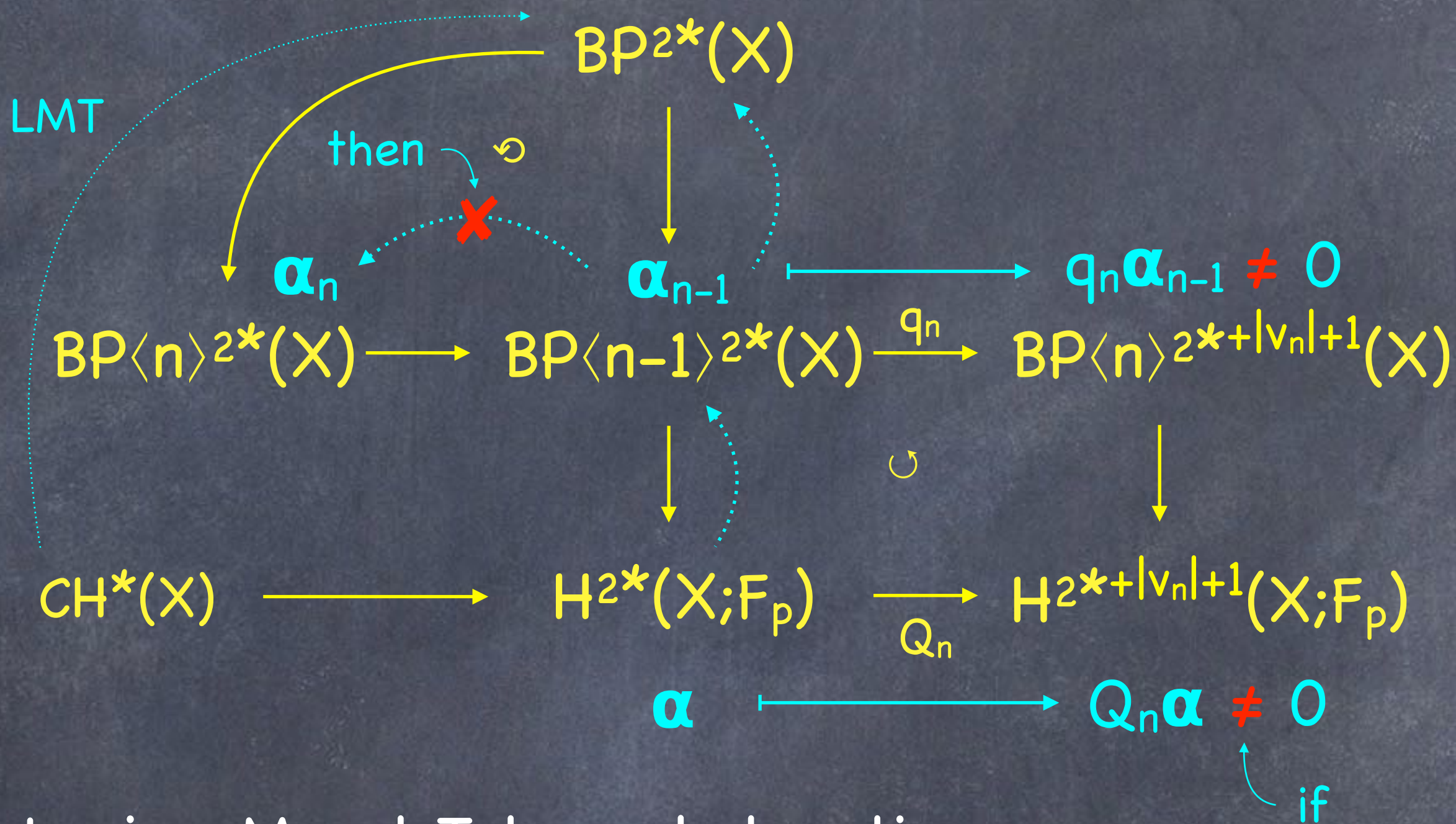
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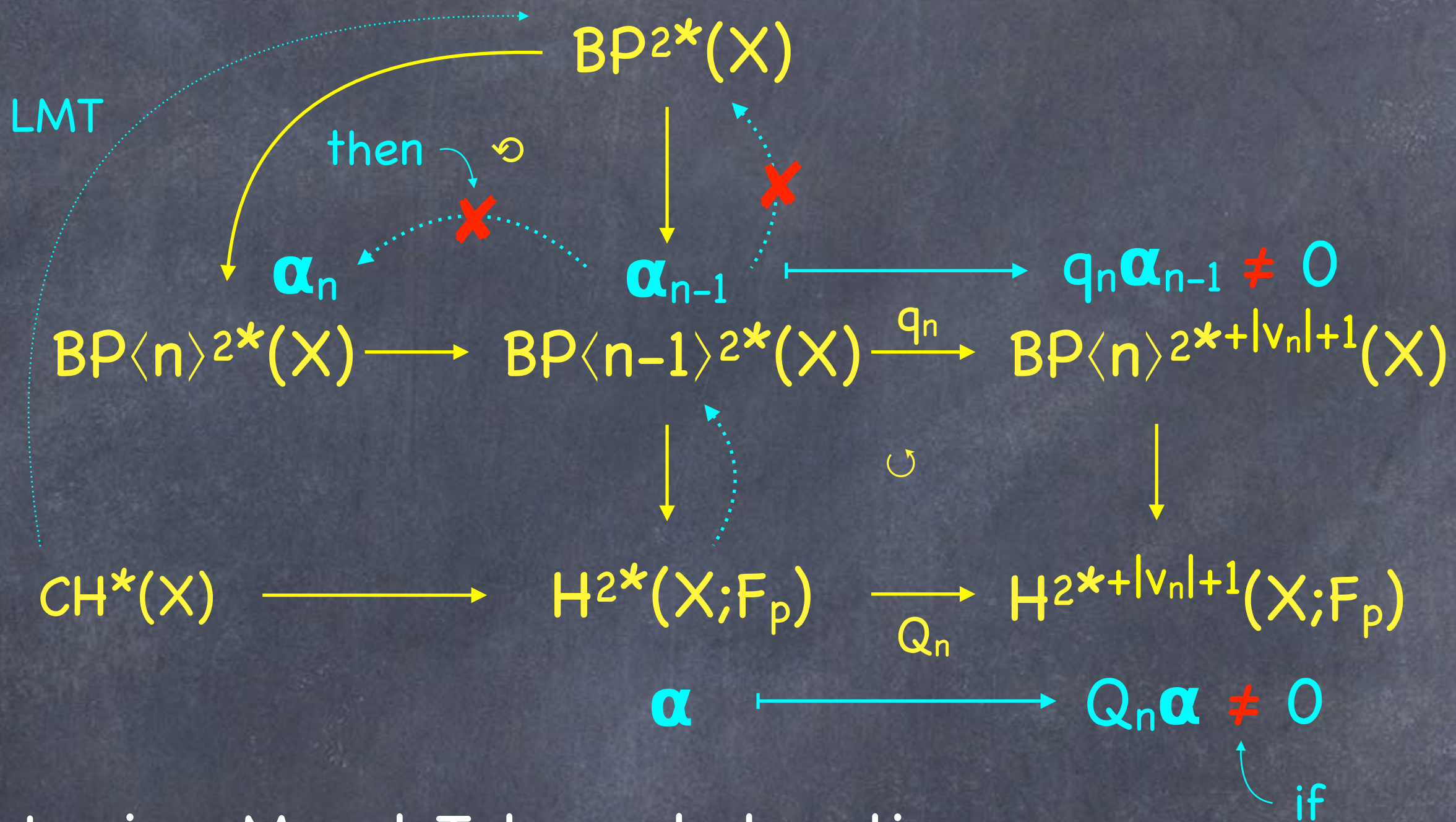
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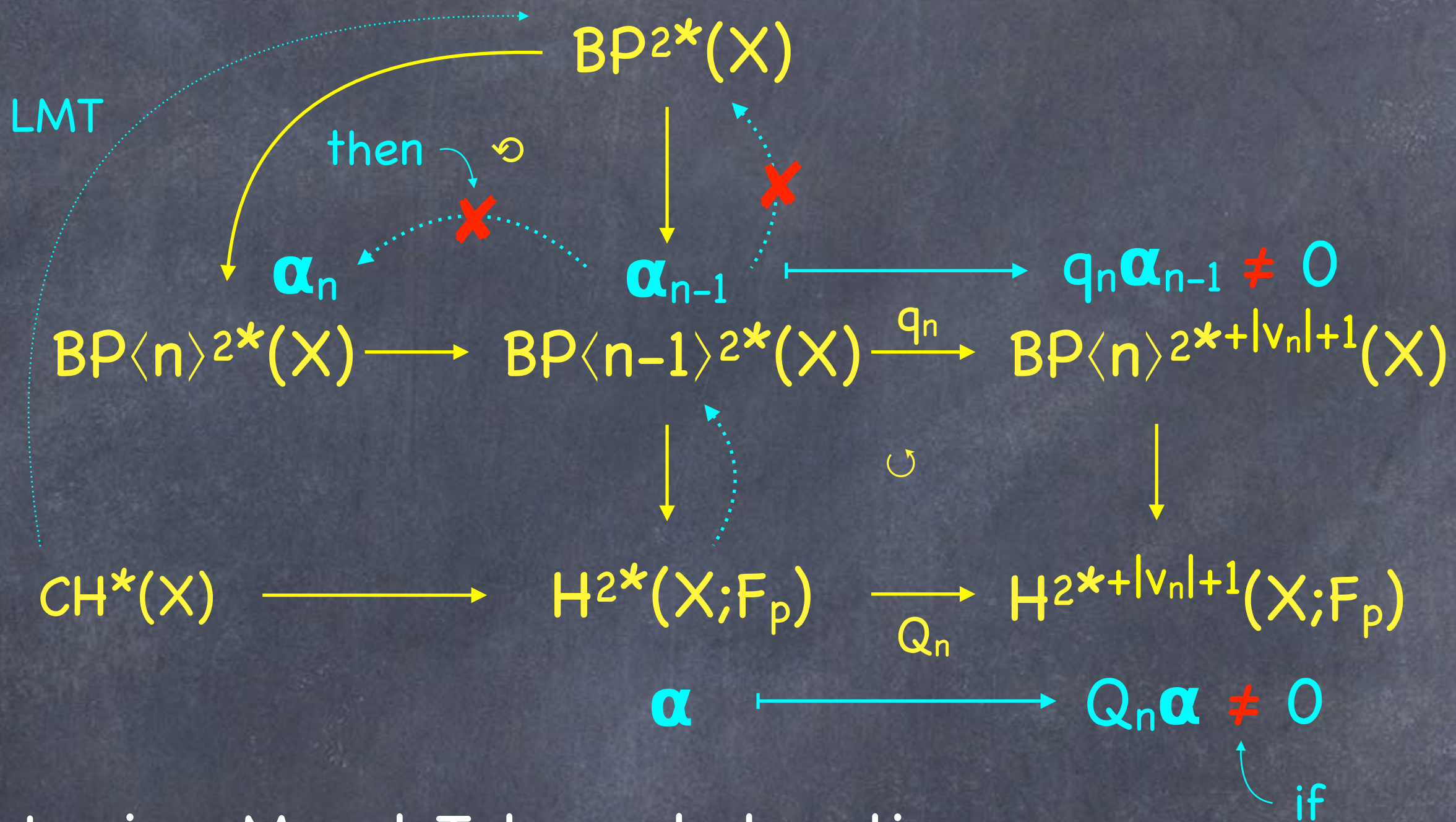
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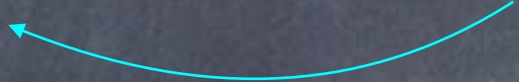
Voevodsky's motivic Milnor operations:

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$$Q_n^{\text{mot}} \in \mathcal{A}^{2p^n-1, p^n-1}$$

mod p -motivic
Steenrod algebra



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For a smooth complex variety X :

$$H_{\text{mot}}^{i,j}(X; \mathbb{F}_p) \xrightarrow{Q_n^{\text{mot}}} H_{\text{mot}}^{i+2p^n-1, j+p^n-1}(X; \mathbb{F}_p)$$

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Recall: $H_{\text{mot}}^{2i,i}(X; \mathbb{F}_p) = \text{CH}^i(X; \mathbb{Z}/p)$ and

$$H_{\text{mot}}^{i,j}(X; \mathbb{F}_p) = 0 \quad \text{if } i > 2j.$$

Obstructions revisited:

X smooth complex variety

$$\begin{array}{ccc} H_{\text{mot}}^{2i,i}(X; \mathbb{F}_p) & \xrightarrow{Q_n^{\text{mot}}} & H_{\text{mot}}^{2i+2p^{n-1}, i+p^{n-1}}(X; \mathbb{F}_p) \\ \downarrow & \curvearrowright & \downarrow \\ H^{2i}(X; \mathbb{F}_p) & \xrightarrow{Q_n} & H^{2i+2p^{n-1}}(X; \mathbb{F}_p) \end{array}$$

topological realization

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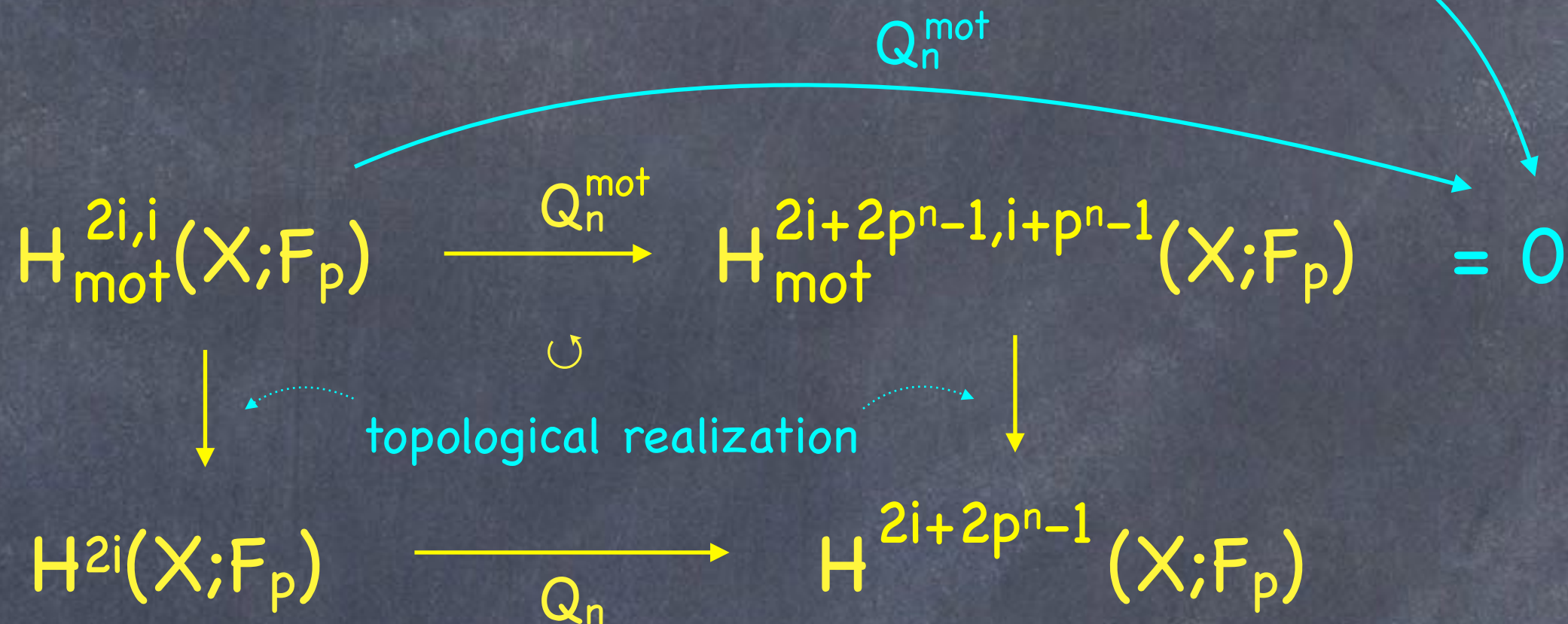
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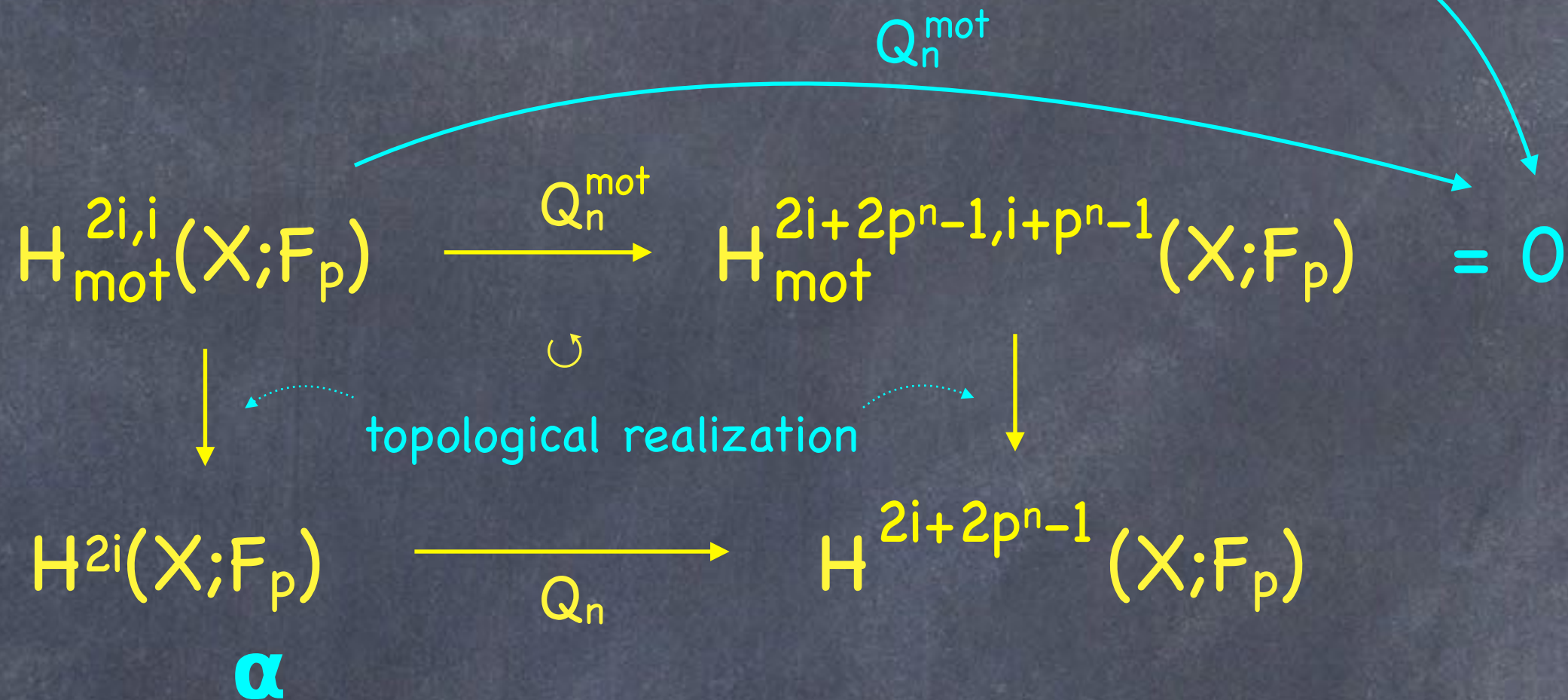
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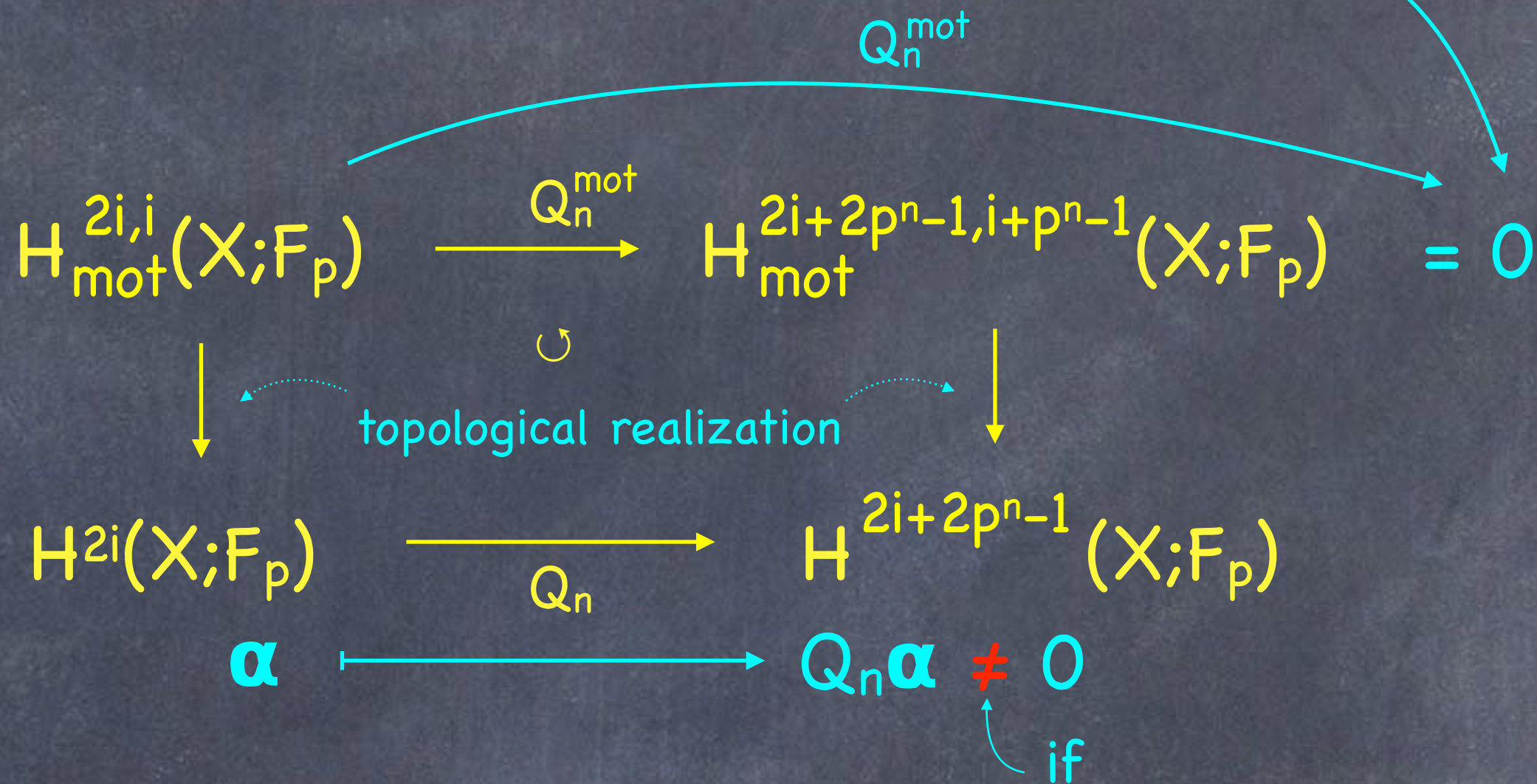
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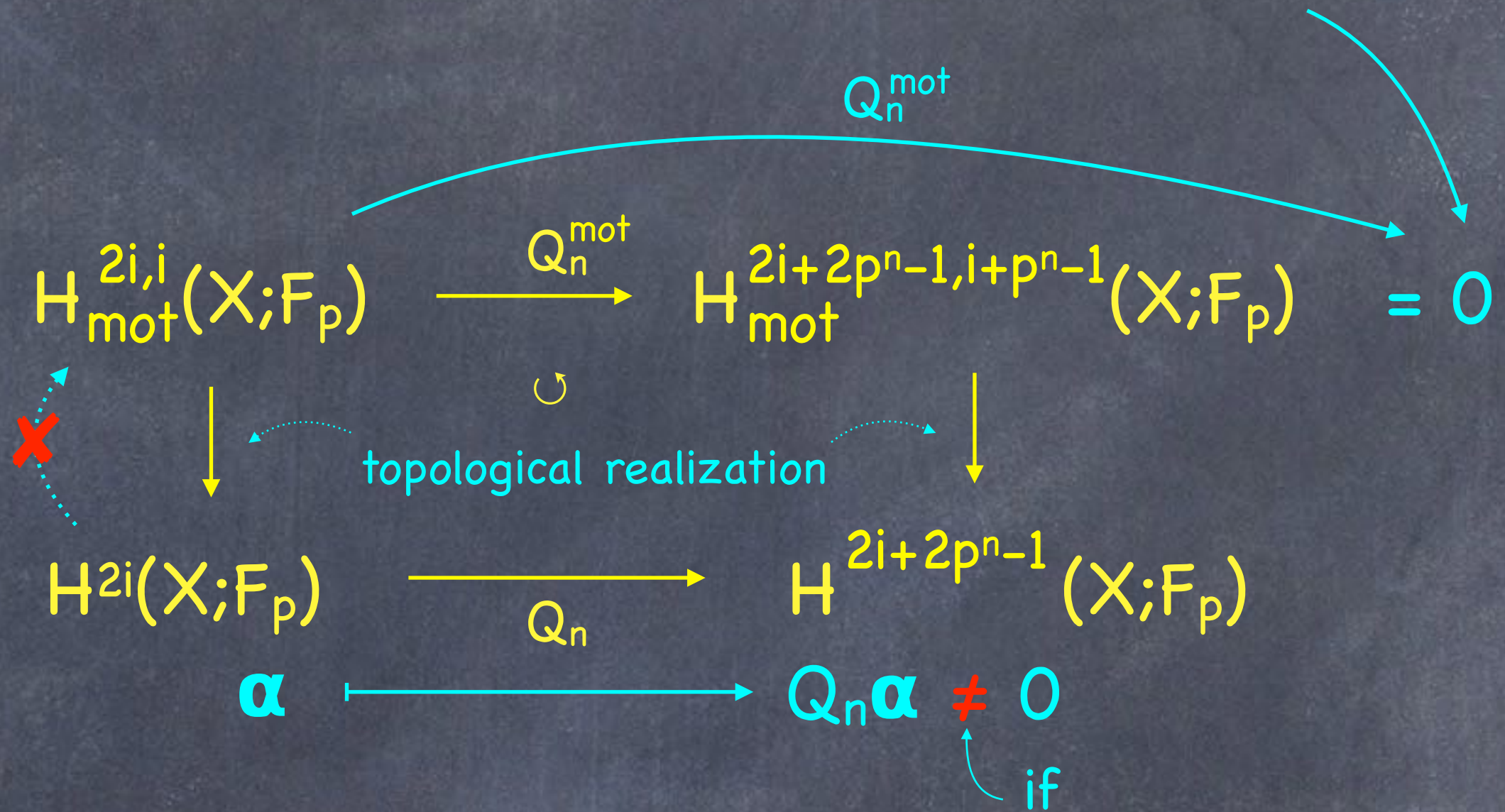
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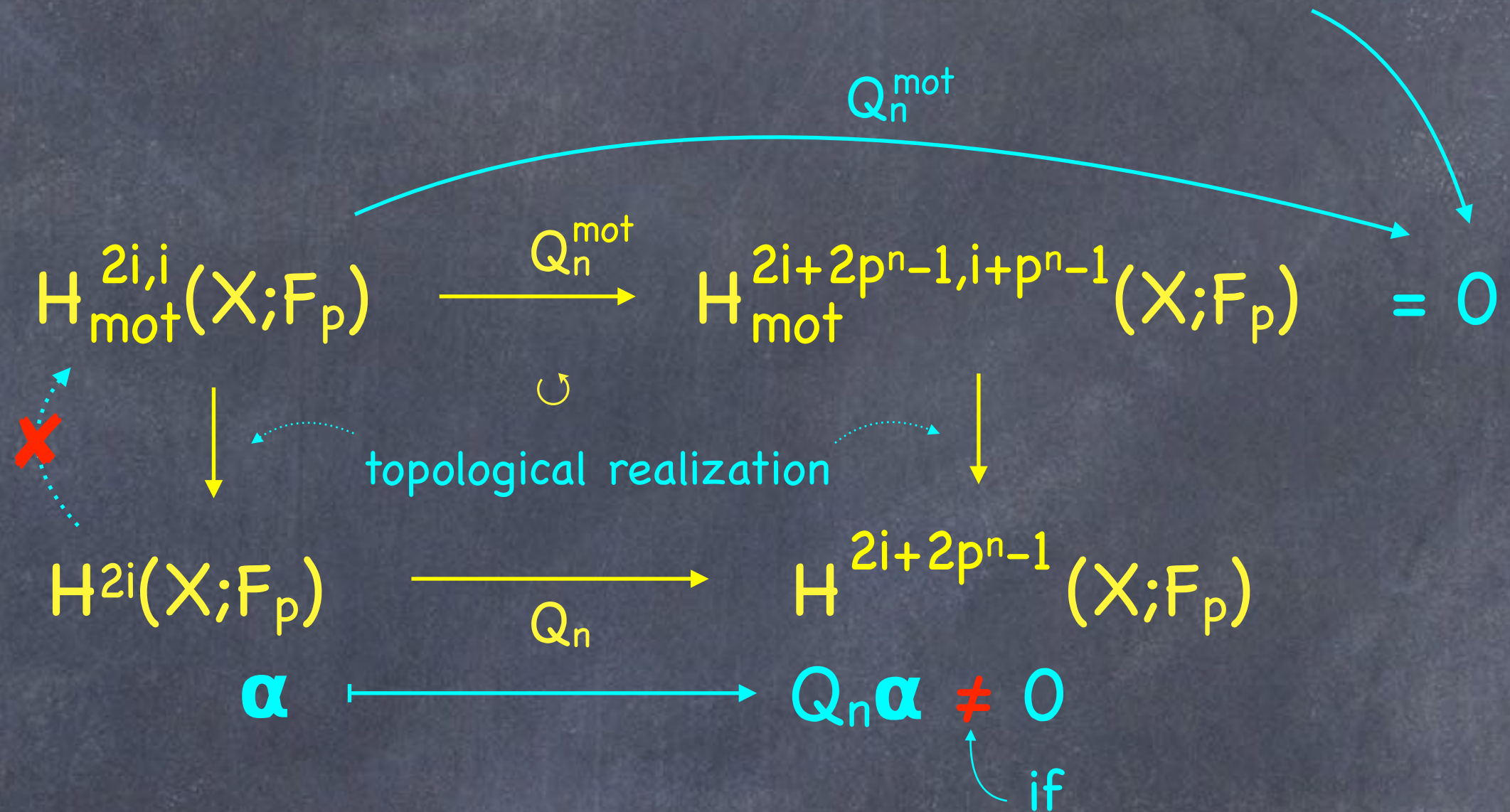
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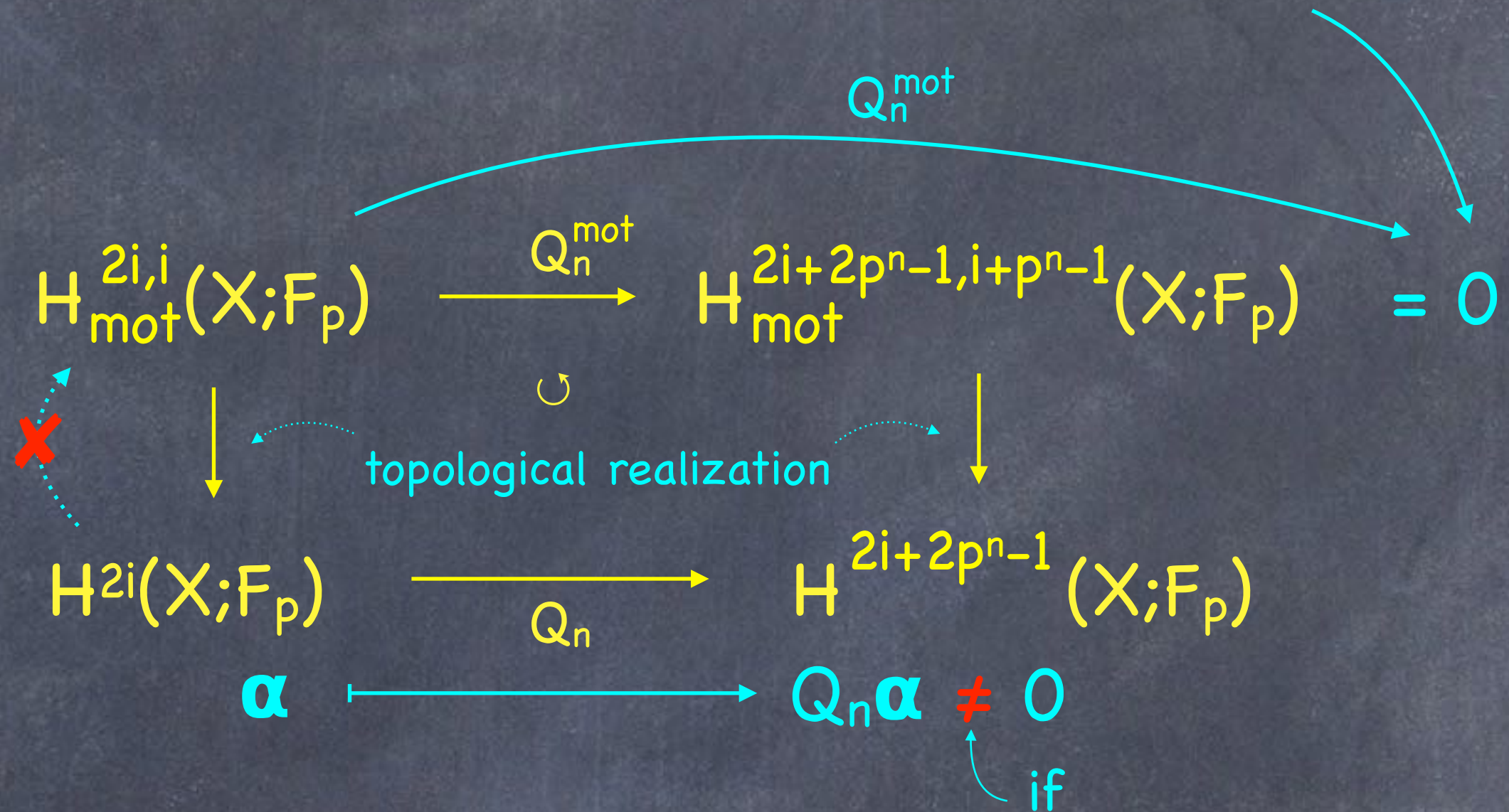
Obstructions revisited:

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Observation: The LMT-obstruction is particular to smooth varieties and bidegrees $(2i,i)$.

Obstructions revisited: \times smooth complex variety



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Example: $Q_n \mathbf{1} \neq 0$ for $\mathbf{1}$ the fundamental class of a suitable Eilenberg-MacLane space, though $\mathbf{1}$ is algebraic.

Back to our task:

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Study $\text{Alg}_E^{2*}(X)$ and its complement in $E_{\text{top}}^{2*}(X)$.

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Recall: BP and $\text{BP}\langle n \rangle$ exist in the **motivic** world (e.g. Vezzosi, Hopkins, Hu-Kriz, Ormsby, Hoyois, Ormsby-Østvær).

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Recall: BP and $\text{BP}\langle n \rangle$ exist in the **motivic** world (e.g. Vezzosi, Hopkins, Hu-Kriz, Ormsby, Hoyois, Ormsby-Østvær).

Question: How can we produce non-algebraic elements in $\text{BP}\langle n \rangle_{\text{top}}^{2*}(X)$?

will drop the "top" again

Back to the cofibre sequence:

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$$\vdots$$

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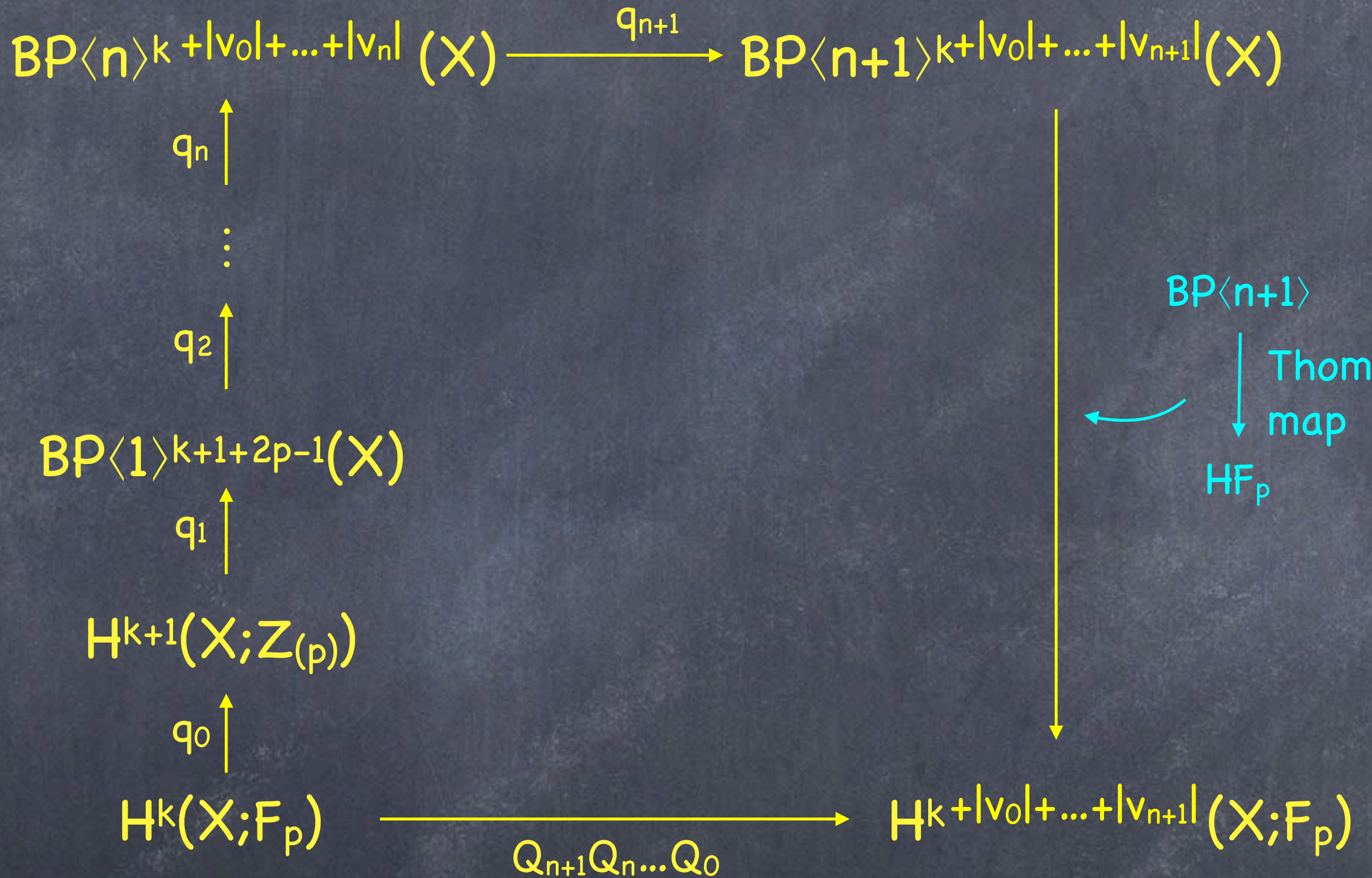
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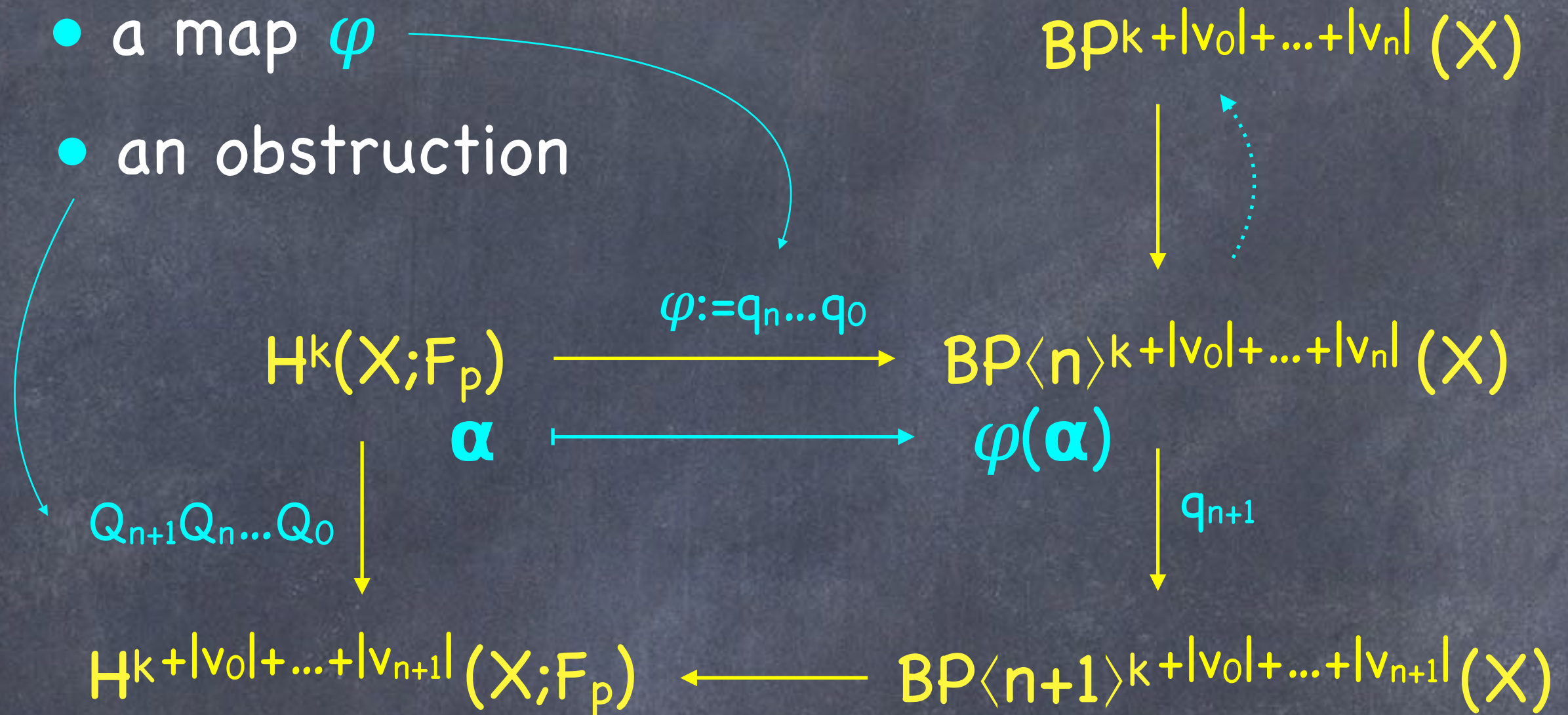
- a map φ

$$\begin{array}{ccc} H^k(X; \mathbb{F}_p) & \xrightarrow{\varphi := q_n \dots q_0} & \mathrm{BP}\langle n \rangle_{k+|v_0|+\dots+|v_n|}(X) \\ \alpha & \xrightarrow{\quad} & \varphi(\alpha) \end{array}$$

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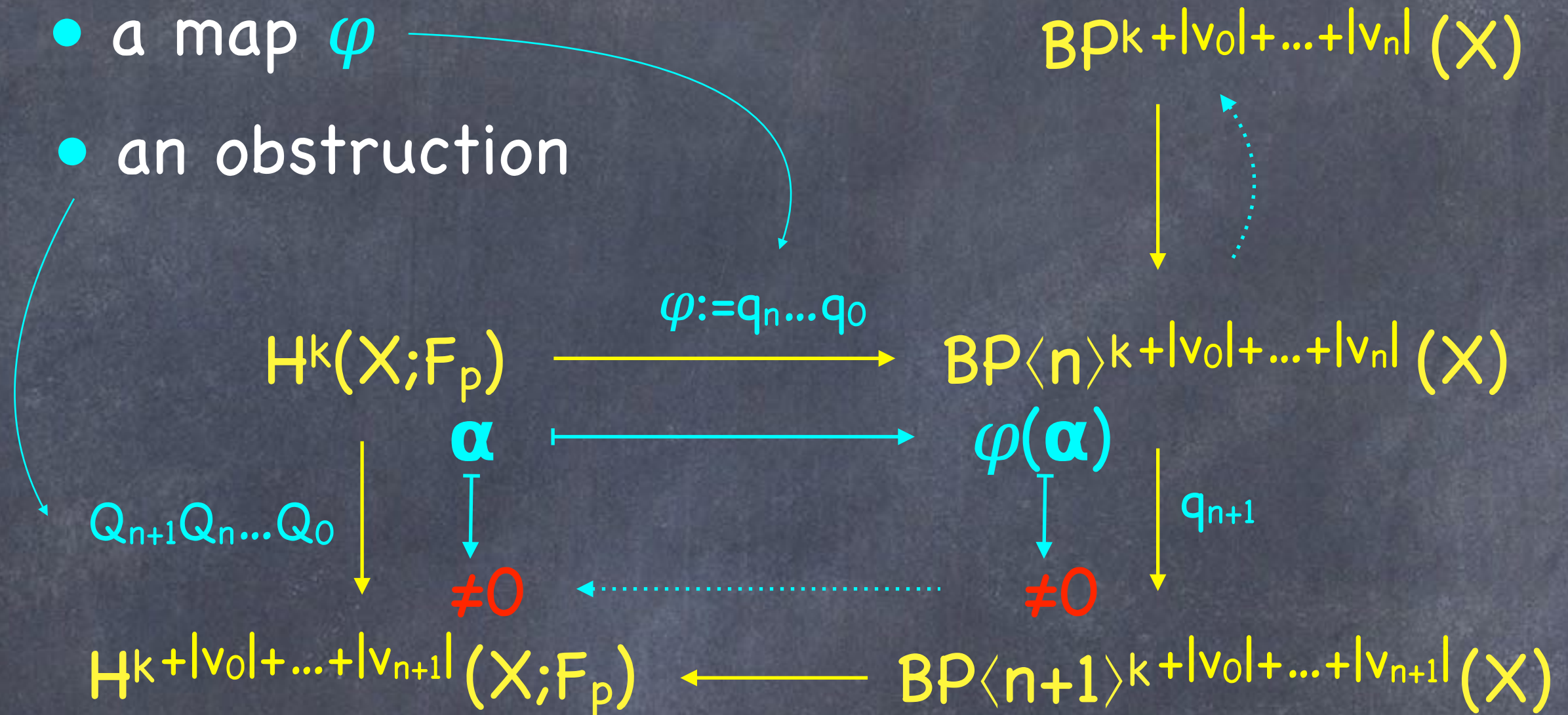
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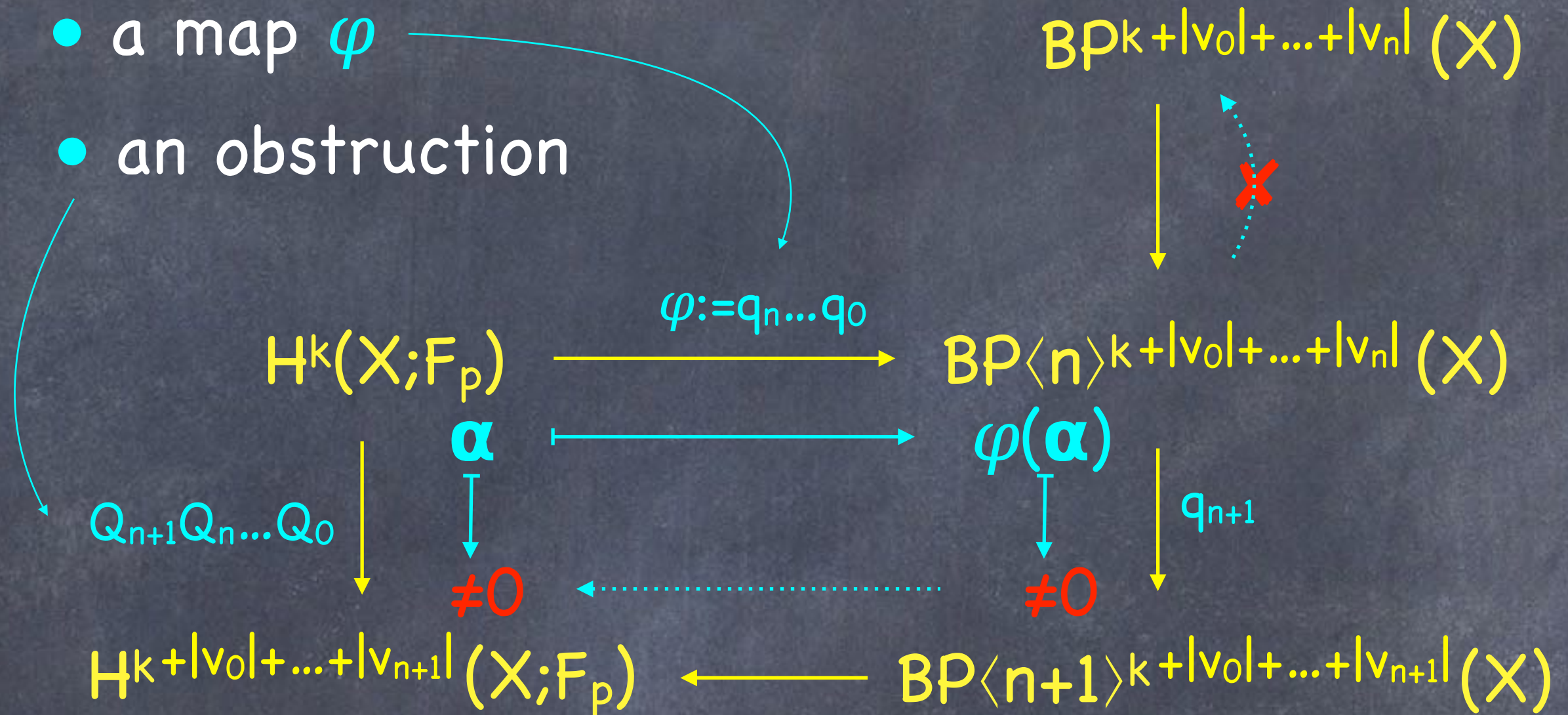
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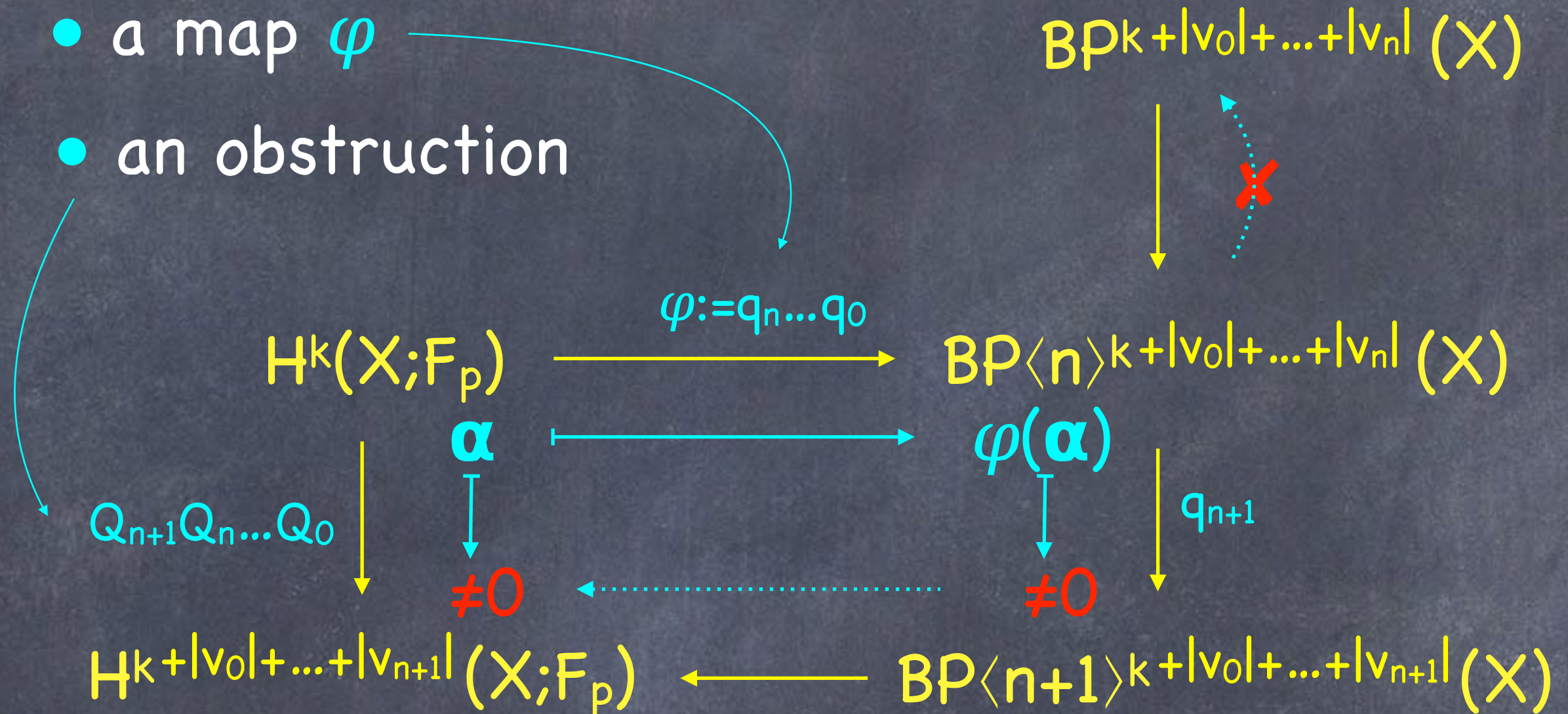
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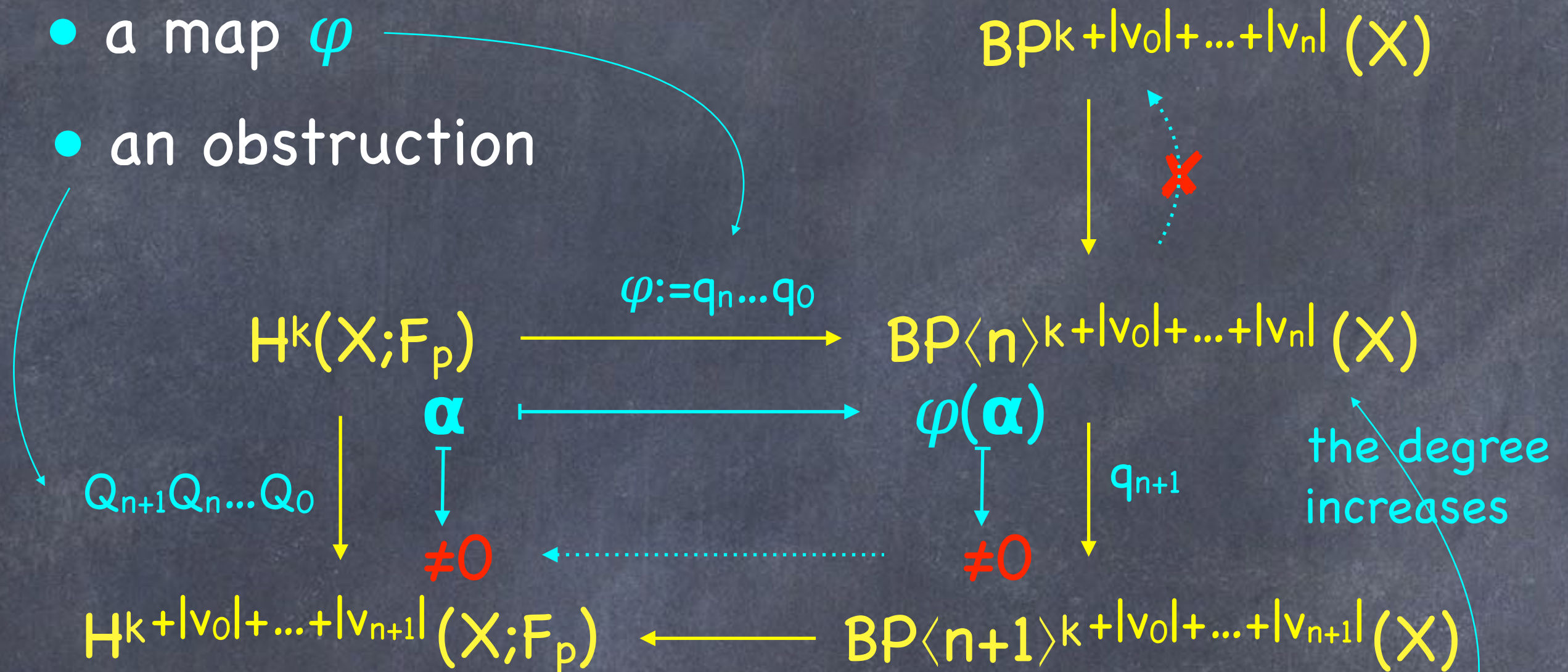


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But we also pay a price...

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Finally, set $X =$ **Godeaux-Serre** variety associated to the group G_{n+3} and pullback x via

$$X \longrightarrow BG_{n+3} \times CP^\infty.$$

a $2(p^{n+1} + \dots + 1) + 1$ -connected map

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