Spherical geometry and Euler's formel

Abel competition
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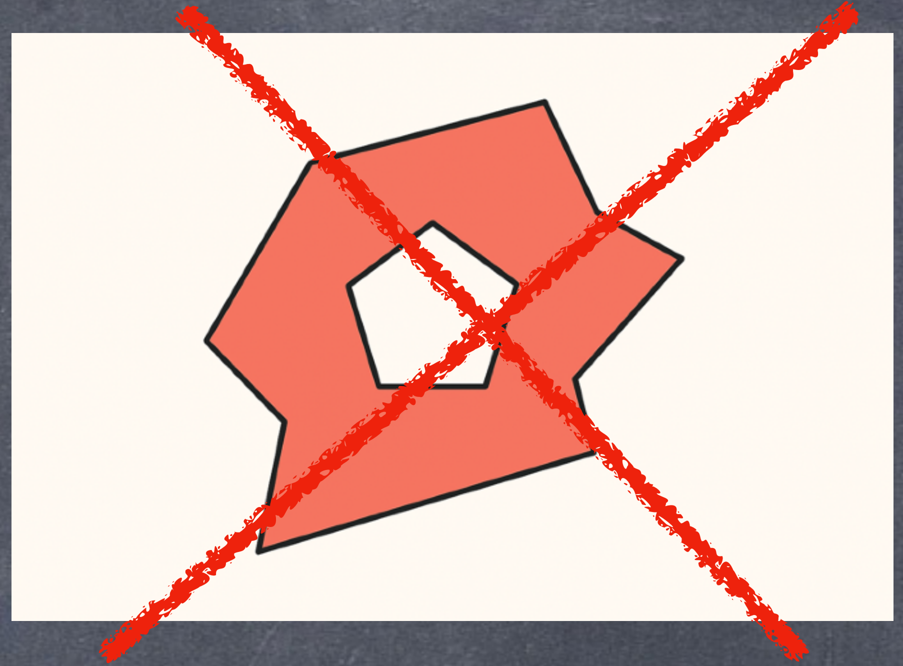
Leonhard Euler (1707–1783)

Euler's formula for polyhedra:

A **polyhedron** is a geometric object with **vertices**, straight **edges** and flat **surfaces**.

- polygons (triangles, rectangles, ...)
- no holes
- each edge meets exactly two faces
- 3-dimensional



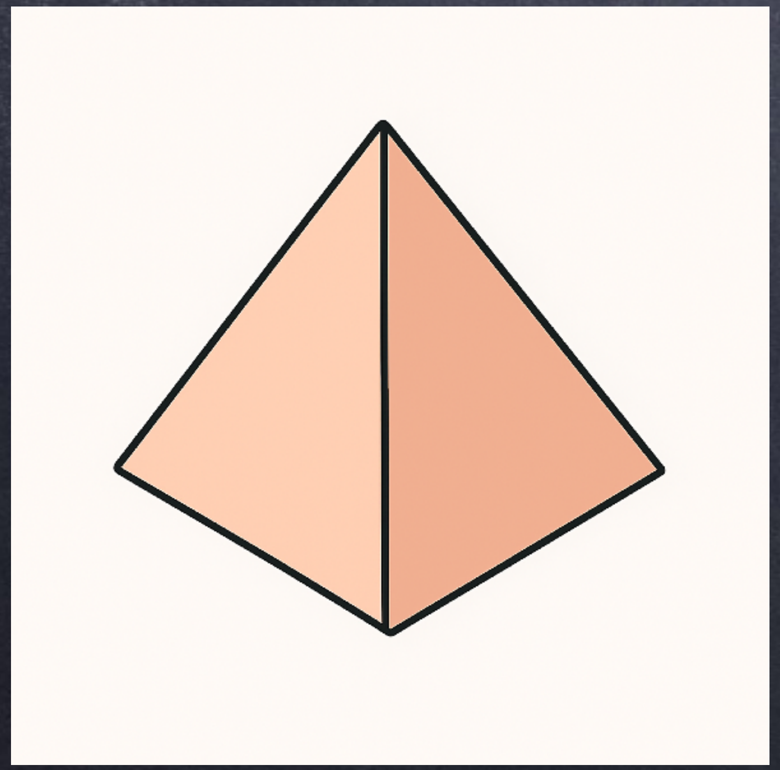
Definition:

Take n points $v_1, \ldots, v_n$ in $\mathbb{R}^3$ which are not in the same plane and no three points lie on the same line.

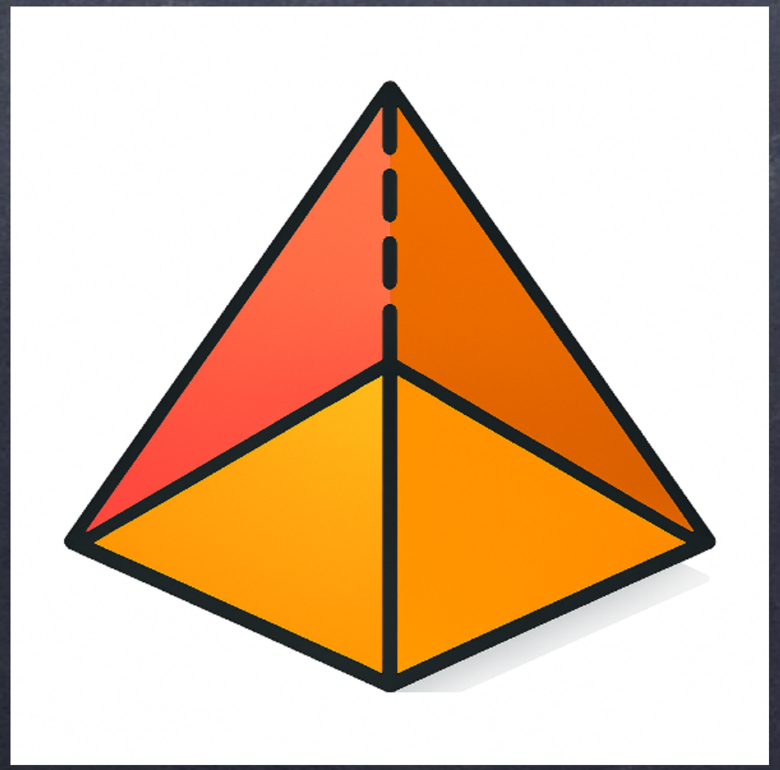
A (convex) polyhedron with vertices $v_1, \ldots, v_n$ is the subset i $\mathbb{R}^3$ given by

$$\{(x_1 v_1 + \cdots + x_n v_n) \in \mathbb{R}^3 : \text{for real numbers } x_i \geq 0 \text{ with } x_1 + \cdots + x_n = 1\}.$$

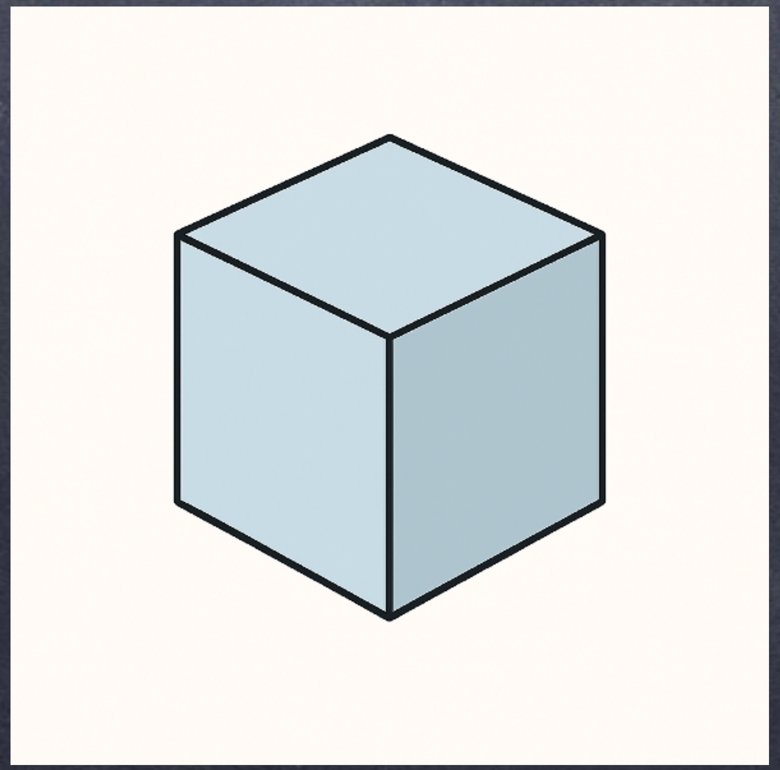
Examples:



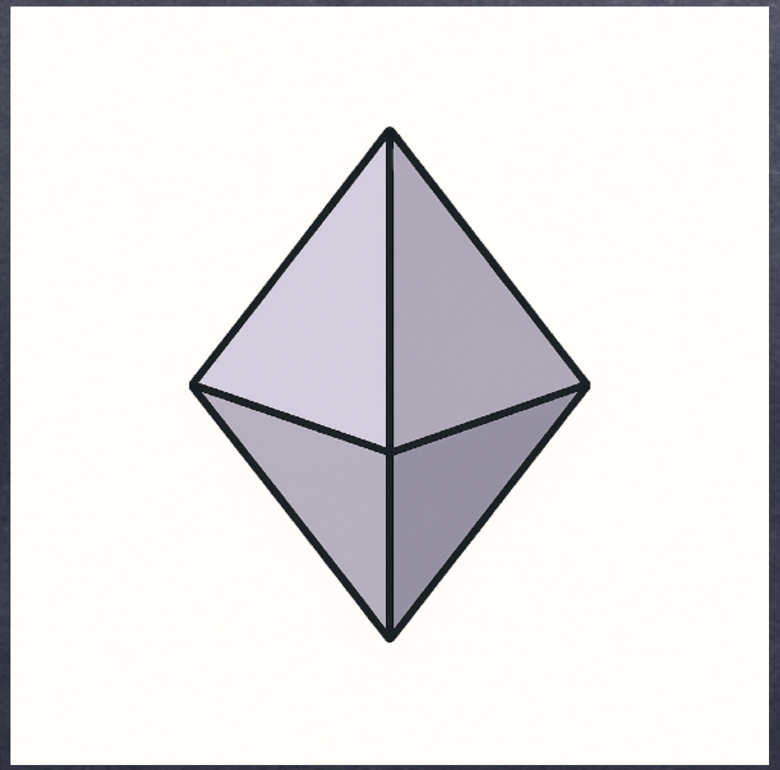
tetrahedron



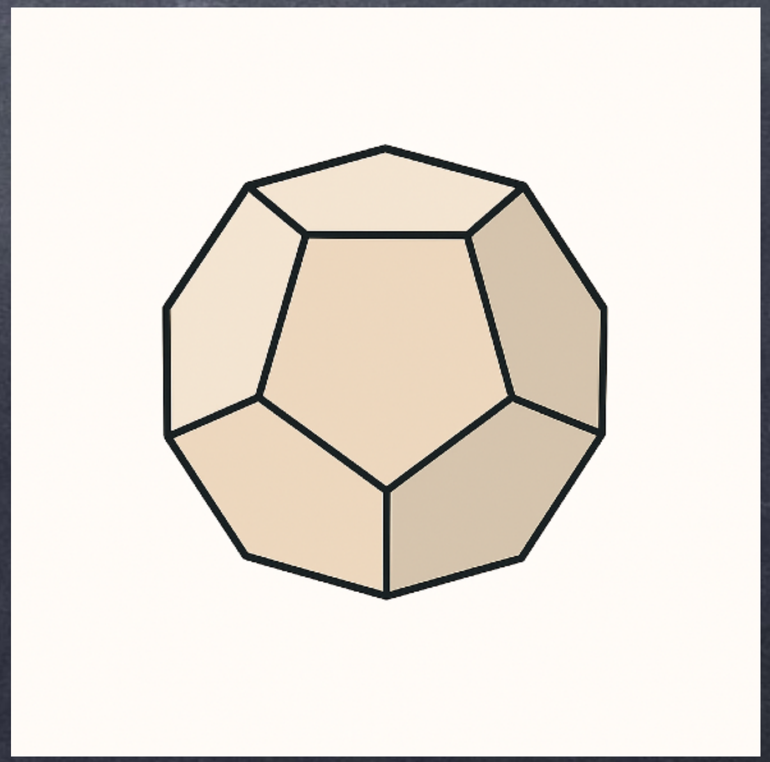
pyramid



hexahedron



octahedron



dodecahedron

...

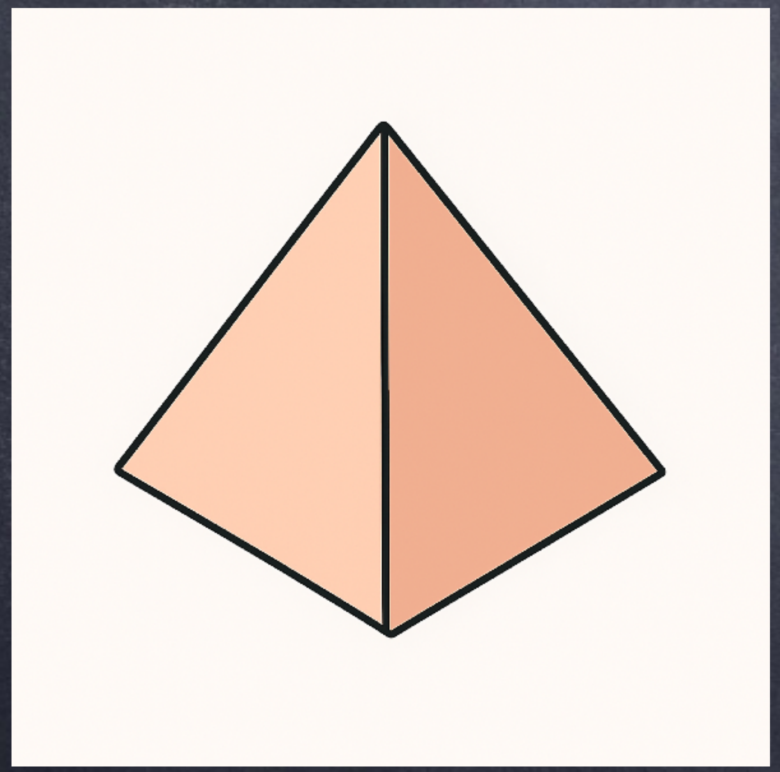
Euler's formula for polyhedra:

A polyhedron is a geometric object with vertices, straight edges and flat surfaces.

set $V = \#$ vertices, $E = \#$ edges, $F = \#$ faces.

Euler's formula: $V - E + F = 2$

Examples:

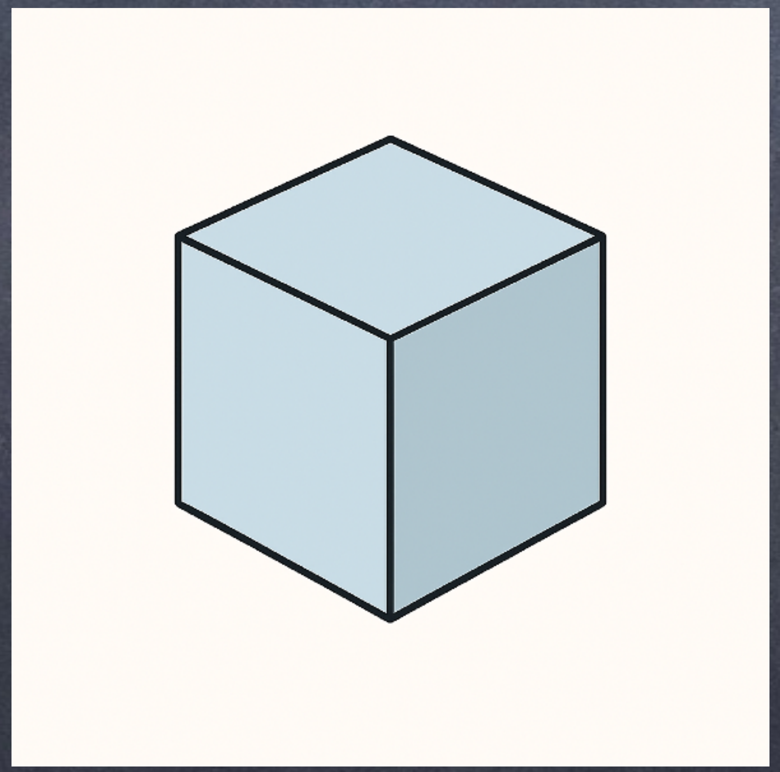


tetrahedron

$V = 4,\ E = 6,$
$F = 4$

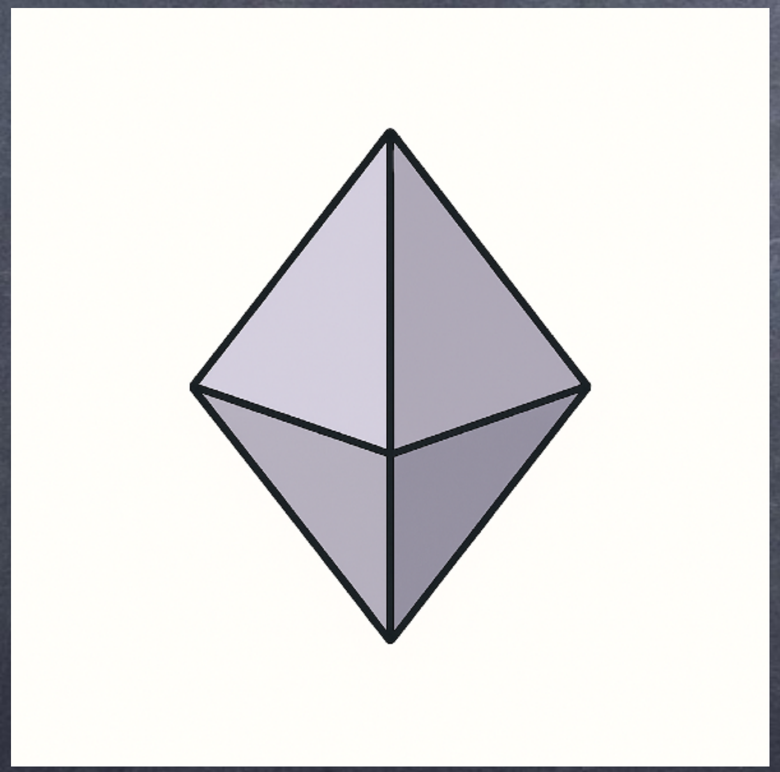
pyramid

$V = 5,\ E = 8,$
$F = 5$



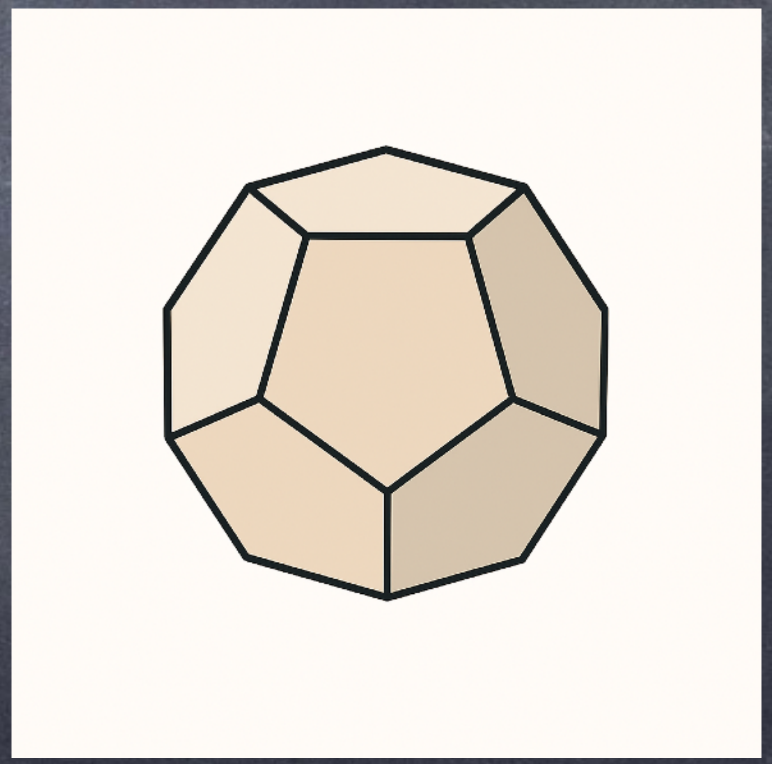
hexahedron

$V = 8,\ E = 12,$
$F = 6$



octahedron

$V = 6,\ E = 12,$
$F = 8$



dodecahedron

$V = 20,\ E = 30,$
$F = 12$

...

Legendre's proof of Euler's formula
via spherical geometry:

Adrien-Marie Legendre (1752-1833)

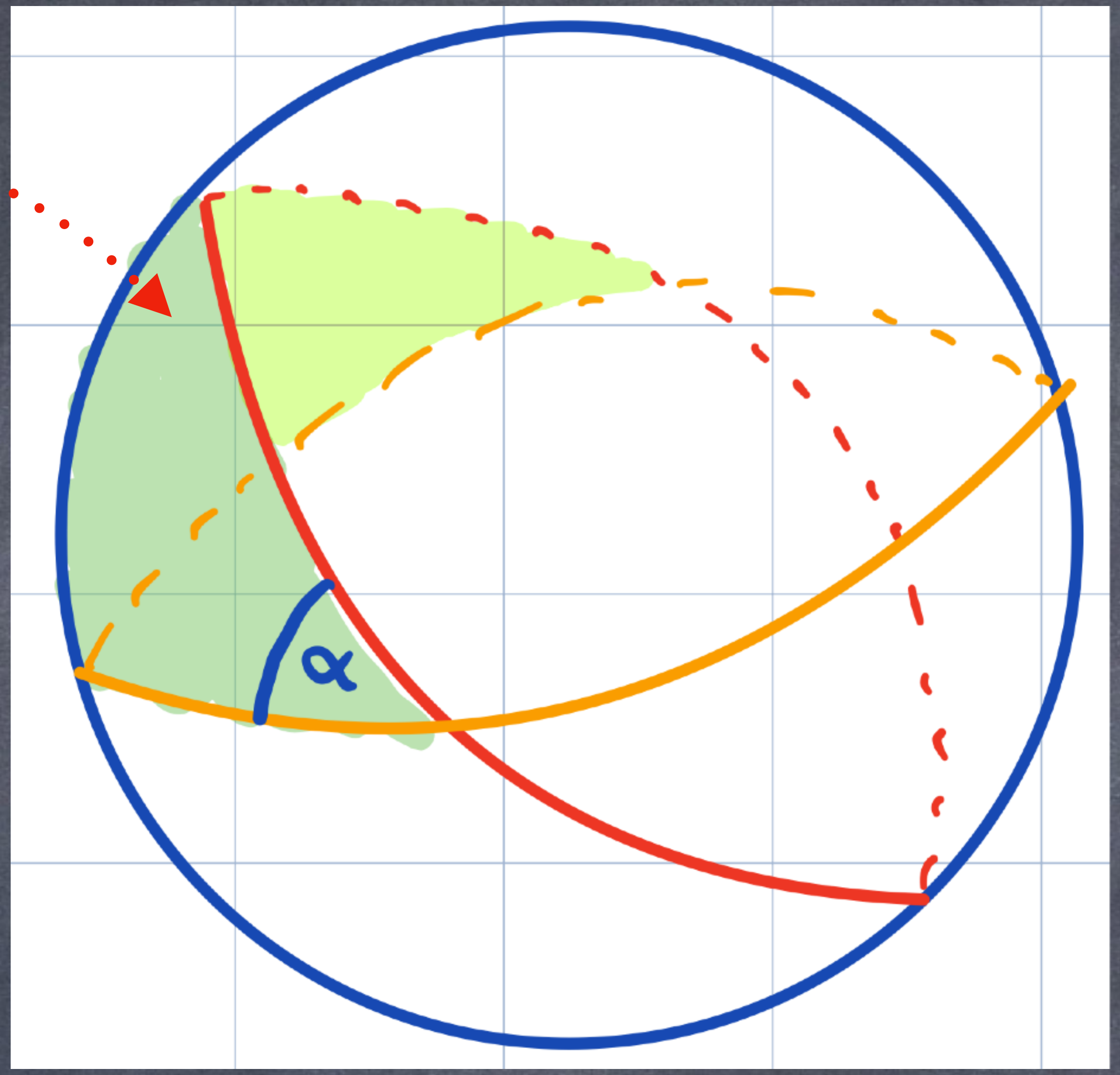
A glimpse at spherical geometry:

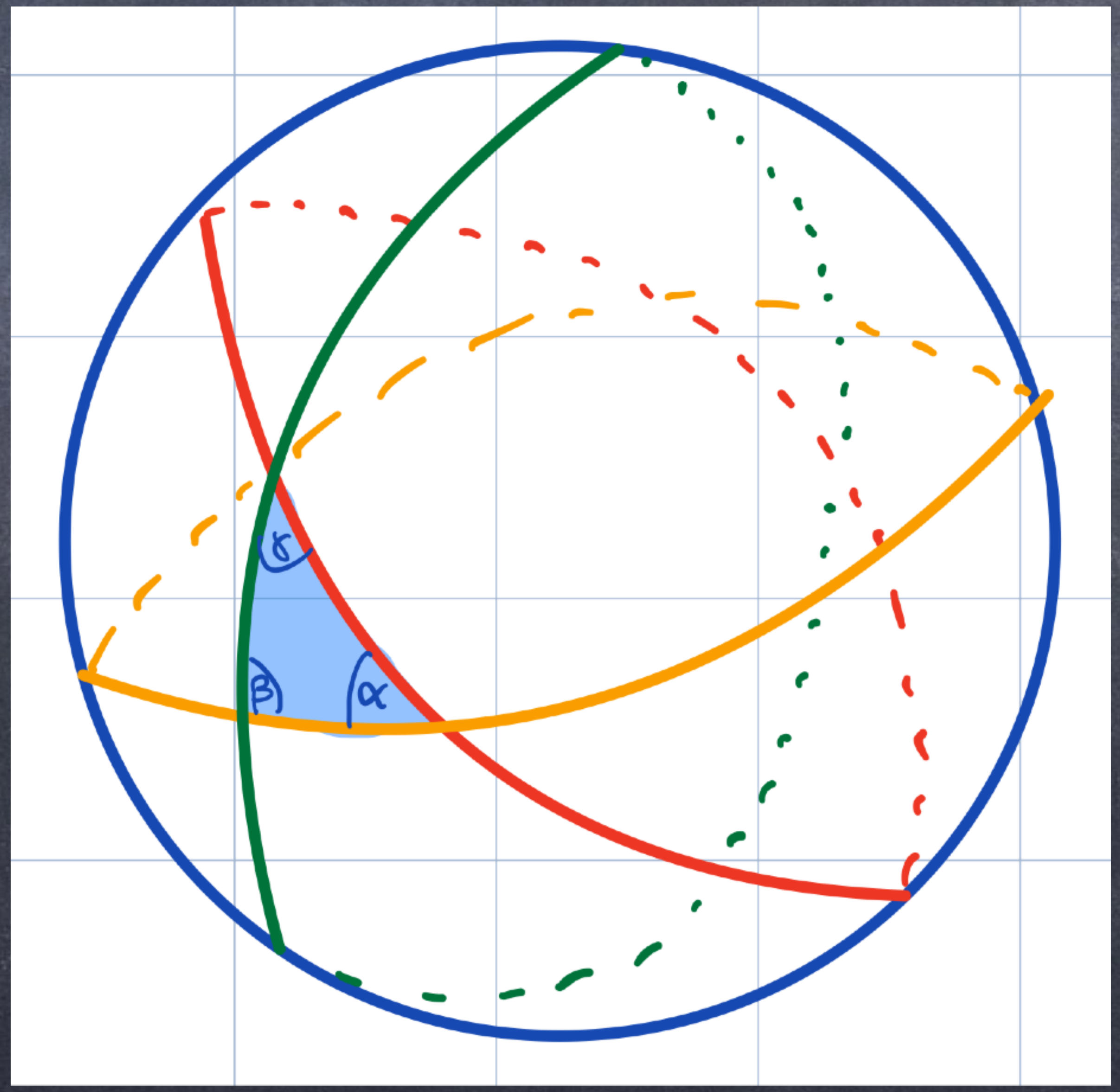
great circles intersect on a sphere with radius 1 with angle α

green area A equals 2α

since $\dfrac{\alpha}{2\pi} = \dfrac{A}{4\pi}$

area of the sphere with radius 1





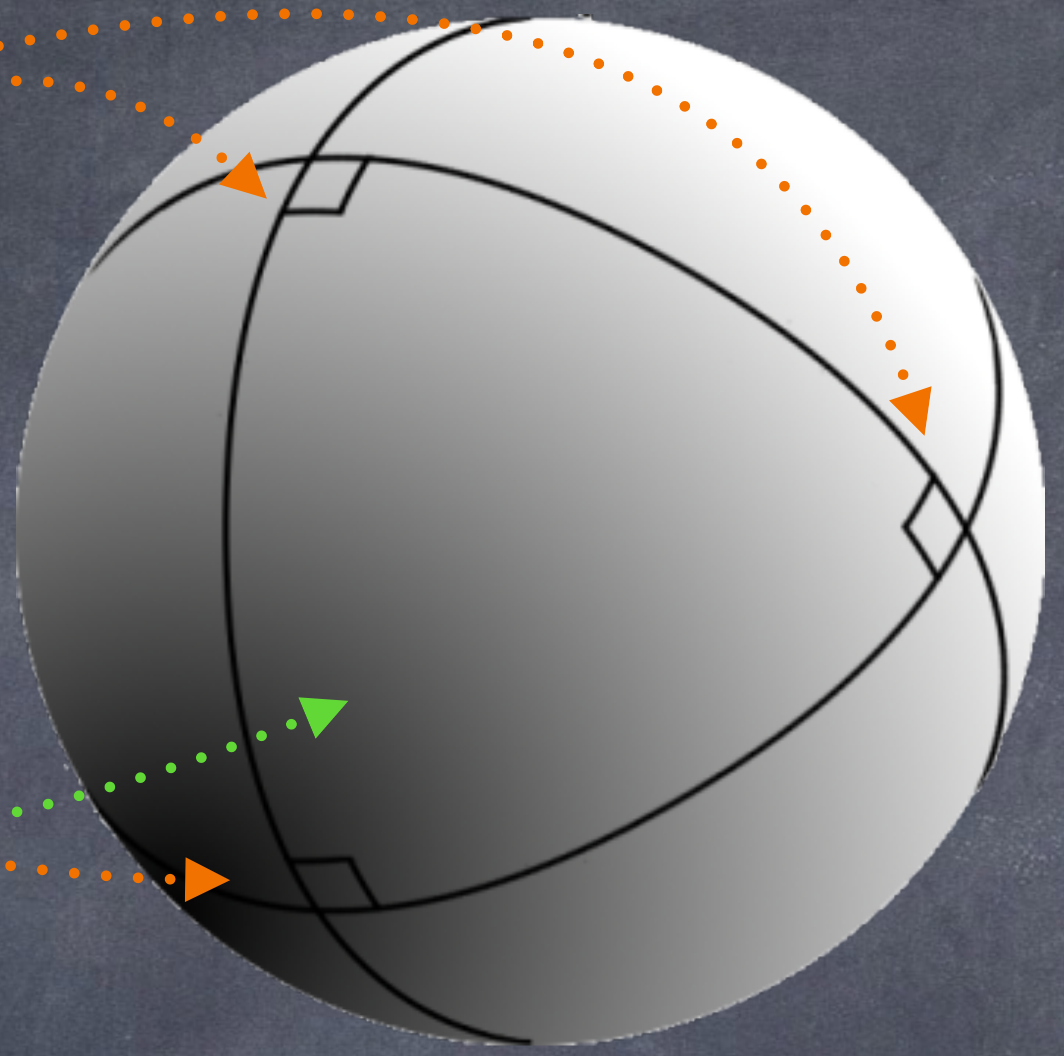
- What is the relation between the area and the angles of the spherical triangle?
- What is the sum of the angles in a spherical triangle?

Albert Girard (1595-1632)

Thomas Harriot (1560-1621)

Girard's theorem:

- Example: all angles equal $\pi/2$



area of the spherical triangle is $4\pi/8 = \pi/2$

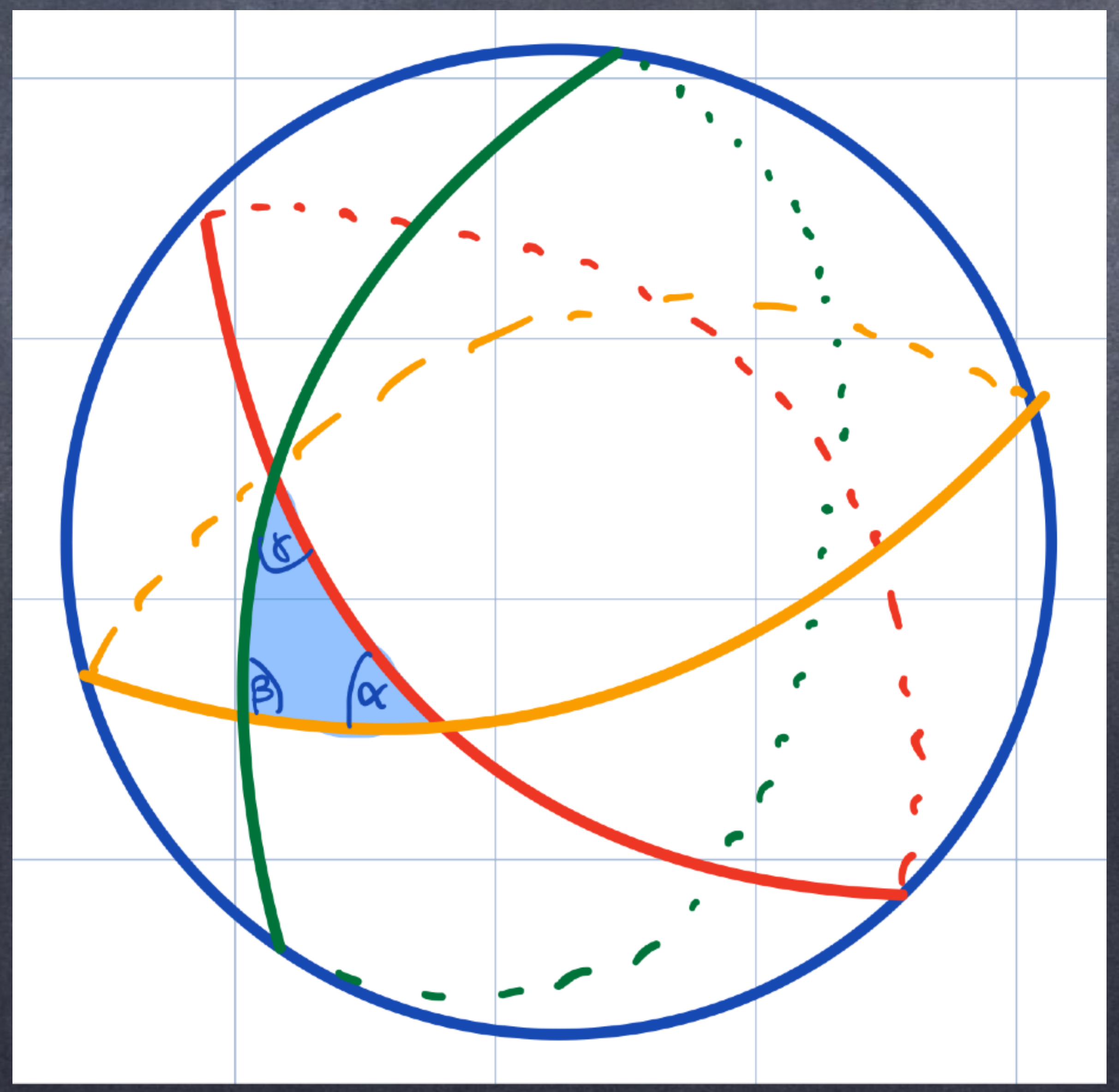
8 such triangles cover the sphere

Albert Girard (1595-1632)

- **Theorem of Girard-Harriot:**

The area A of a spherical triangle on a sphere of radius 1 satisfies

$$A + \pi = \alpha + \beta + \gamma.$$



Thomas Harriot (1560-1621)

• Theorem of Girard-Harriot:

The area A of a spherical triangle satisfies $A + \pi = \alpha + \beta + \gamma$.

Idea: we compute the area again using great circles

• **Proof:** $4\pi =$ $2 \times$ area red/orange $= 4\alpha$

areal of a sphere with radius 1

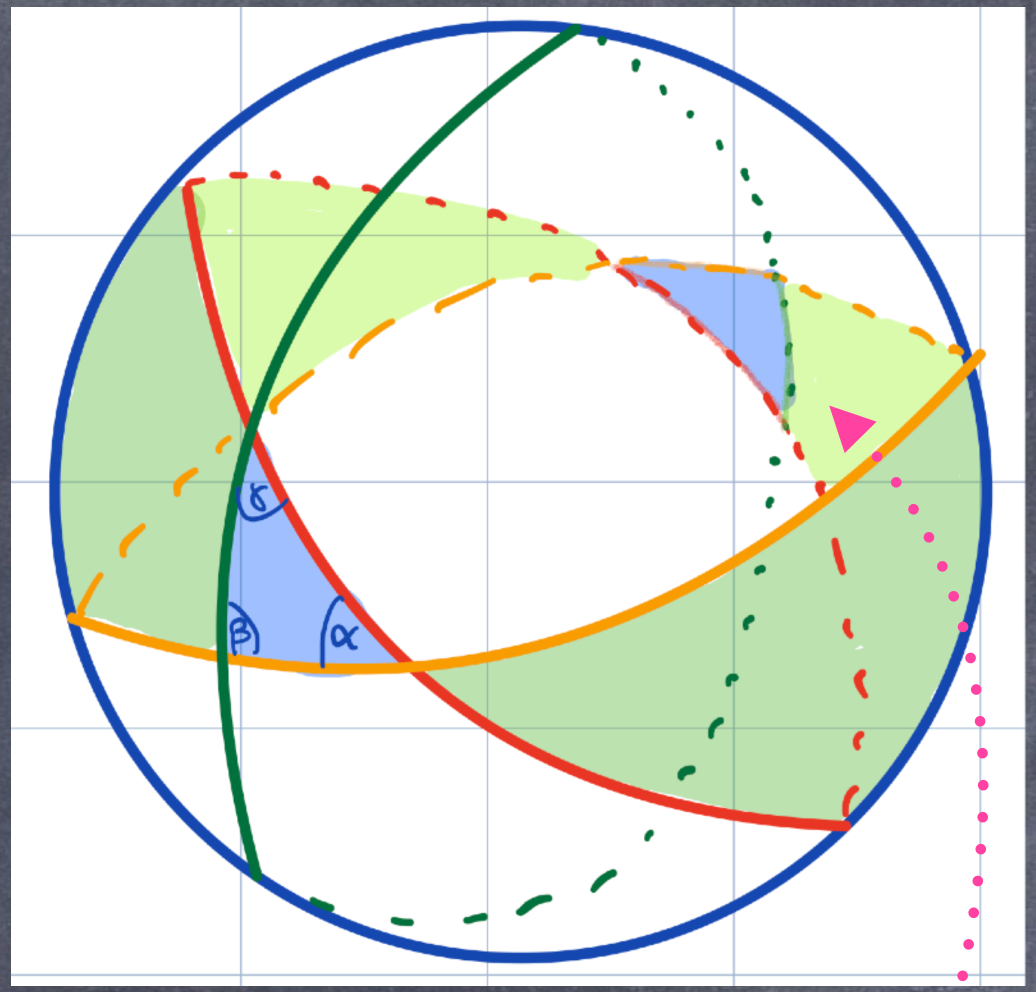
$+ \; 2 \times$ area green/red $= 4\gamma$

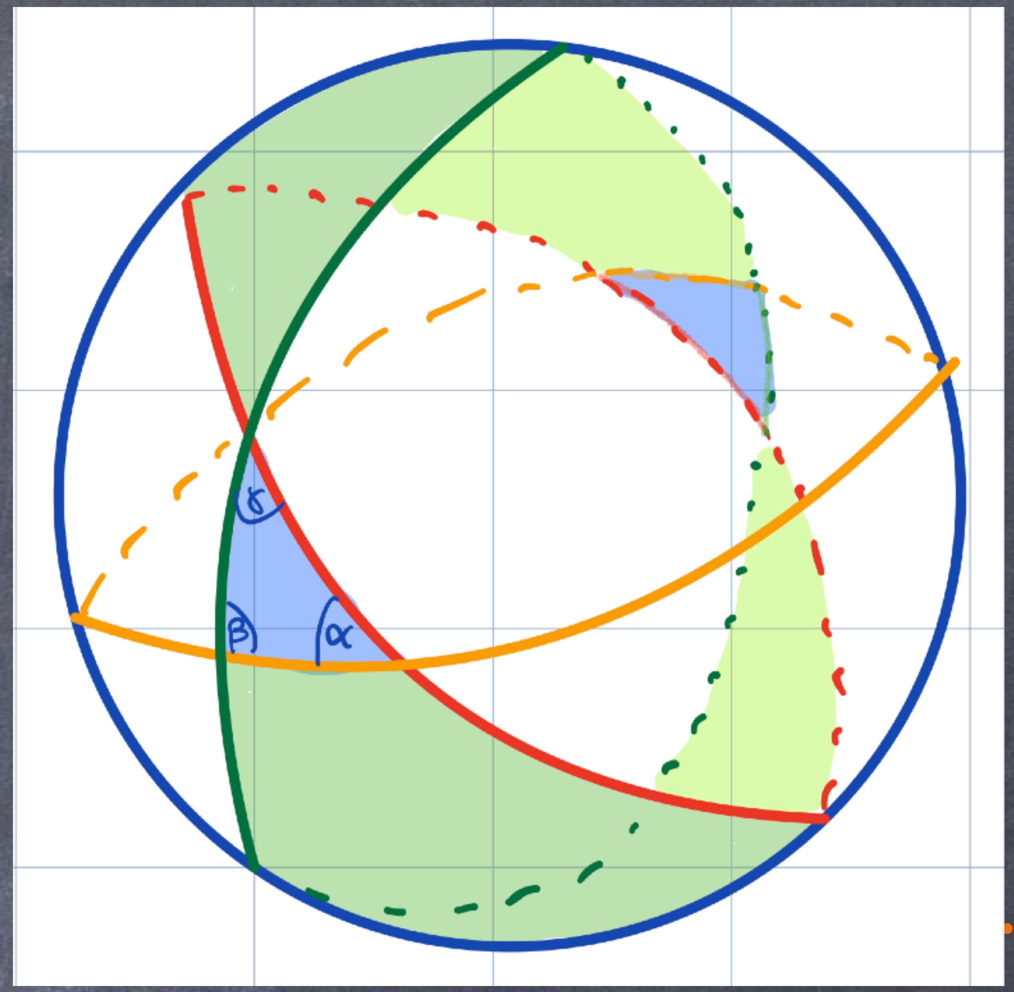
$+ \; 2 \times$ area orange/green $= 4\beta$

$- \; 4 \times$ area triangle

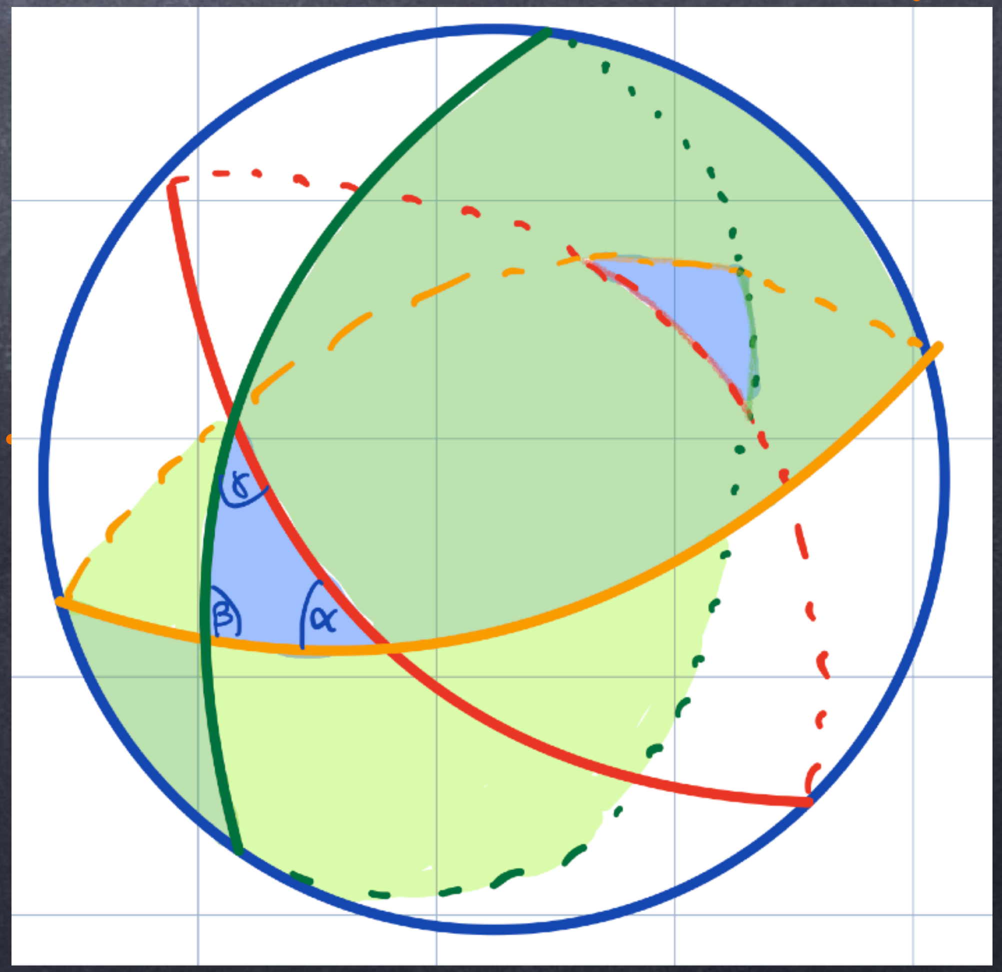
we covered the area of the triangle in total 6 times, that is 4 times too often

$$4\pi = 4(\alpha + \beta + \gamma) - 4A$$



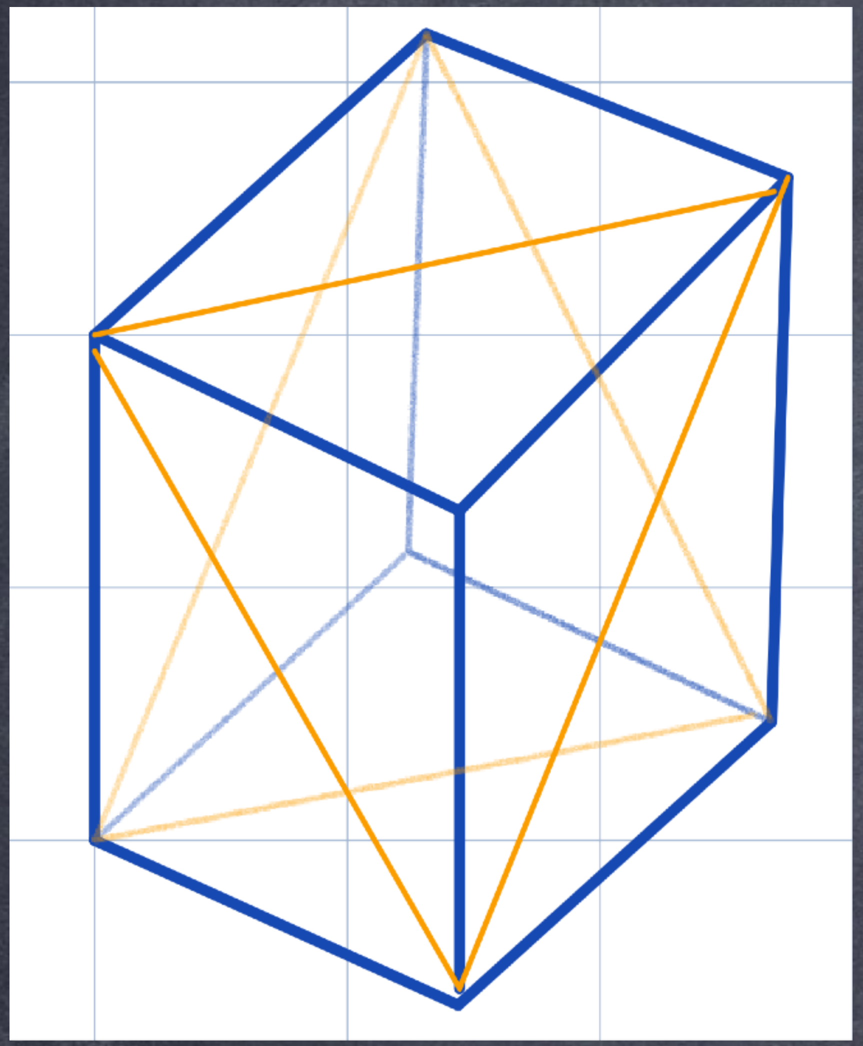


kongruent triangle on the backside

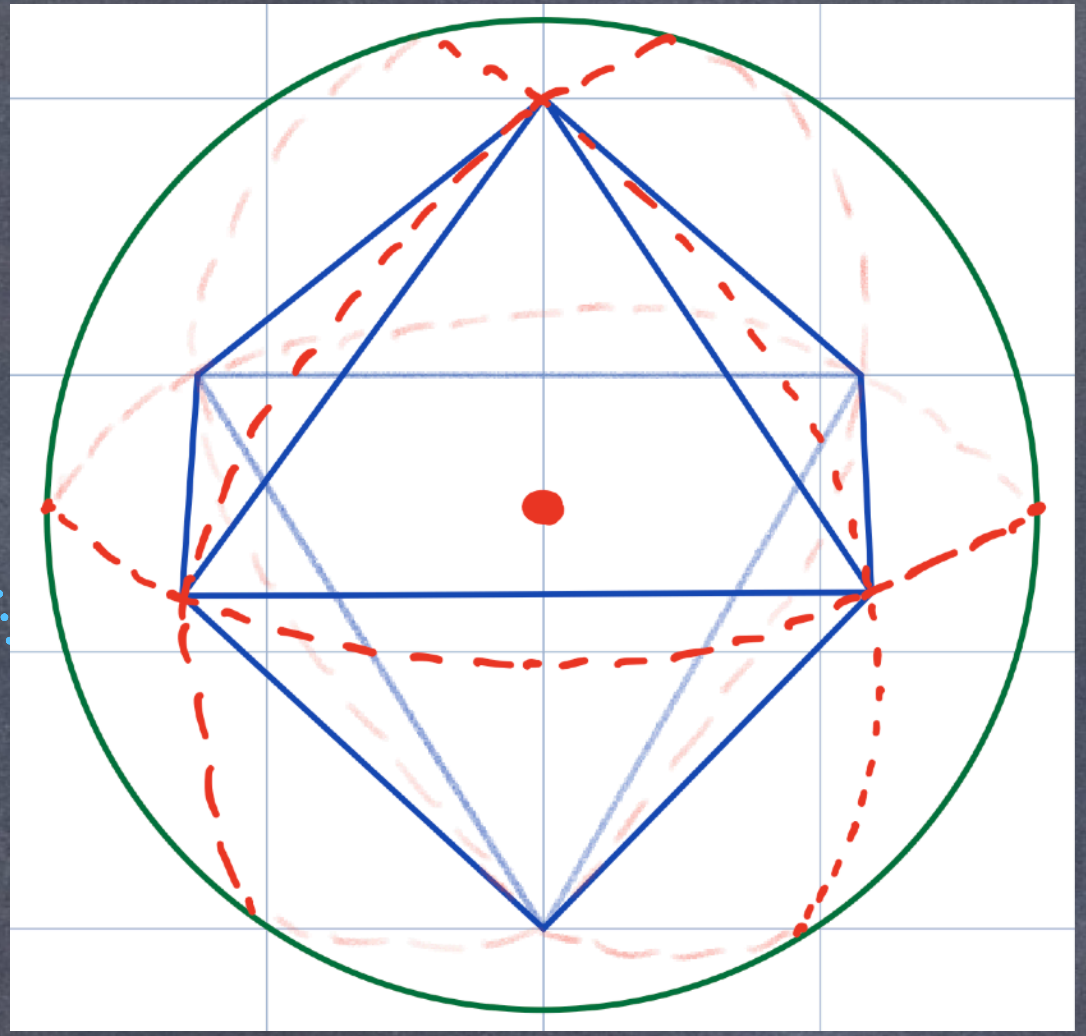


Legendre's proof of Euler's formula via spherical geometry:

- we triangulate the polygons:



- we project the triangulated polyhedron onto a spherical screen of radius 1:



- we take the sum of the angles at all vertices in two different ways:

- $V + F - E$ remains unchanged

- $2E = 3F$

- $F + 4 = 2V$

each edge meets two triangles

$$V - E + F = 2$$

area of the sphere with radius 1 equals the sum of the areas of all triangles

Girard-Harriot: each triangle contributes one π

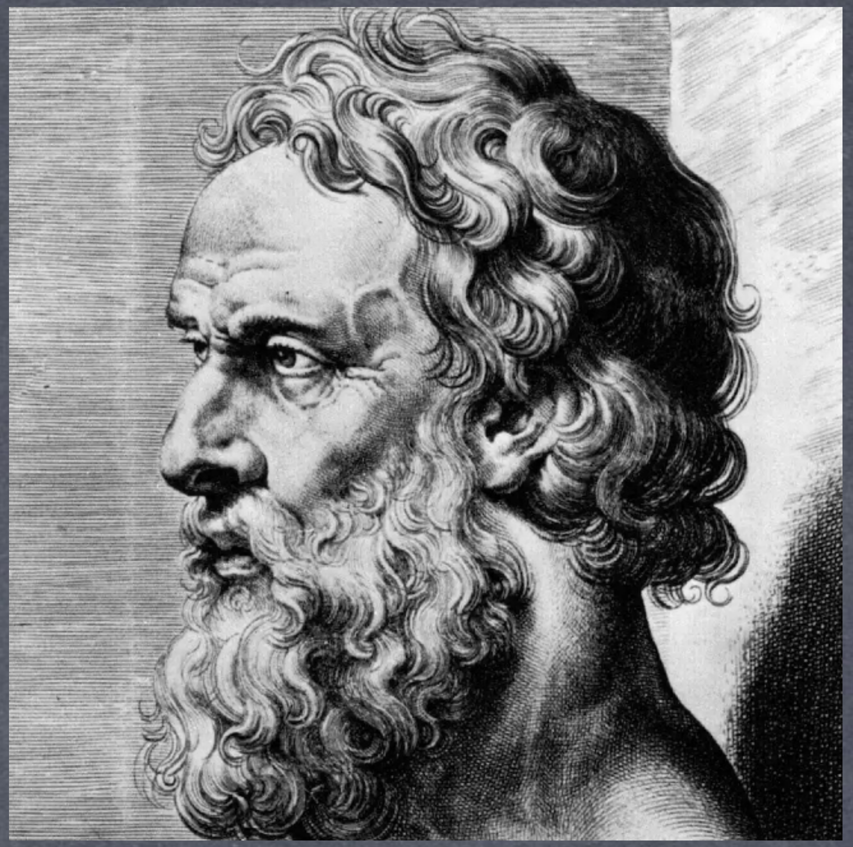
$$4\pi + F\pi \quad = \quad 2V\pi$$

at each vertex we get an angle sum of 2π

A Platonic solid is a convex polyhedron where all faces are congruent regular polygons and each vertex meets the same number of faces.

alle polygonene er like

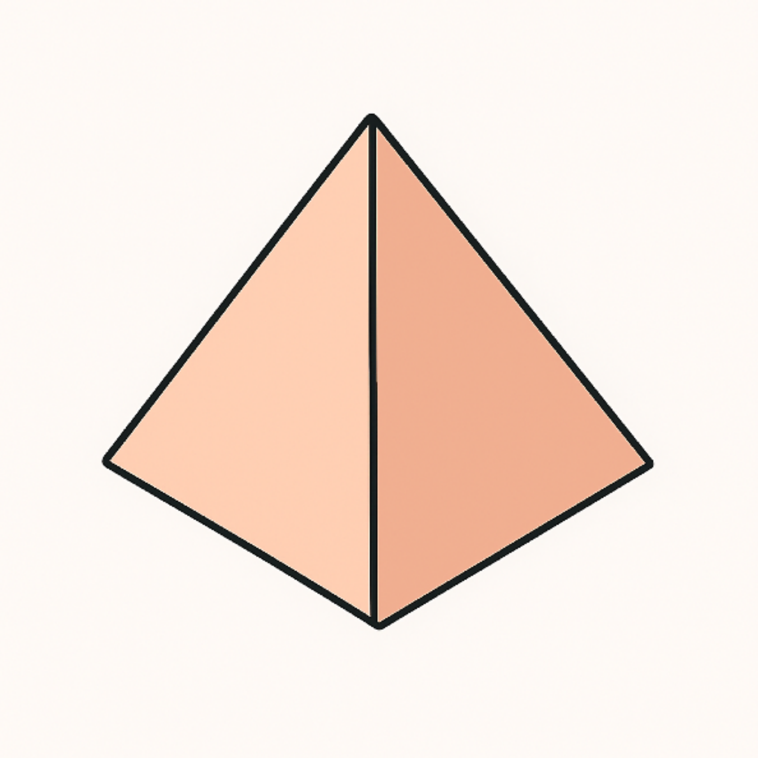
all angles are equal and all edges have the same length



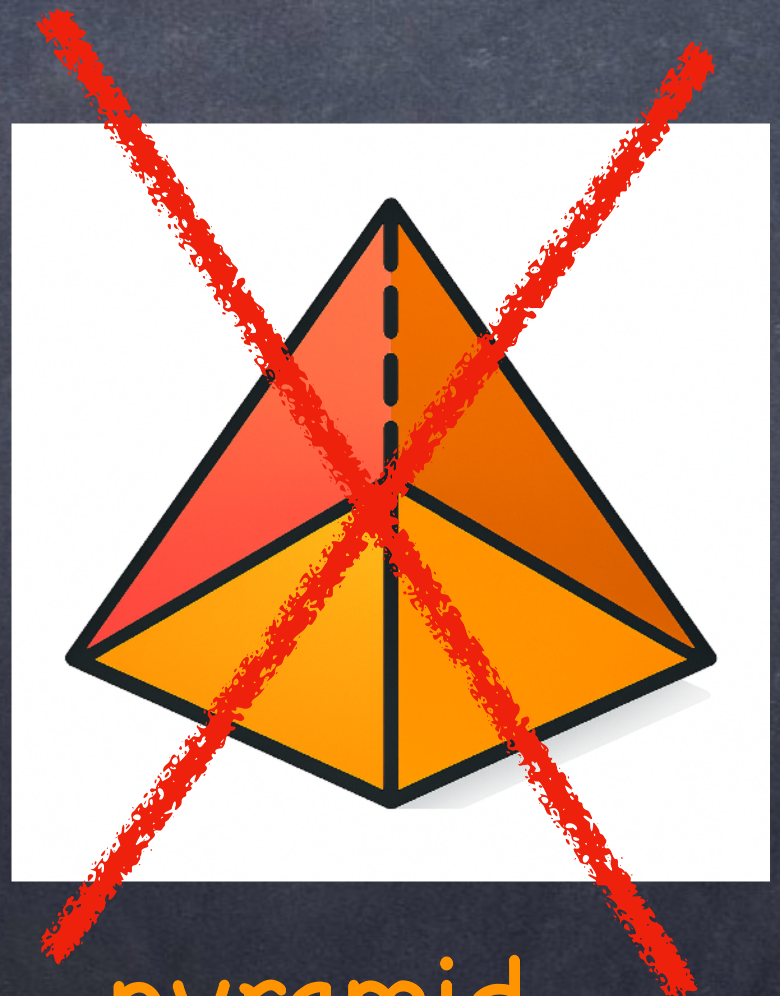
Plato (ca 428-348 BC)

Are there more?

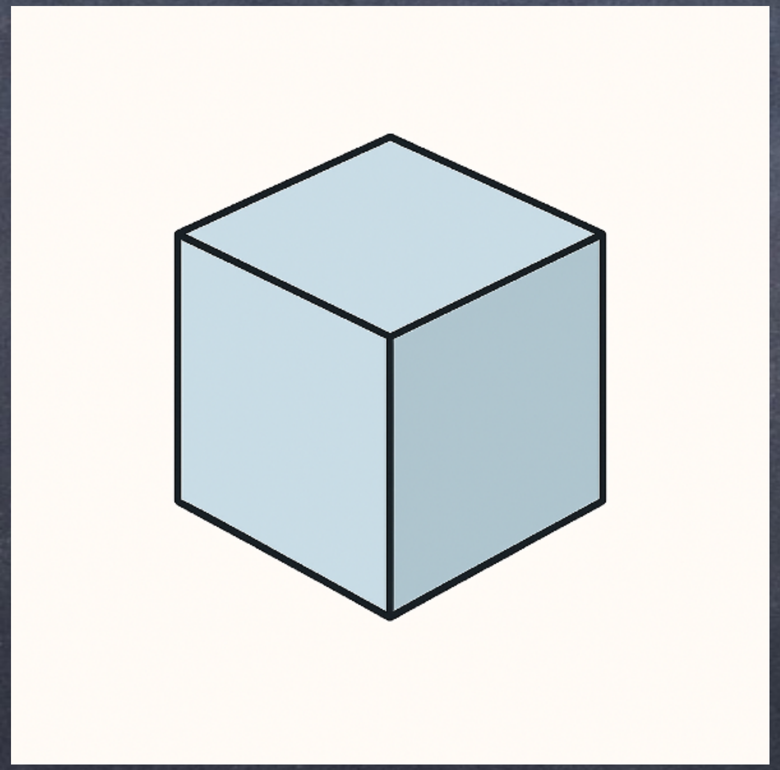
Examples:



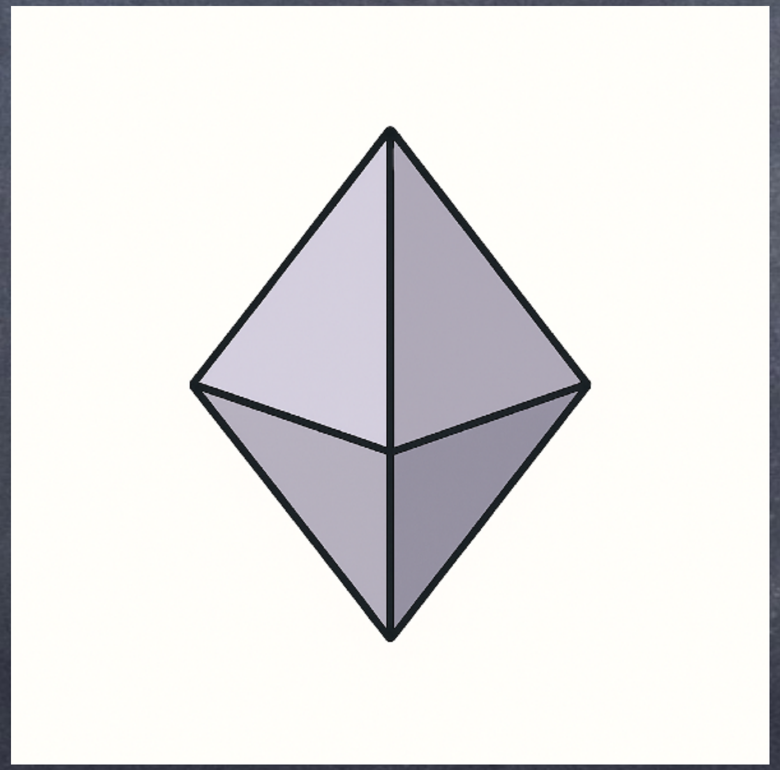
tetrahedron ✓



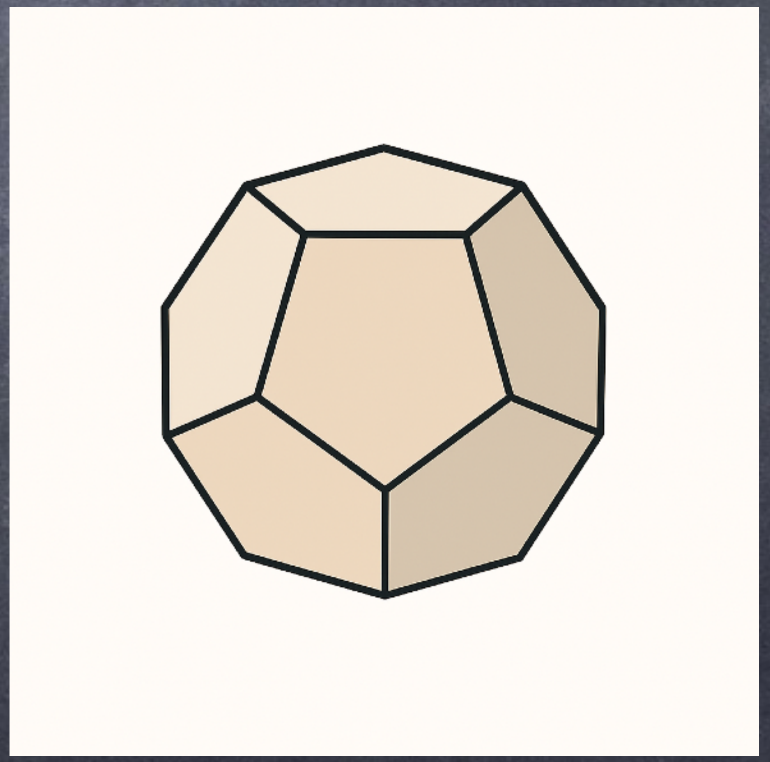
pyramid



hexahedron ✓



octahedron ✓



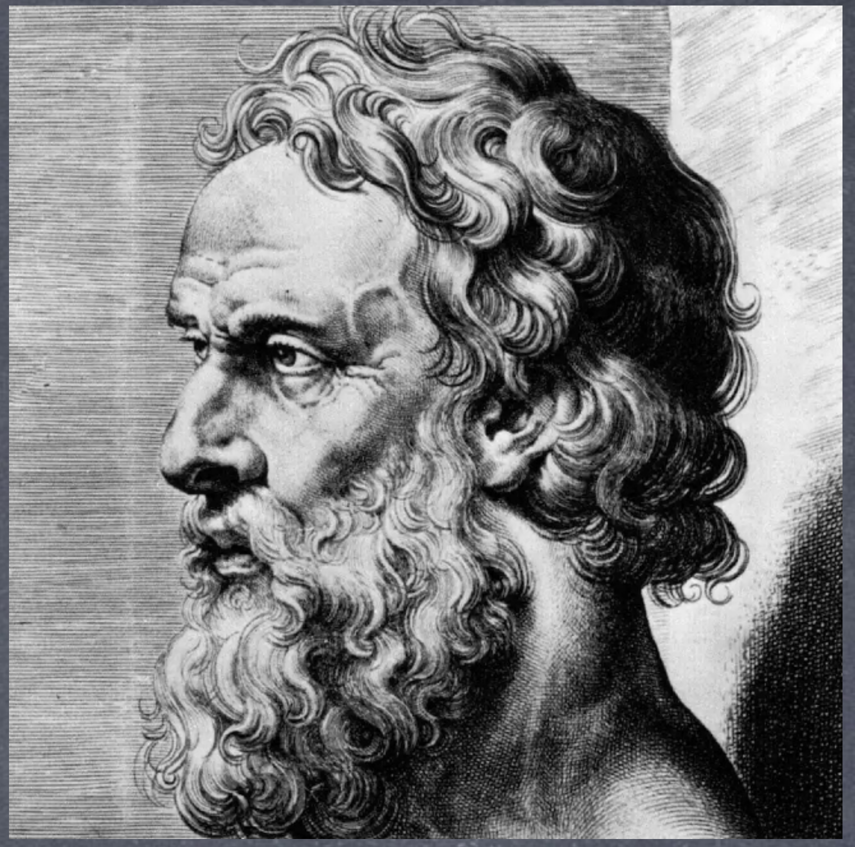
dodecahedron ✓

... ?

A Platonic solid is a convex polyhedron where all faces are congruent regular polygons and each vertex meets the same number of faces.

set p = number of edges each polygon has

set q = number of faces that meet at each vertex

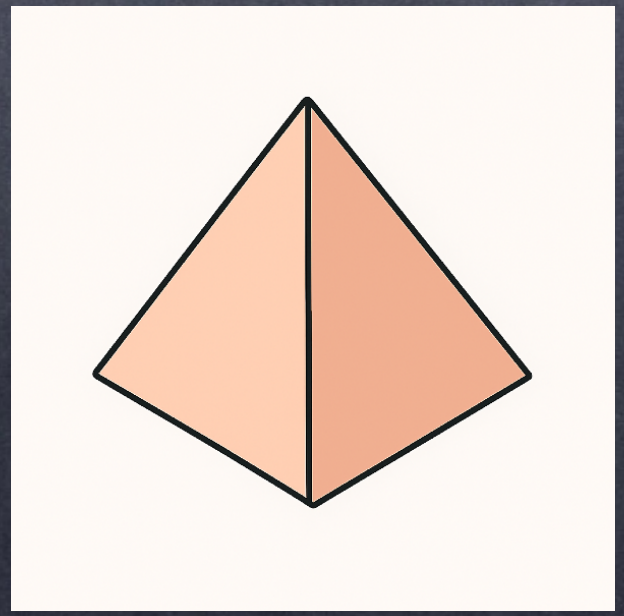


Plato (ca 428–348 BC)

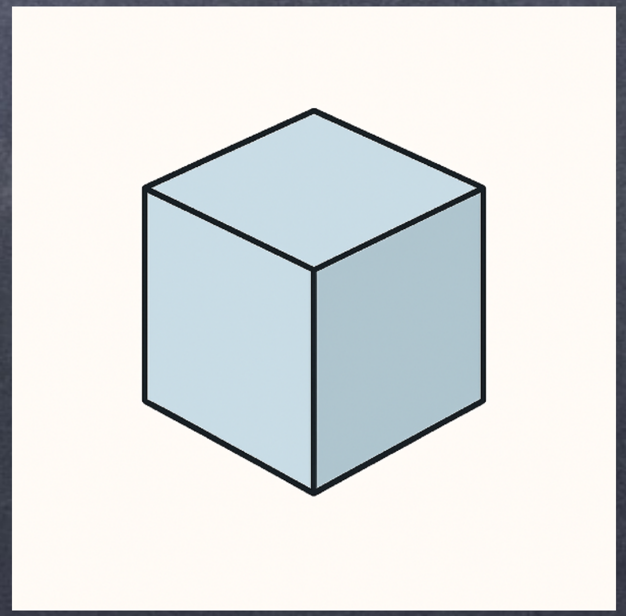
$$pF = 2E = qV$$

each edge meets two faces, that is, we count each edge twice

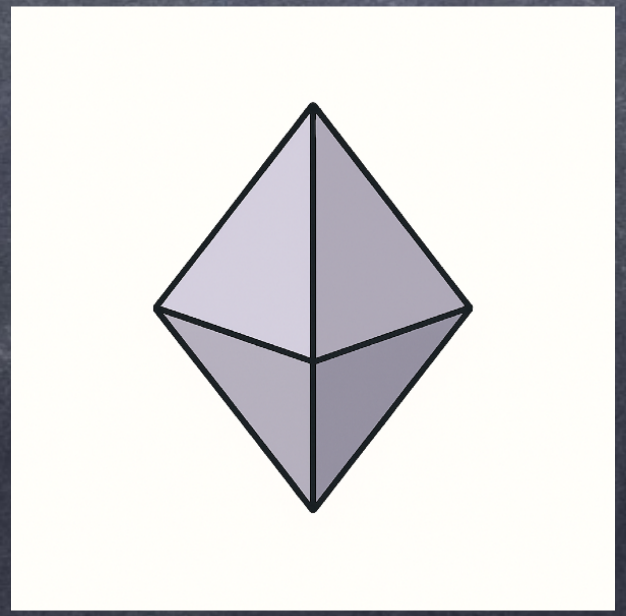
Examples:



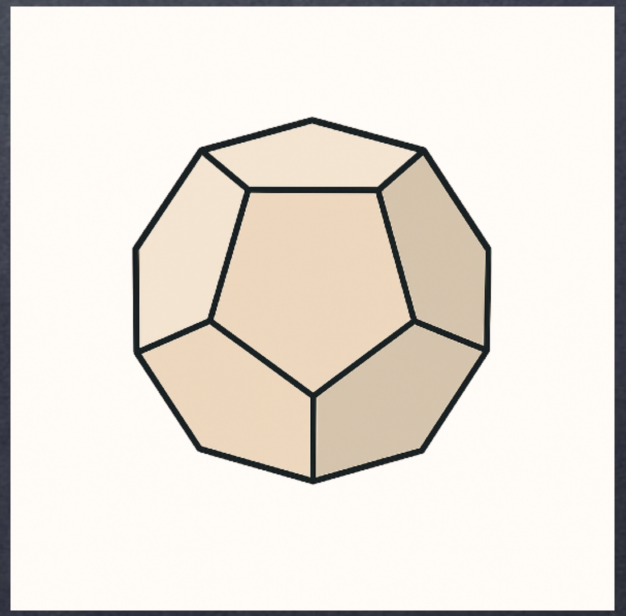
$p = 3,\ q = 3$



$p = 4,\ q = 3$



$p = 3,\ q = 4$

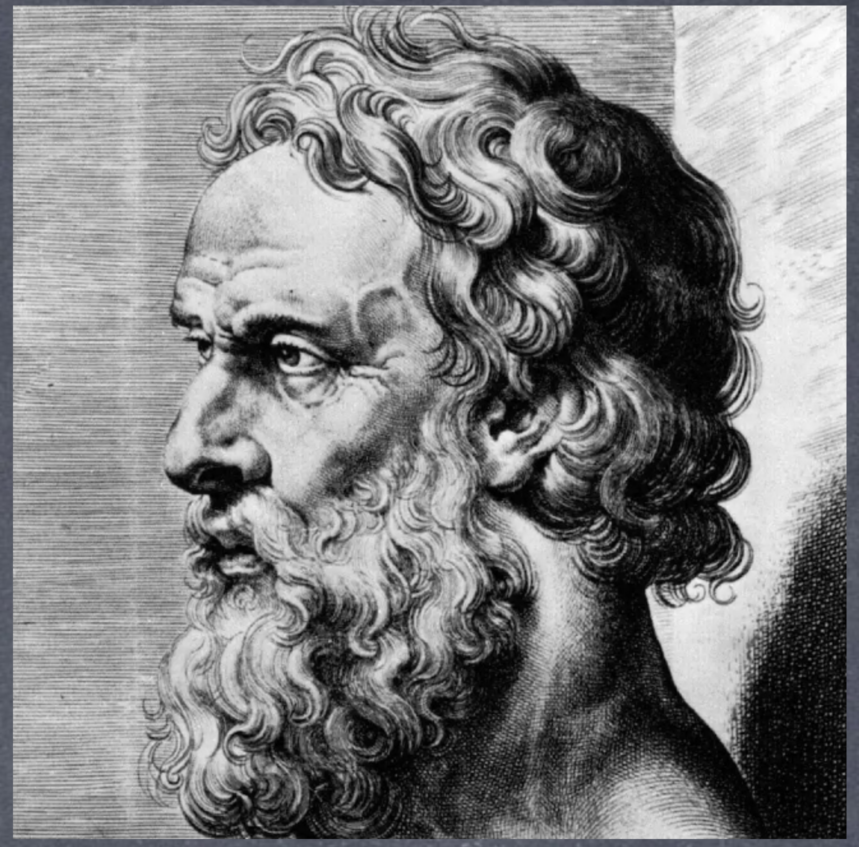


$p = 5,\ q = 3$

A Platonic solid is a convex polyhedron where all faces are congruent regular polygons and each vertex meets the same number of faces.

set p = number of edges each polygon has

set q = number of faces that meet at each vertex



Plato (ca 428-348 BC)

$$pF = 2E = qV$$

Eulers Formel: $V - E + F = 2$

know $\frac{1}{E} > 0$

each edge meets two faces, that is, we count each edge twice

$$\frac{2E}{q} - E + \frac{2E}{p} = 2$$

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{2} + \frac{1}{E} > \frac{1}{2}$$

(p, q) må være $(3,3), (4,3), (3,4), (5,3), (3,5)$

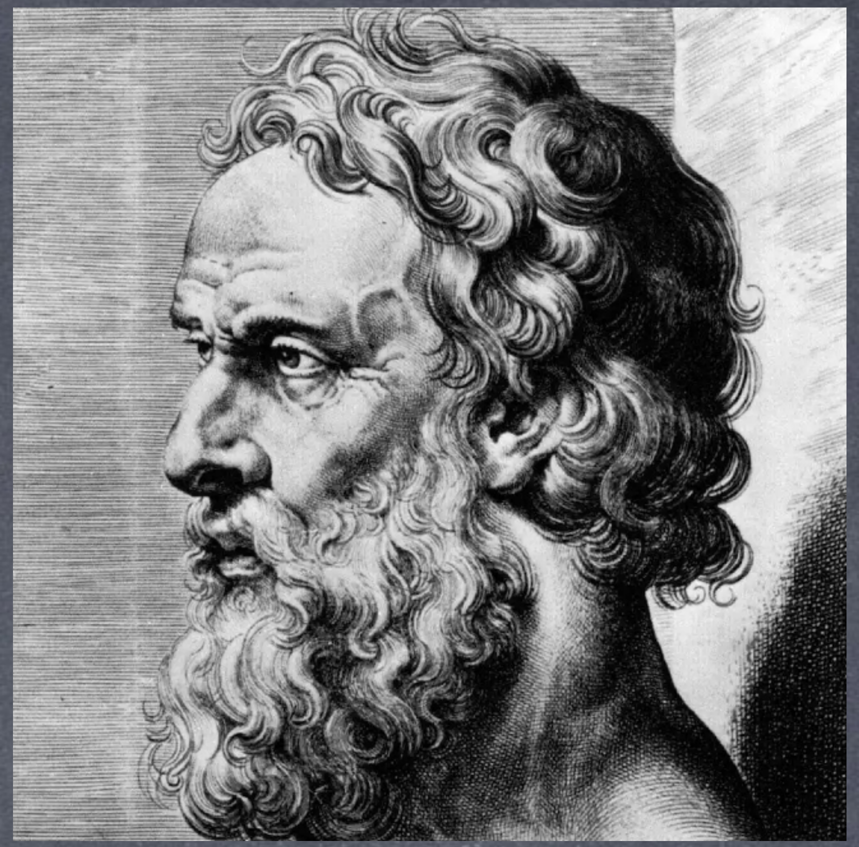
we also know that p and q are at least 3

There are exactly 5 Platonic solds!

A Platonic solid is a convex polyhedron where all faces are congruent regular polygons and each vertex meets the same number of faces.

set p = number of edges each polygon has

set q = number of faces that meet at each vertex



Plato (ca 428-348 BC)

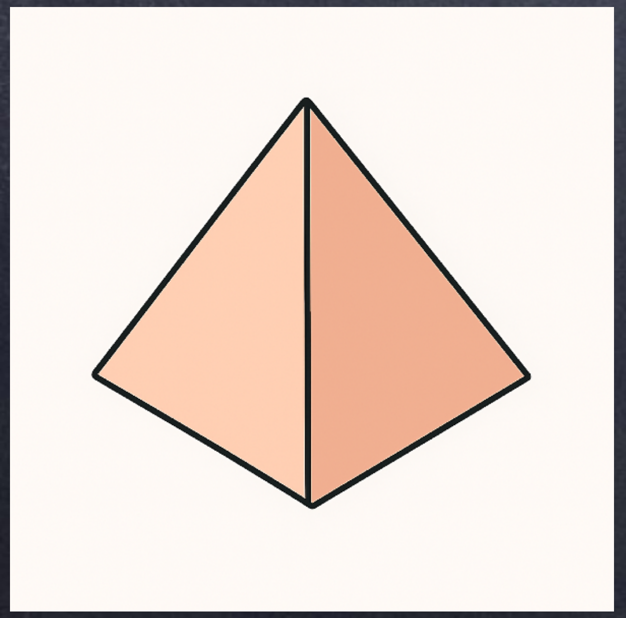
(p, q) må være $(3,3), (4,3), (3,4), (5,3), (3,5)$

There are exactly 5 Platonic solds!

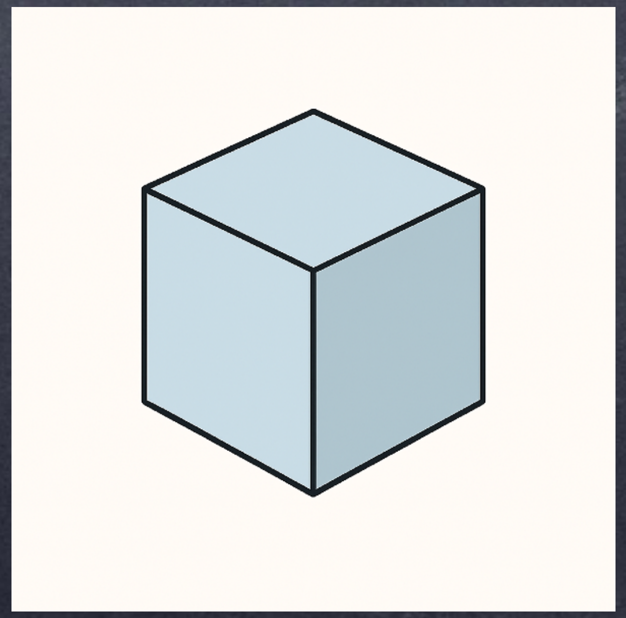
$V = 12,\ E = 30,$

$F = 20$

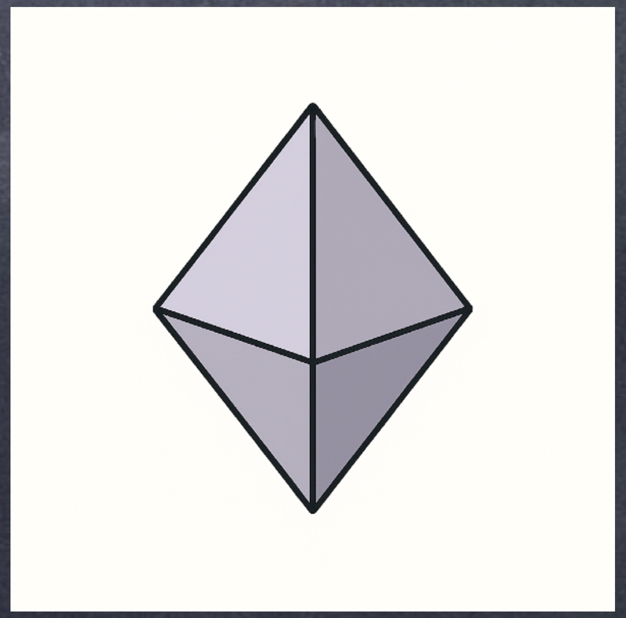
icosahedron



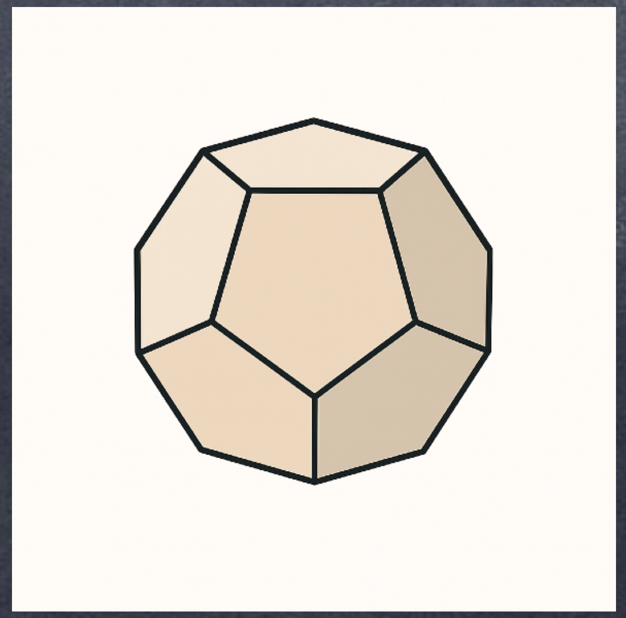
$p = 3,\ q = 3$



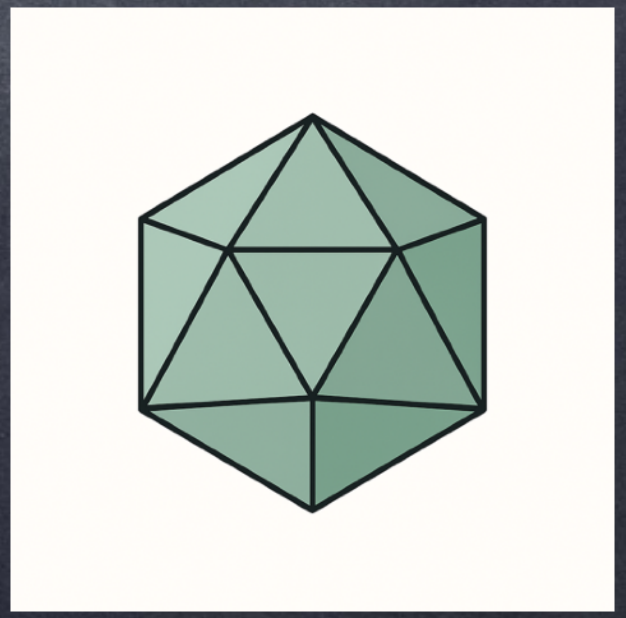
$p = 4,\ q = 3$



$p = 3,\ q = 4$



$p = 5,\ q = 3$



$p = 3,\ q = 5$

P = # pentagons = ?

H = # hexagons = ?

$V - E + F = 2$ Euler

$F = P + H$

$E = (5P + 6H)/2$

$V = (5P + 6H)/3$

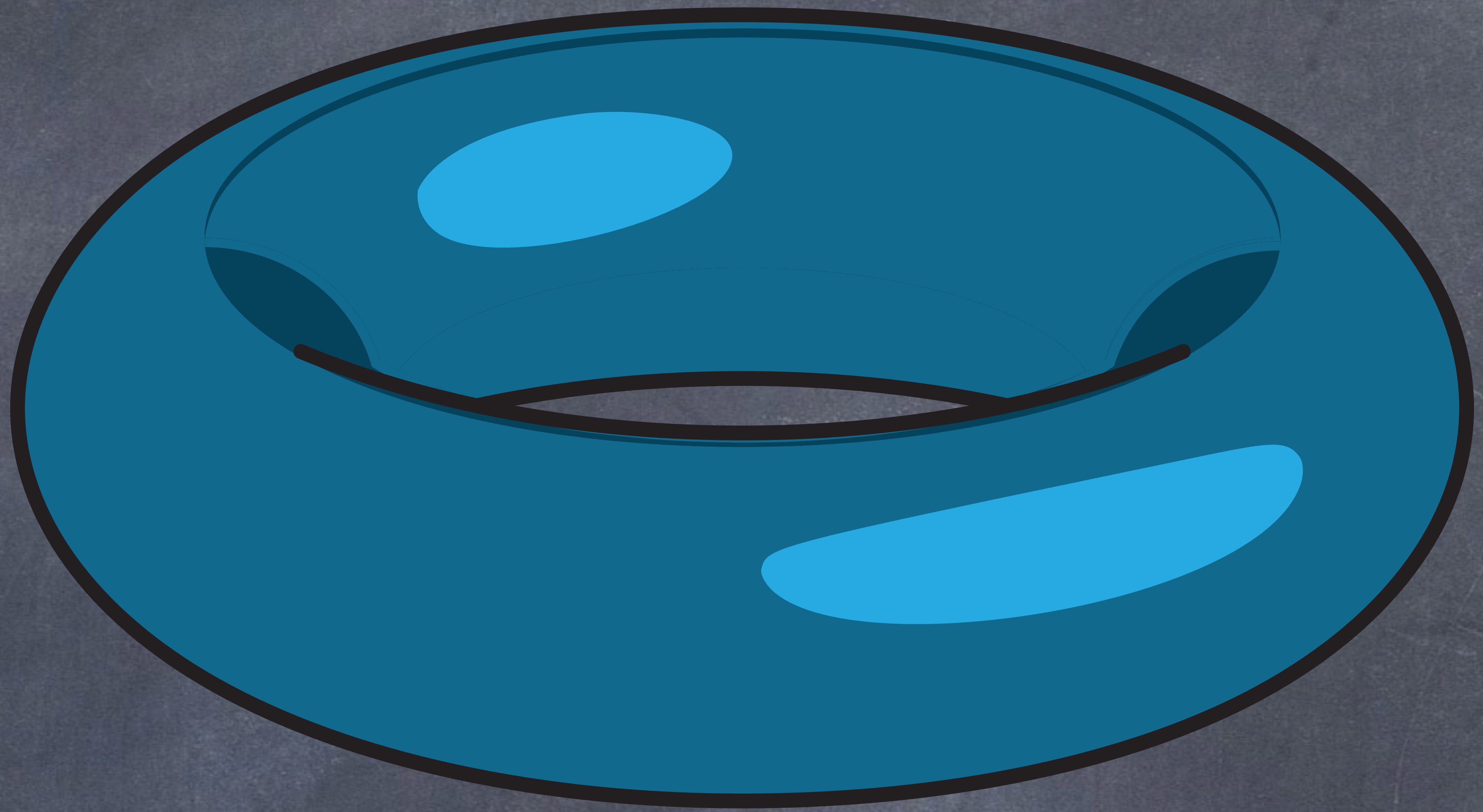
$P = 12$

$H = 20$

$H = (5P)/3$

Sphere

Torus



...

$$V - E + F = 2$$

$$V - E + F = 0$$

Euler-characteristic of a manifold

a lot of exciting mathematics to explore

Gauss-Bonnet-Formula, Poincaré-Hopf Index-Theorem, ...

Thank you!