A_3-formality and Massey products for Demushkin groups

Topology Seminar
Universitat de Barcelona
16 September 2026

Gereon Quick
NTNU

This is joint work with Ambrus Pál

Norm Residue Isomorphism Theorem:

F field with $\mathrm{char}(F) \neq p$ containing primitive p-th root of unity

Voevodsky, Rost, ...

Milnor K-theory
$T(F^{\times})/(u \otimes (1-u), u \neq 0,1)$

continuous cohomology of absolute Galois group G_F of F with trivial action on $\mathbb{F}_p$

$$K_{\bullet}^{M}(F)/p \xrightarrow{\cong} H^{\bullet}(G_F, \mathbb{F}_p)$$

quadratic algebra

- generators in degree 1
- relations in degree 2

strong restriction on which $\mathbb{F}_p$-algebras can occur as the Galois cohomology of a field

Which quadratic algebras occur as $H^{\bullet}(G_F, \mathbb{F}_p)$?

Additional properties? $H^\bullet(G_F, \mathbb{F}_p)$ quadratic algebra

- For example, are there nontrivial Massey products?

- many vanishing Massey products in arithm. & alg. geometry: Deligne-Griffiths-Morgan-Sullivan, but also many non-vanishing ones: e.g. Ekedahl, Morishita, Sharifi, Vogel, Gärtner, Bleher-Chinburg-Gillibert, Deninger,...

elements in $H^1(G_F, \mathbb{F}_2)$

- Hopkins and Wickelgren: $\langle a_1, a_2, a_3 \rangle \neq \varnothing \iff 0 \in \langle a_1, a_2, a_3 \rangle$

  $p = 2$, F a global field of $\mathrm{char}(F) \neq 2$

  is defined, i.e., $a_1 \cup a_2 = 0 = a_2 \cup a_3$

  Massey product vanishes

Massey products provide an obstruction to formality

- Is $\mathscr{C}^\bullet(G_F, \mathbb{F}_p)$ a formal dga?

continuous Galois cochains

- Do at least all Massey products vanish when they are defined?

Massey vanishing conjecture of $\boxed{\text{Mináč–Tân:}}$

for every field F, all $n \geq 3$, all primes p

✓ **Conjecture:** For $a_1, \ldots, a_n \in H^1(G_F, \mathbb{F}_p)$,

$$\langle a_1, \ldots, a_n \rangle \neq \varnothing \iff 0 \in \langle a_1, \ldots, a_n \rangle.$$

- Efrat-Matzri, Mináč–Tân: all fields, all primes, $n = 3$

  - new examples of profinite groups which are not absolute Galois groups

  - Example: S = free pro-2 group on generators $x_1, \ldots, x_5$

commutator:
$[x, y] = x^{-1}y^{-1}xy$

Mináč–Tân: and relation $1 = [x_4, x_5][[x_2, x_3]x_1]$

For $G = S/\langle r \rangle$, $\mathscr{C}^\bullet(G, \mathbb{F}_2)$ has non-vanishing triple Massey products, and hence G is not the maximal pro-2 quotient of an absolute Galois group

Massey vanishing conjecture of $\boxed{\text{Mináč–Tân}}$:

for every field F, all $n \geq 3$, all primes p

Conjecture: For $a_1, \ldots, a_n \in H^1(G_F, \mathbb{F}_p)$:

$$\langle a_1, \ldots, a_n \rangle \neq \varnothing \iff 0 \in \langle a_1, \ldots, a_n \rangle.$$

✓

- Efrat-Matzri, Mináč-Tân: $\boxed{\text{all fields, all primes,}}$ $n = 3$
- Merkurjev-Scavia: $\boxed{\text{all fields,}}$ $p = 2$, $n = 4$
- Harpaz-Wittenberg: $\boxed{\text{all number fields,}}$ all primes, all $n \geq 3$
- Pál-Szabó: fields with $\mathrm{vcd} \leq 1$
- Quadrelli: elementary type pro-p-groups
- Maire-Mináč-Ramakrishna-Tân:

  number fields with $\mu_p \not\subset K$

all primes, all $n \geq 3$

strong Massey vanishing:

$$0 \in \langle a_1, \ldots, a_n \rangle \iff a_i \cup a_{i+1} = 0$$

for all $i = 1, \ldots, n-1$

Hopkins-Wickelgren Question:

for every field F and all primes p?

Question: Is $\mathscr{C}^\bullet(G_F, \mathbb{F}_p)$ formal?

we then say that F is formal at p

- Yes, for example, for pseudo-algebraically closed fields

The answer is $\boxed{\text{no}}$ in general

- Positselski: $\mathbb{Q}_\ell$, for odd $\ell \neq p$, is not formal at p
- Harpaz-Wittenberg: $\mathbb{Q}$ is not formal for $p = 2$

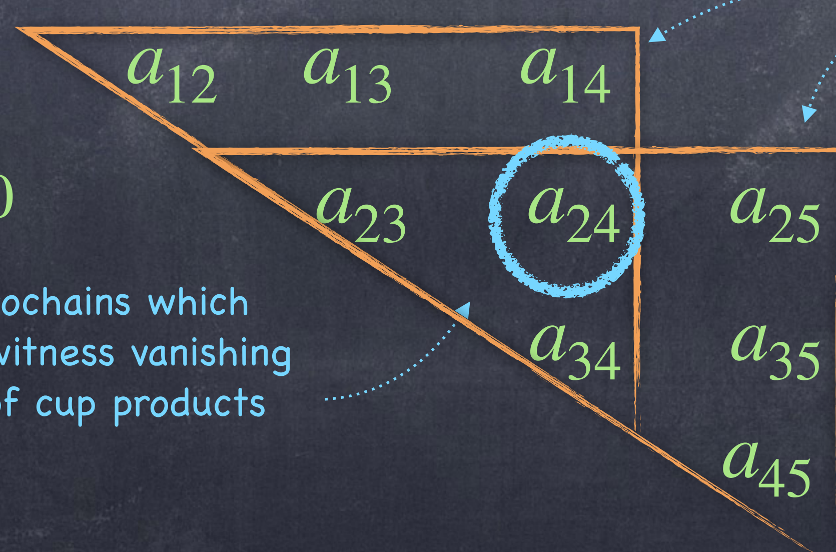
- formality implies all Massey products are defined and vanish when neighbouring cup products are zero

- exist $a_1, a_2, a_3, a_4 \in H^1(\mathbb{Q}, \mathbb{F}_2)$

with $a_1 \cup a_2 = a_2 \cup a_3 = a_3 \cup a_4 = 0$

but $\langle a_1, a_2, a_3, a_4 \rangle$ not defined

cochains witnessing vanishing of triple Massey products



cochains which witness vanishing of cup products

- $\langle a_1, a_2, a_3, a_4 \rangle$ is defined if both $\langle a_1, a_2, a_3 \rangle$ and $\langle a_2, a_3, a_4 \rangle$ vanish ...

with the same choice of a_{24}

Hopkins-Wickelgren Question:

for every field F and all primes p?

$\boxed{\text{Question: Is } \mathscr{C}^\bullet(G_F, \mathbb{F}_p) \text{ formal?}}$

we then say that F is formal at p

- Positselski: $\mathbb{Q}_\ell$, for odd $\ell \neq p$, is not formal at p
- Harpaz-Wittenberg: $\mathbb{Q}$ is not formal for $p = 2$
- Merkurjev-Scavia:
  - every $F/\mathbb{Q}$ of odd degree is not formal for $p = 2$
  - every F of char. $\neq p$ is in some L which is not formal at p

However, there are also important positive cases...

- Theorem (Pál-Q.): If F has virtual cohomological dimension ≤ 1, then $H^\bullet(G_F, \mathbb{F}_p)$ is intrinsically formal.

every dga $\mathscr{C}^\bullet$ over $\mathbb{F}_p$ with $H^\bullet \cong H^\bullet(G_F, \mathbb{F}_p)$ is formal

A_3-formality and Massey products:

we assume all dgas are over a field $\mathbb{F}$

Theorem (Kadeishvili; Merkulov): $\mathscr{C}^\bullet$ a dga with cohomology $H^\bullet$.

Then there is an A_∞-algebra structure $(H^\bullet, \{m_n\}_{n\geq 1})$ such that $m_1 = 0$ and a quasi-isomorphism of A_∞-algebras $H^\bullet \xrightarrow{\simeq} \mathscr{C}^\bullet$.

minimal model for $\mathscr{C}^\bullet$

Recall: A dga $\mathscr{C}^\bullet$ is A_∞-formal if its minimal model satisfies $m_n = 0$ for all $n \geq 3$.

Definition: We say that the dga $\mathscr{C}^\bullet$ is A_3-formal if its minimal model can be chosen such that $m_3 = 0$.

e.g. Buijs—Moreno-Fernández—Murillo

all triple Massey products in $\mathscr{C}^\bullet$ vanish when defined and if $a_1 \cup a_2 = a_2 \cup a_3 = a_3 \cup a_4 = 0$ then $\langle a_1, a_2, a_3, a_4 \rangle$ is defined $\Longleftarrow$ $\mathscr{C}^\bullet$ is A_3-formal $\Longleftarrow$ $\mathscr{C}^\bullet$ is A_∞-formal

- provides a stronger obstruction than vanishing of triple Massey products, which vanishes for certain classes of profinite groups...

Demushkin groups: p a prime number, G a pro-p group

Definition: G is called a Demushkin group if

- $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty$,
- $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1$,
- the cup-product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

- the only finite Demushkin group is $\mathbb{Z}/2\mathbb{Z}$
- set $d := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ and $q := 0$ if $G^{\mathrm{ab}} \cong \mathbb{Z}_p^d$ and $q := p^f$ if $G^{\mathrm{ab}} \cong \mathbb{Z}/p^f\mathbb{Z} \times \mathbb{Z}_p^{d-1}$
- d and q are invariants determining G

Theorem (Pál-Q.): Let $d \geq 2$ and G be a pro-p Demushkin group with invariant q:

- If $q \neq 3$, then $\mathscr{C}^\bullet(G, \mathbb{F}_p)$ is A_3-formal.
- If $q = p = 3$, then $\mathscr{C}^\bullet(G, \mathbb{F}_3)$ is not A_3-formal.

- However, by Positselski such G is not A_∞-formal for $q \neq 2$ and $d = 2$

Demushkin groups: p a prime number, G a pro-p group

Definition: G is called a Demushkin group if
- $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty$,
- $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1$,
- the cup-product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

Classification:
- $d := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ and $q = 0$ if $G^{\mathrm{ab}} \cong \mathbb{Z}_p^d$ and $q = p^f$ if $G^{\mathrm{ab}} \cong \mathbb{Z}/p^f\mathbb{Z} \times \mathbb{Z}_p^{d-1}$

Theorem (Demushkin, Serre, Labute): G is isomorphic to the pro-p group with generators $x_1, \ldots, x_d$ subject to a single relation r:

commutator: $[x, y] = x^{-1}y^{-1}xy$

- $q \neq 2$: $d \geq 2$ even, $r = x_1^q[x_1, x_2][x_3, x_4]\cdots[x_{d-1}, x_d]$
- $q = 2$: $d \geq 3$ odd, $r = x_1 x_2^{2^f}[x_2, x_3]\cdots[x_{d-1}, x_d]$ with $f \geq 2$ or $2^f = 1$
- $q = 2$: $d \geq 2$ even, $r = x_1^{2+2^f}[x_1, x_2]\cdots[x_{d-1}, x_d]$ with $f \geq 2$ or $2^f = 1$
- $q = 2$: $d \geq 4$ even, $r = x_1^2[x_1, x_2]x_3^{2^f}[x_3, x_4]\cdots[x_{d-1}, x_d]$ with $f \geq 2$

Demushkin groups: p a prime number, G a pro-p group

Definition: G is called a Demushkin group if

- $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty$,
- $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1$,
- the cup-product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

Examples:

- $d := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ and $q = 0$ if $G^{\mathrm{ab}} \cong \mathbb{Z}_p^d$ and $q = p^f$ if $G^{\mathrm{ab}} \cong \mathbb{Z}/p^f\mathbb{Z} \times \mathbb{Z}_p^{d-1}$

- $G = \mathbb{Z}_p \times \mathbb{Z}_p$ with $q = 0$ (this G is even formal)

- pro-p completion of the fundamental group of compact orientable surface of genus g, $d = 2g$ and $q = 0$ is A_3-formal

- $G = \mathbb{Z}_p \rtimes \mathbb{Z}_p = \langle x_1, x_2 \rangle / (x_1^q [x_2, x_2])$ is A_3-formal iff $q \neq 3$

for this type of G

- However, for every $q \neq 2$, $H^\bullet(G, \mathbb{F}_p)$ is an exterior algebra on two generators

Demushkin groups: p a prime number, G a pro-p group

Definition: G is called a Demushkin group if

- $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty$,
- $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1$,
- the cup-product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

Examples:

- $d := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ and $q = 0$ if $G^{\mathrm{ab}} \cong \mathbb{Z}_p^d$ and $q = p^f$ if $G^{\mathrm{ab}} \cong \mathbb{Z}/p^f\mathbb{Z} \times \mathbb{Z}_p^{d-1}$

- $K/\mathbb{Q}_\ell$ of finite degree, $G_K(p)$:= maximal pro-p quotient of G_K, $\ell \neq p$

  Then $G_K(p)$ is Demushkin if $\mu_p \subset K$ with

  - $d = 2$, $q = p^f$ maximal power such that a primitive qth root of unity is in K

  $$G_K(p) = \langle x_1, x_2 \rangle / (x_1^q [x_2, x_2])$$

- $K/\mathbb{Q}_p$ of finite degree n; if $\mu_p \subset K$ then $G_K(p)$ is Demushkin with

  - $d = n + 2$, $q = p^f$ maximal power such that a primitive qth root of unity is in K

  two examples with $q = 2$:

  - $G_{\mathbb{Q}_2}(2) = \langle x_1, x_2, x_3 \rangle / (x_1^2 x_2^4 [x_2, x_2])$
  - $G_{\mathbb{Q}_2(\sqrt{-2})}(2) = \langle x_1, x_2, x_3, x_4 \rangle / (x_1^{2+4} [x_2, x_2][x_3, x_4])$

Demushkin groups: p a prime number, G a pro-p group

Definition: G is called a Demushkin group if
- $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty$,
- $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1$,
- the cup-product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

Examples:

Efrat's Elementary Type Conjecture

- Demushkin groups are the building blocks for groups of elementary type in number theory

i.e., $\cong G_F(p)$ for a field containing μ_p

- not known whether all Demushkin groups are realisable
- $G = \langle x_1, x_2, x_3 \rangle / (x_1^2 [x_2, x_3])$ not known to be realisable
- $G = \langle x_1, x_2, x_3, x_4 \rangle / (x_1^3 [x_1, x_2][x_3, x_4])$ only Demushkin group of rank four known to be realisable: $G \cong G_{\mathbb{Q}_3}(3)$ (Efrat, Koenigsmann)

A_3-formality for Demushkin groups:

Theorem (Pál-Q.): Let $d \geq 2$ and G be a pro-p Demushkin group with invariant q:
- If $q \neq 3$, then $\mathscr{C}^\bullet(G, \mathbb{F}_p)$ is A_3-formal.
- If $q = p = 3$, then $\mathscr{C}^\bullet(G, \mathbb{F}_3)$ is not A_3-formal.

Strategy for the proof:

also considered by Benson-Krause-Schwede for different purposes

- $H^\bullet := H^\bullet(G, \mathbb{F}_p)$ is Koszul algebra (Mináč-Pasini-Quadrelli-Tân)
- $\mathscr{C}^\bullet(G, \mathbb{F}_p)$ is A_3-formal if and only if $[m_3] \in \mathrm{HH}^{3,-1}(H^\bullet)$ is zero.

canonical class of $\mathscr{C}^\bullet(G, \mathbb{F}_p)$

- use Dwyer's Theorem relating Massey products and unipotent representations of G to compute $[m_3] \in \mathrm{HH}^{3,-1}(H^\bullet)$
- use the classification of Demushkin groups and single relation

Thank you!