A_3-formality and Massey products for Demushkin groups

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This is joint work with
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Norm Residue Isomorphism Theorem:

F field with $\mathrm{char}(F) \neq p$ containing primitive p-th root of unity

Voevodsky, Rost, ...

Milnor K-theory
$T(F^\times)/(u \otimes (1-u), u \neq 0,1)$

continuous cohomology of absolute Galois group G_F of F with trivial action on $\mathbb{F}_p$

$$K_\bullet^M(F)/p \xrightarrow{\cong} H^\bullet(G_F, \mathbb{F}_p)$$

quadratic algebra

- generators in degree 1
- relations in degree 2

strong restriction on which $\mathbb{F}_p$-algebras can occur as the Galois cohomology of a field

Which quadratic algebras occur as $H^\bullet(G_F, \mathbb{F}_p)$?

Additional properties? $H^\bullet(G_F, \mathbb{F}_p)$ quadratic algebra

- For example, are there nontrivial Massey products?

- many vanishing Massey products in arithm. & alg. geometry: Deligne-Griffiths-Morgan-Sullivan, but also many non-vanishing ones: e.g. Ekedahl, Morishita, Sharifi, Vogel, Gärtner, Bleher-Chinburg-Gillibert, Deninger,...

- Hopkins and Wickelgren: $\langle a_1, a_2, a_3\rangle \neq \emptyset \iff 0 \in \langle a_1, a_2, a_3\rangle$

  $p = 2$, F a global field of $\mathrm{char}(F) \neq 2$

  is defined, i.e., $a_1 \cup a_2 = 0 = a_2 \cup a_3$

  Massey product vanishes

elements in $H^1(G_F, \mathbb{F}_2)$

Massey products provide an obstruction to formality

- Is $\mathscr{C}^\bullet(G_F, \mathbb{F}_p)$ a formal dga?

  continuous Galois cochains

- Do at least all Massey products vanish when they are defined?

Massey vanishing conjecture of Mináč-Tân:

for every field F, all $n \geq 3$, all primes p

Conjecture: For $a_1, \ldots, a_n \in H^1(G_F, \mathbb{F}_p)$,

$$\langle a_1, \ldots, a_n \rangle \neq \emptyset \iff 0 \in \langle a_1, \ldots, a_n \rangle.$$

✓

- Efrat-Matzri, Mináč-Tân: all fields, all primes, $n = 3$

- new examples of profinite groups which are not absolute Galois groups

- Example: S = free pro-2 group on generators $x_1, \ldots, x_5$

Mináč-Tân: and relation $1 = [x_4, x_5][[x_2, x_3]x_1]$

commutator: $[x, y] = x^{-1}y^{-1}xy$

For $G = S/\langle r \rangle$, $\mathscr{C}^\bullet(G, \mathbb{F}_2)$ has non-vanishing triple Massey products, and hence G is not the maximal pro-2 quotient of an absolute Galois group

Massey vanishing conjecture of Miná$\check{c}$-Tân:

for every field F, all $n \geq 3$, all primes p

Conjecture: For $a_1, \ldots, a_n \in H^1(G_F, \mathbb{F}_p)$:

$$\langle a_1, \ldots, a_n \rangle \neq \emptyset \iff 0 \in \langle a_1, \ldots, a_n \rangle.$$

✓

- Efrat-Matzri, Miná$\check{c}$-Tân: all fields, all primes, $n = 3$
- Merkurjev-Scavia: all fields, $p = 2$, $n = 4$
- Harpaz-Wittenberg: all number fields, all primes, all $n \geq 3$
- Pál-Szabó: fields with $\mathrm{vcd} \leq 1$
- Quadrelli: elementary type pro-p-groups
- Maire-Miná$\check{c}$-Ramakrishna-Tân: number fields with $\mu_p \not\subset K$

all primes, all $n \geq 3$

strong Massey vanishing:

$0 \in \langle a_1, \ldots, a_n \rangle$

$\iff a_i \cup a_{i+1} = 0$

for all $i = 1, \ldots, n-1$

Hopkins-Wickelgren Question:

for every field F and all primes p?

Question: Is $\mathscr{C}^\bullet(G_F, \mathbb{F}_p)$ formal?

we then say that F is formal at p

- Yes, for example, for pseudo-algebraically closed fields

The answer is **no** in general

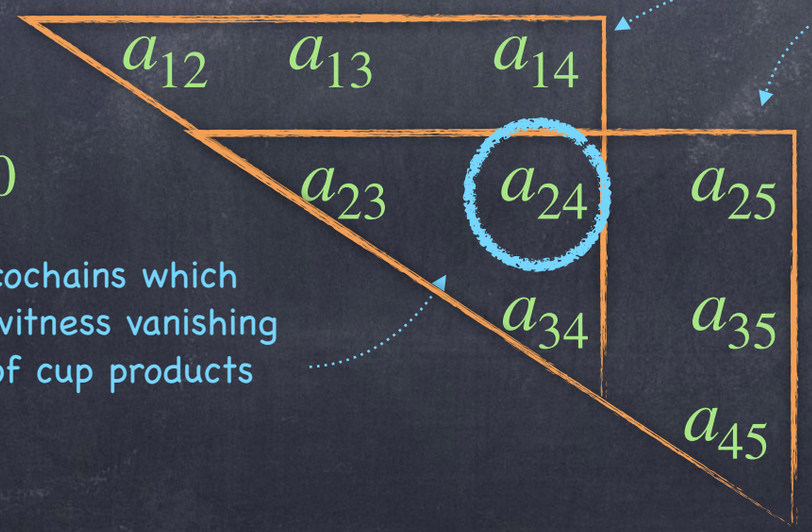
- Positselski: $\mathbb{Q}_\ell$, for odd $\ell \neq p$, is not formal at p
- Harpaz-Wittenberg: $\mathbb{Q}$ is not formal for $p = 2$

- formality implies all Massey products are defined and vanish when neighbouring cup products are zero

- exist $a_1, a_2, a_3, a_4 \in H^1(\mathbb{Q}, \mathbb{F}_2)$

with $a_1 \cup a_2 = a_2 \cup a_3 = a_3 \cup a_4 = 0$

but $\langle a_1, a_2, a_3, a_4 \rangle$ not defined



cochains witnessing vanishing of triple Massey products

cochains which witness vanishing of cup products

- $\langle a_1, a_2, a_3, a_4 \rangle$ is defined if both $\langle a_1, a_2, a_3 \rangle$ and $\langle a_2, a_3, a_4 \rangle$ vanish ...

with the same choice of a_{24}

Hopkins-Wickelgren Question:

for every field F and all primes p?

Question: Is $\mathscr{C}^\bullet(G_F, \mathbb{F}_p)$ formal?

we then say that F is formal at p

- Positselski: $\mathbb{Q}_\ell$, for odd $\ell \neq p$, is not formal at p
- Harpaz-Wittenberg: $\mathbb{Q}$ is not formal for $p = 2$
- Merkurjev-Scavia:
  - every $F/\mathbb{Q}$ of odd degree is not formal for $p = 2$
  - every F of char. $\neq p$ is in some L which is not formal at p

However, there are also important positive cases...

- Theorem (Pál-Q.): If F has virtual cohomological dimension ≤ 1, then $H^\bullet(G_F, \mathbb{F}_p)$ is intrinsically formal.

every dga $\mathscr{C}^\bullet$ over $\mathbb{F}_p$ with $H^\bullet \cong H^\bullet(G_F, \mathbb{F}_p)$ is formal

A_3-formality and Massey products:

we assume all dgas are over a field $\mathbb{F}$

Theorem (Kadeishvili; Merkulov): $\mathscr{C}^\bullet$ a dga with cohomology $H^\bullet$.

Then there is an A_∞-algebra structure $(H^\bullet, \{m_n\}_{n\geq 1})$ such that $m_1 = 0$ and a quasi-isomorphism of A_∞-algebras $H^\bullet \xrightarrow{\simeq} \mathscr{C}^\bullet$.

minimal model for $\mathscr{C}^\bullet$

Recall: A dga $\mathscr{C}^\bullet$ is A_∞-formal if its minimal model satisfies $m_n = 0$ for all $n \geq 3$.

Definition: We say that the dga $\mathscr{C}^\bullet$ is A_3-formal if its minimal model can be chosen such that $m_3 = 0$.

all triple Massey products in $\mathscr{C}^\bullet$ vanish when defined and if $a_1 \cup a_2 = a_2 \cup a_3 = a_3 \cup a_4 = 0$ then $\langle a_1, a_2, a_3, a_4\rangle$ is defined

e.g. Buijs–Moreno-Fernández–Murillo

$\mathscr{C}^\bullet$ is A_3-formal $\Longleftarrow$ $\mathscr{C}^\bullet$ is A_∞-formal

- provides a stronger obstruction than vanishing of triple Massey products, which vanishes for certain classes of profinite groups...

Demushkin groups: p a prime number, G a pro-p group

Definition: G is called a Demushkin group if
- $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty$,
- $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1$,
- the cup-product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

- the only finite Demushkin group is $\mathbb{Z}/2\mathbb{Z}$
- set $d := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ and $q := 0$ if $G^{\mathrm{ab}} \cong \mathbb{Z}_p^d$ and $q := p^f$ if $G^{\mathrm{ab}} \cong \mathbb{Z}/p^f\mathbb{Z} \times \mathbb{Z}_p^{d-1}$
- d and q are invariants determining G

Theorem (Pál-Q.): Let $d \geq 2$ and G be a pro-p Demushkin group with invariant q:
- If $q \neq 3$, then $\mathscr{C}^\bullet(G, \mathbb{F}_p)$ is A_3-formal.
- If $q = p = 3$, then $\mathscr{C}^\bullet(G, \mathbb{F}_3)$ is not A_3-formal.

- However, by Positselski such G is not A_∞-formal for $q \neq 2$ and $d = 2$

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Classification: • $d := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ and $q = 0$ if $G^{\mathrm{ab}} \cong \mathbb{Z}_p^d$ and $q = p^f$ if $G^{\mathrm{ab}} \cong \mathbb{Z}/p^f\mathbb{Z} \times \mathbb{Z}_p^{d-1}$

Theorem (Demushkin, Serre, Labute): G is isomorphic to the pro-p group with generators $x_1, \ldots, x_d$ subject to a single relation r:

commutator: $[x, y] = x^{-1}y^{-1}xy$

- $q \neq 2$: $d \geq 2$ even, $r = x_1^q[x_1, x_2][x_3, x_4]\cdots[x_{d-1}, x_d]$
- $q = 2$: $d \geq 3$ odd, $r = x_1x_2^{2^f}[x_2, x_3]\cdots[x_{d-1}, x_d]$ with $f \geq 2$ or $2^f = 1$
- $q = 2$: $d \geq 2$ even, $r = x_1^{2+2^f}[x_1, x_2]\cdots[x_{d-1}, x_d]$ with $f \geq 2$ or $2^f = 1$
- $q = 2$: $d \geq 4$ even, $r = x_1^2[x_1, x_2]x_3^{2^f}[x_3, x_4]\cdots[x_{d-1}, x_d]$ with $f \geq 2$

Demushkin groups: p a prime number, G a pro-p group

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- the cup-product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

Examples: • $d := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ and $q = 0$ if $G^{\mathrm{ab}} \cong \mathbb{Z}_p^d$ and $q = p^f$ if $G^{\mathrm{ab}} \cong \mathbb{Z}/p^f\mathbb{Z} \times \mathbb{Z}_p^{d-1}$

- $G = \mathbb{Z}_p \times \mathbb{Z}_p$ with $q = 0$ (this G is even formal)
- pro-p completion of the fundamental group of compact orientable surface of genus g, $d = 2g$ and $q = 0$ is A_3-formal
- $G = \mathbb{Z}_p \rtimes \mathbb{Z}_p = \langle x_1, x_2 \rangle / (x_1^q [x_2, x_2])$ is A_3-formal iff $q \neq 3$

for this type of G

- However, for every $q \neq 2$, $H^\bullet(G, \mathbb{F}_p)$ is an exterior algebra on two generators

Demushkin groups: p a prime number, G a pro-p group

Definition: G is called a Demushkin group if
- $\dim_{\mathbb{F}_p} H^1(G,\mathbb{F}_p) < \infty$,
- $\dim_{\mathbb{F}_p} H^2(G,\mathbb{F}_p) = 1$,
- the cup-product $H^1(G,\mathbb{F}_p)\times H^1(G,\mathbb{F}_p)\to H^2(G,\mathbb{F}_p)$ is a non-degenerate bilinear form.

Examples:
- $d := \dim_{\mathbb{F}_p} H^1(G,\mathbb{F}_p)$ and $q=0$ if $G^{\mathrm{ab}} \cong \mathbb{Z}_p^d$ and $q=p^f$ if $G^{\mathrm{ab}} \cong \mathbb{Z}/p^f\mathbb{Z}\times\mathbb{Z}_p^{d-1}$
- $K/\mathbb{Q}_\ell$ of finite degree, $G_K(p)$:= maximal pro-p quotient of G_K, $\ell \neq p$

Then $G_K(p)$ is Demushkin if $\mu_p \subset K$ with
- $d=2$, $q=p^f$ maximal power such that a primitive qth root of unity is in K

$G_K(p) = \langle x_1, x_2\rangle/(x_1^q[x_2,x_2])$

- $K/\mathbb{Q}_p$ of finite degree n; if $\mu_p \subset K$ then $G_K(p)$ is Demushkin with
  - $d=n+2$, $q=p^f$ maximal power such that a primitive qth root of unity is in K

two examples with $q=2$:

- $G_{\mathbb{Q}_2}(2) = \langle x_1,x_2,x_3\rangle/(x_1^2x_2^4[x_2,x_2])$
- $G_{\mathbb{Q}_2(\sqrt{-2})}(2) = \langle x_1,x_2,x_3,x_4\rangle/(x_1^{2+4}[x_2,x_2][x_3,x_4])$

Demushkin groups: p a prime number, G a pro-p group

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- the cup-product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

Examples:

Efrat's Elementary Type Conjecture

- Demushkin groups are the building blocks for groups of elementary type in number theory

i.e., $\cong G_F(p)$ for a field containing μ_p

- not known whether all Demushkin groups are realisable
- $G = \langle x_1, x_2, x_3 \rangle / (x_1^2[x_2, x_3])$ not known to be realisable
- $G = \langle x_1, x_2, x_3, x_4 \rangle / (x_1^3[x_1, x_2][x_3, x_4])$ only Demushkin group of rank four known to be realisable: $G \cong G_{\mathbb{Q}_3}(3)$ (Efrat, Koenigsmann)

A_3-formality for Demushkin groups:

Theorem (Pál-Q.): Let $d \geq 2$ and G be a pro-p Demushkin group with invariant q:
- If $q \neq 3$, then $\mathscr{C}^\bullet(G, \mathbb{F}_p)$ is A_3-formal.
- If $q = p = 3$, then $\mathscr{C}^\bullet(G, \mathbb{F}_3)$ is not A_3-formal.

Strategy for the proof:

- $H^\bullet := H^\bullet(G, \mathbb{F}_p)$ is Koszul algebra (Mináč-Pasini-Quadrelli-Tân)
- $\mathscr{C}^\bullet(G, \mathbb{F}_p)$ is A_3-formal if and only if $[m_3] \in \mathrm{HH}^{3,-1}(H^\bullet)$ is zero.

canonical class of $\mathscr{C}^\bullet(G, \mathbb{F}_p)$

- use Dwyer's Theorem relating Massey products and unipotent representations of G to compute $[m_3] \in \mathrm{HH}^{3,-1}(H^\bullet)$

- use the classification of Demushkin groups and single relation

also considered by Benson-Krause-Schwede for different purposes

The canonical class:

Assume that $\mathscr{C}^\bullet$ is a dga over $\mathbb{F}_p$ with $H^\bullet$ a Koszul algebra, i.e. $K(A) \hookrightarrow B(A)$ is a quasi-iso.

- $H^\bullet = T(H^1)/(R)$ with $R \subset H^1 \otimes H^1$
- $K_3(H^\bullet) = (H^1 \otimes R) \cap (R \otimes H^1) \subset (H^1)^{\otimes 3}$

There are $\mathbb{F}_p$-linear maps

- $f_1 : H^1 \to \operatorname{Ker} \mathscr{C}^1$ which induces $\mathrm{id}_{H^\bullet}$ on cohomology
- $f_2 : R \to \mathscr{C}^1$ st $\delta f_2(x,y) = -f_1(x) \cup f_1(y)$

can check that formula always yields a cocycle

Then we can define the $\mathbb{F}_p$-linear map
- $\kappa_3 : K_3(H^\bullet) \to \operatorname{Ker}(\delta) \subset \mathscr{C}^2$
  by $\kappa_3(x,y,z) = -f_1(x) \cup f_2(y,z) - f_2(x,y) \cup f_1(z)$

and we can check that the induced map $[\kappa_3] : K_3(H^\bullet) \to H^2$ is a cocycle for $\partial : \operatorname{Hom}_{\mathbb{F}_p}(K_\bullet(H^\bullet), H^\bullet[-1])$ which computes Hochschild cohomology $\mathrm{HH}^{\bullet,-1}(H^\bullet)$

Theorem (Kadeishvili): $\mathscr{C}^\bullet$ is A_3-formal if and only if $[\kappa_3] \in \mathrm{HH}^{3,-1}(H^\bullet)$ is zero.

canonical class of $\mathscr{C}^\bullet$

also considered by Benson-Krause-Schwede for different purposes

Dwyer's Theorem:

p a prime, G a pro-p group, $a_1, a_2, a_3 \in H^1(G, \mathbb{F}_p)$

set of defining systems M for $\langle a_1, a_2, a_3 \rangle$

bijection

set of continuous group homomorphisms $\bar{\rho}_M \colon G \to \bar{U}_4(\mathbb{F}_p)$ such that $(i, i+1)$-component = $-a_i$

cont. homomorphisms $G \to \mathbb{F}_p$

cont. maps $G \to \mathbb{F}_p$

$$M = \{a_{12}, a_{23}, a_{34}, a_{13}, a_{24}\}$$

(and $\langle a_1, a_2, a_3 \rangle_M = a_{12} \cup a_{24} + a_{13} \cup a_{34}$)

$$\bar{\rho}_M = \begin{pmatrix} 1 & -a_{12} & -a_{13} & * \\ & 1 & -a_{23} & -a_{24} \\ & & 1 & -a_{34} \\ & & & 1 \end{pmatrix}$$

and $0 = \langle a_1, a_2, a_3 \rangle_M$ if and only if $\bar{\rho}_M$ lifts to ρ_M

$$\rho_M = \begin{pmatrix} 1 & -a_{12} & -a_{13} & -a_{14} \\ & 1 & -a_{23} & -a_{24} \\ & & 1 & -a_{34} \\ & & & 1 \end{pmatrix}$$



group of unitary (4×4)-matrices

centre in $U_4(\mathbb{F}_p)$

Sketch of proof: p a prime number, G a pro-p Demushkin group

- Assume that G is generated by x_1, x_2 with relation $1 = x_1^q[x_1, x_2]$ and $q \neq 2$.
- Let $\chi_i \colon G \to \mathbb{F}_p$ be the generator of $H^1(G, \mathbb{F}_p) = \mathrm{Hom}(G, \mathbb{F}_p)$ st $\chi_i(x_j) = -\delta_{ij}$.

then $H^\bullet(G, \mathbb{F}_p)$ is an exterior algebra on two generators

such that $H^\bullet = T(H^1)/(R)$

to have fewer signs later

- Then $R = \{\chi_1 \otimes \chi_1, \chi_2 \otimes \chi_2, \chi_1 \otimes \chi_2 + \chi_2 \otimes \chi_1\}$ since $\chi_i \cup \chi_i = 0$ and $\chi_1 \cup \chi_2 = -\chi_2 \cup \chi_1$.

can choose identity on $\mathrm{Hom}(G, \mathbb{F}_p)$

- need to construct $f_1 \colon H^1 \to \mathrm{Ker}\, \mathscr{C}^1$, $f_2 \colon R \to \mathscr{C}^1$, and then $\kappa_3 \colon K_3(H^\bullet) \to \mathrm{Ker}(\delta) \subset \mathscr{C}^2$,
$\kappa_3(x, y, z) = -f_1(x) \cup f_2(y, z) - f_2(x, y) \cup f_1(z)$

can be done via Dwyer's Theorem

here of dimension 4; in general of dimension $d^3 - 2d$

- $K_3(H^\bullet) = (H^1 \otimes R) \cap (R \otimes H^1)$

$$= \{\chi_1 \otimes \chi_1 \otimes \chi_1,\ \chi_2 \otimes \chi_2 \otimes \chi_2,\ \chi_1 \otimes \chi_1 \otimes \chi_2 + \chi_1 \otimes \chi_2 \otimes \chi_1 + \chi_2 \otimes \chi_1 \otimes \chi_1,$$

$$\chi_2 \otimes \chi_2 \otimes \chi_1 + \chi_2 \otimes \chi_1 \otimes \chi_2 + \chi_1 \otimes \chi_2 \otimes \chi_2\}$$

pure tensors leading to triple Massey products

do not correspond directly to triple Massey products

Sketch of proof: p a prime number, G a pro-p Demushkin group, $q \neq 2$

- want to check whether $\kappa_3(\chi_1 \otimes \chi_1 \otimes \chi_1) = -\chi_1 \cup f_2(\chi_1 \otimes \chi_1) - f_2(\chi_1 \otimes \chi_1) \cup f_1(\chi_1)$ is a coboundary in $\mathscr{C}^2(G, \mathbb{F}_p)$

this is a cocycle in a triple Massey product

such that $\delta f_2(\chi_1 \otimes \chi_1) = -\chi_1 \cup \chi_1$

- can use Dwyer's Theorem: $\bar{\rho}\colon G \to \bar{U}_4(\mathbb{F}_p)$ with $\bar{\rho} = \begin{pmatrix} 1 & -\chi_1 & f_2(\chi_1 \otimes \chi_1) & * \\ & 1 & -\chi_1 & f_2(\chi_1 \otimes \chi_1) \\ & & 1 & -\chi_1 \\ & & & 1 \end{pmatrix}$

- define $\bar{\rho}$ by $x_1 \mapsto A := \begin{pmatrix} 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$ and $x_2 \mapsto I_4$ in $\bar{U}_4(\mathbb{F}_p)$

Can we lift $\bar{\rho}$ to a cont. homomorphism $\rho\colon G \to U_4(\mathbb{F}_p)$?

- need to check relation $A^q[A, I_4] = I_4$: ✓ always in $\bar{U}_4(\mathbb{F}_p)$

$$A^q = \begin{pmatrix} 1 & q & \binom{q}{2} & \binom{q}{3} \\ & 1 & q & \binom{q}{2} \\ & & 1 & q \\ & & & 1 \end{pmatrix} \neq 0 \text{ in } \mathbb{F}_3 \text{ if } q = 3 \quad \overset{?}{=} I_4$$

✓ in $U_4(\mathbb{F}_p)$ if and only if $q = 0$ or $q = p^f \geq 4$

- computations are more complicated when $d \geq 4$ ...

- check Hochschild coboundaries vanish on $\chi_1 \otimes \chi_1 \otimes \chi_1$

- obstruction for $q = 3$ for every d

Thank you!