On counting and adding points quadratically

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This is joint work with
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Glen Matthew Wilson UiT

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The Brouwer degree:

unit sphere in the plane
of complex numbers

$$\mathbb{S}^1 \longrightarrow \mathbb{S}^1$$

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for every integer n

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{f_n} & \mathbb{S}^1 \\ z & \longmapsto & z^n \end{array}$$

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f_n $\longleftarrow$ n

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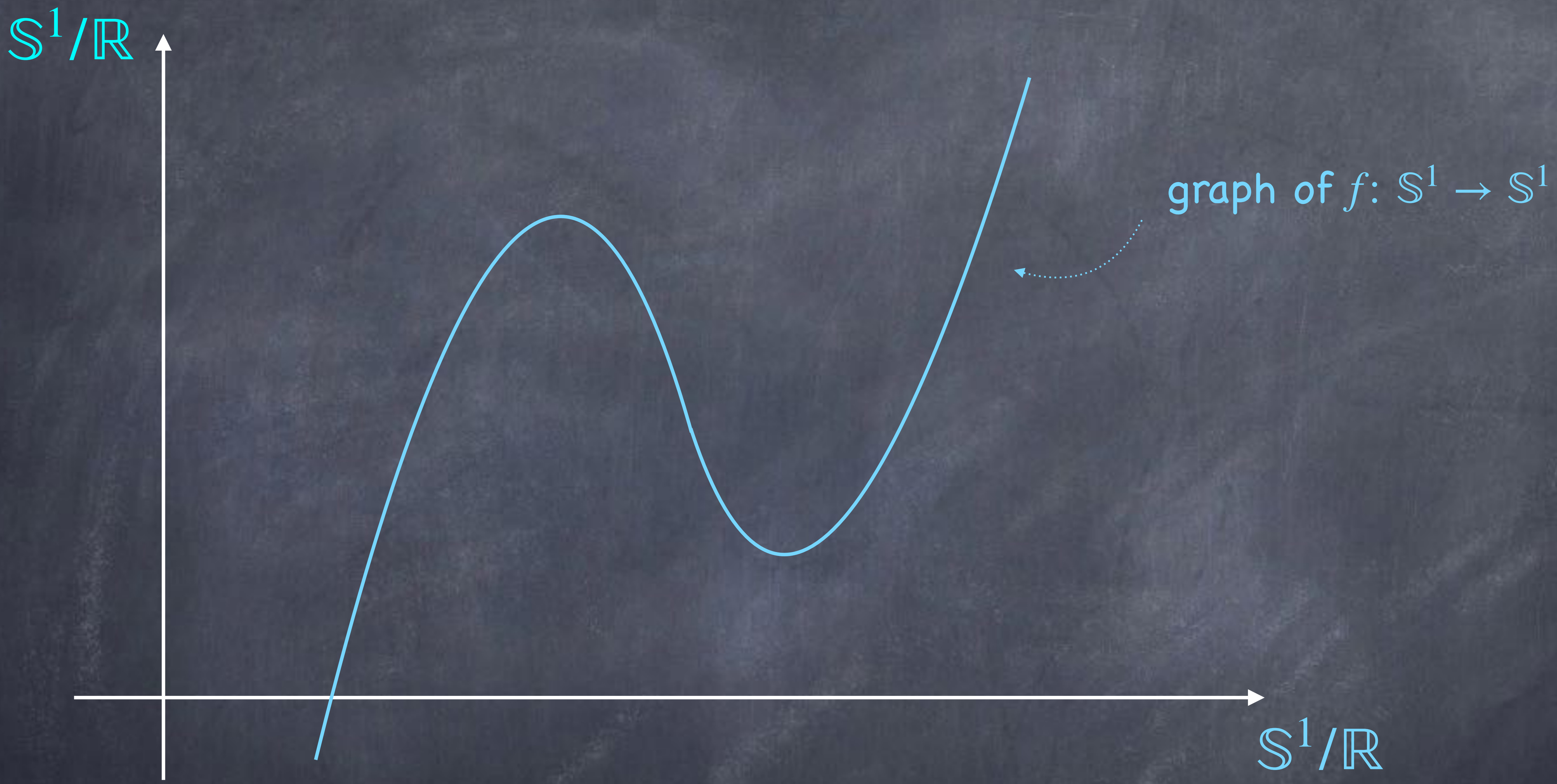
$\boxed{\text{Brouwer degree}}$

Brouwer degree in differential topology:



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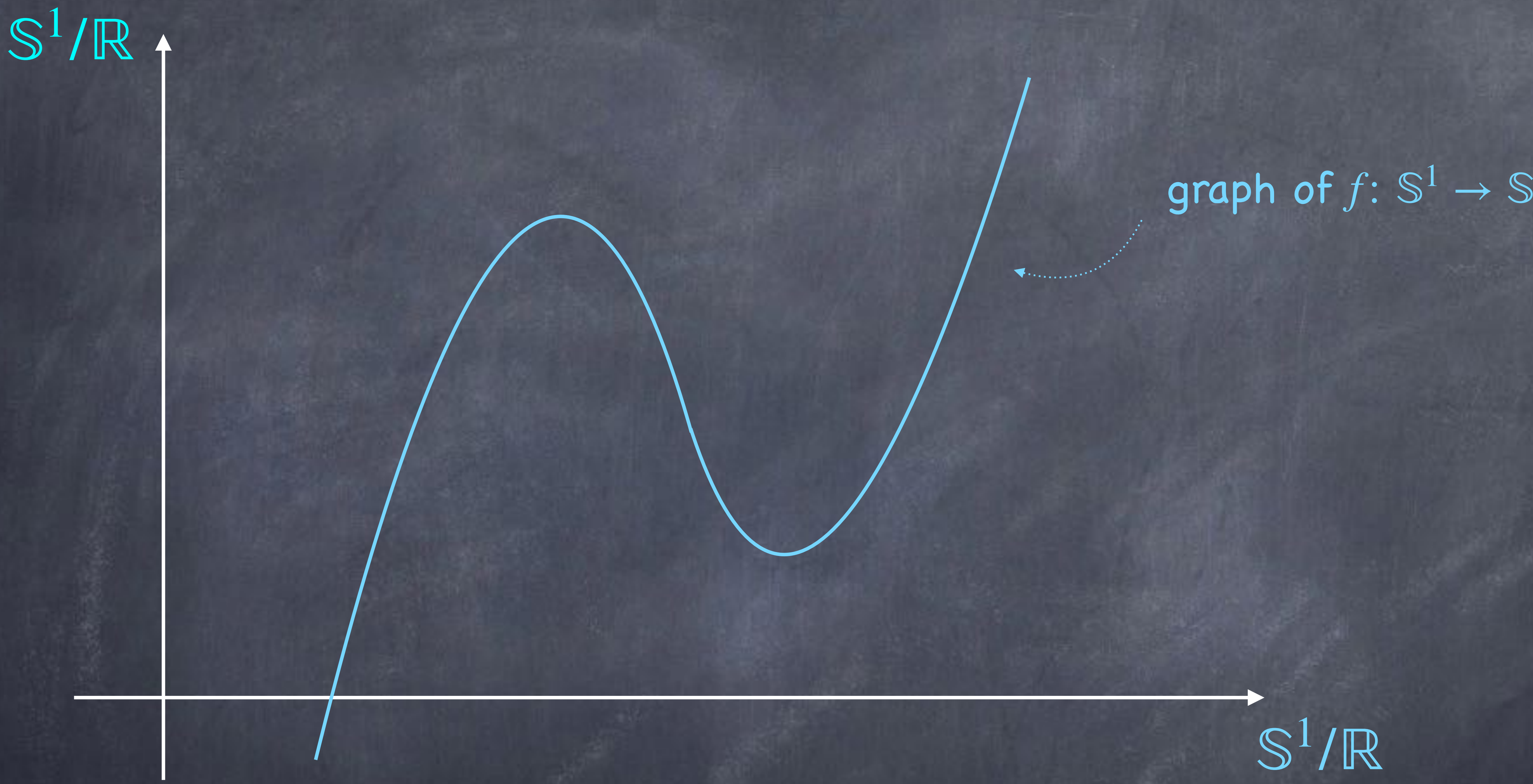

degree of f = $\# f^{-1}(y)$

any regular value

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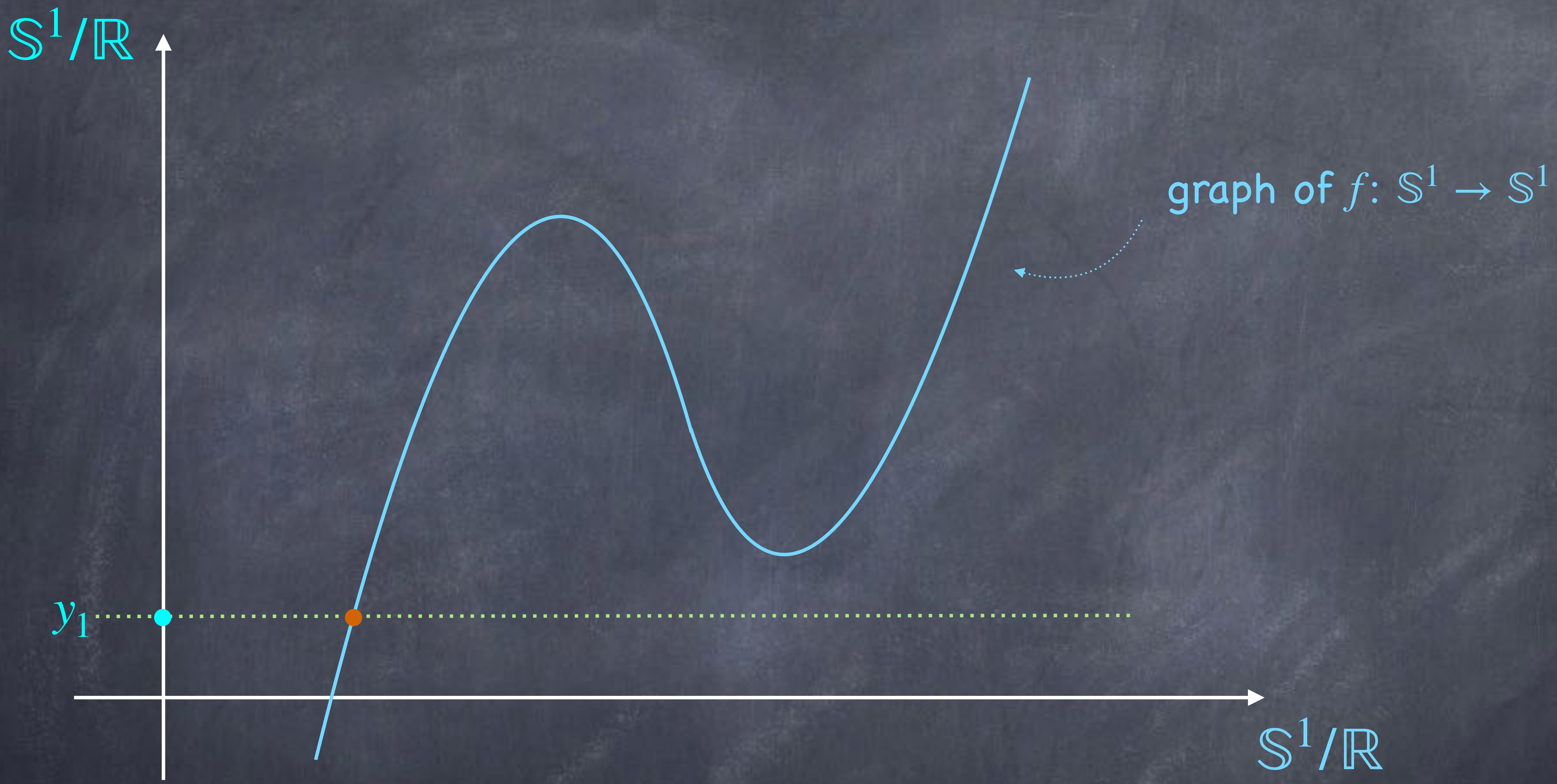

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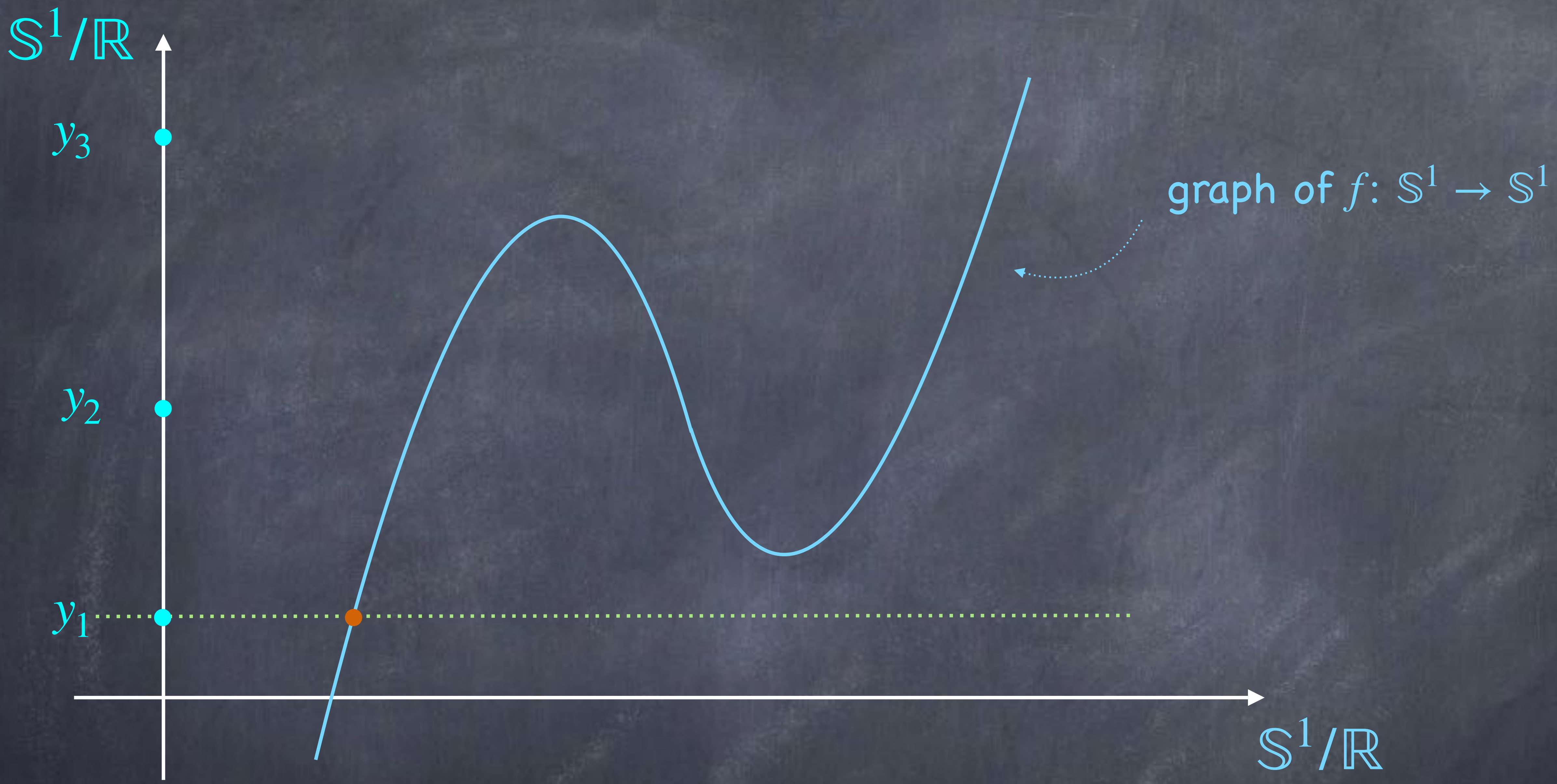

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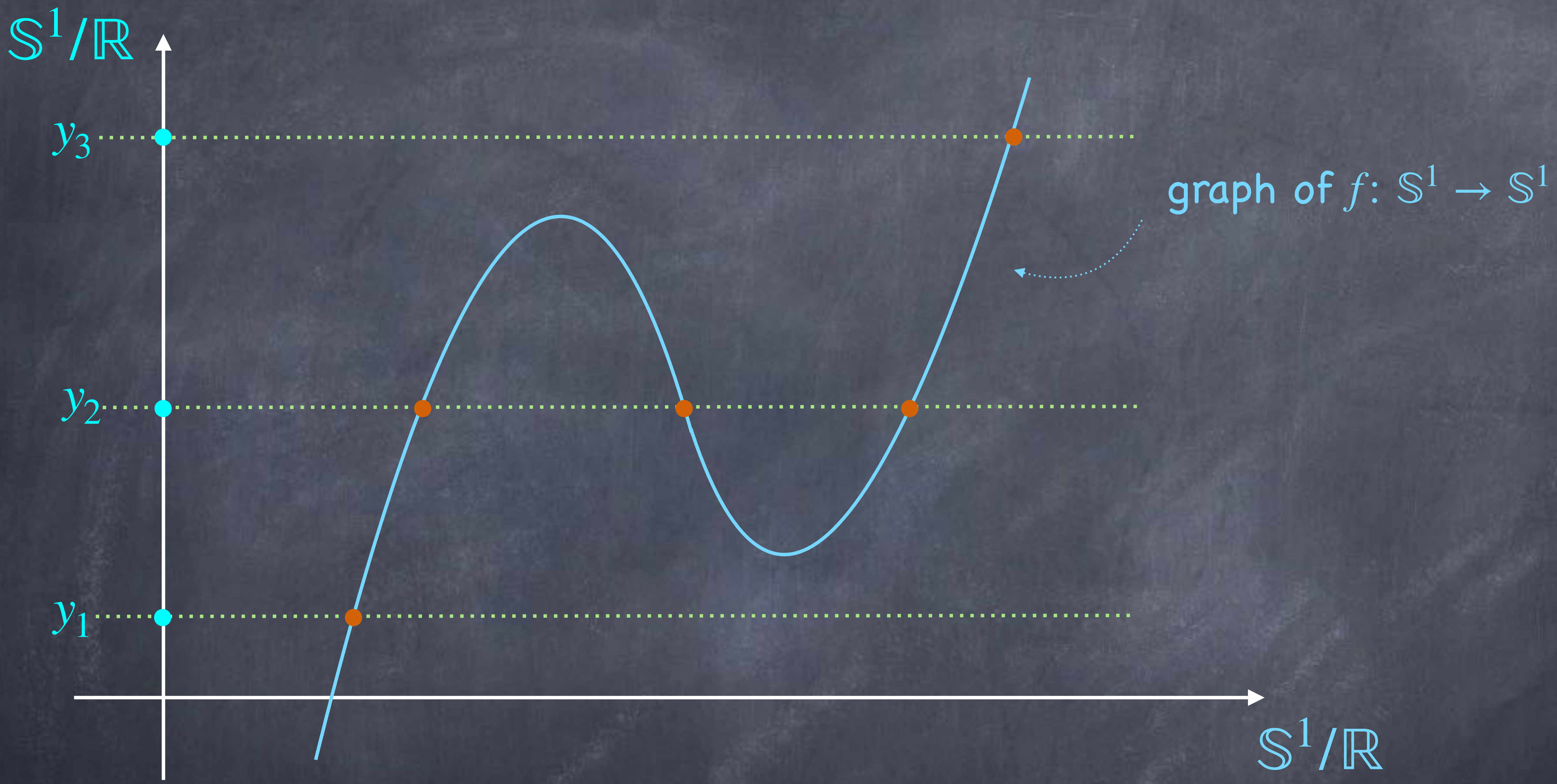

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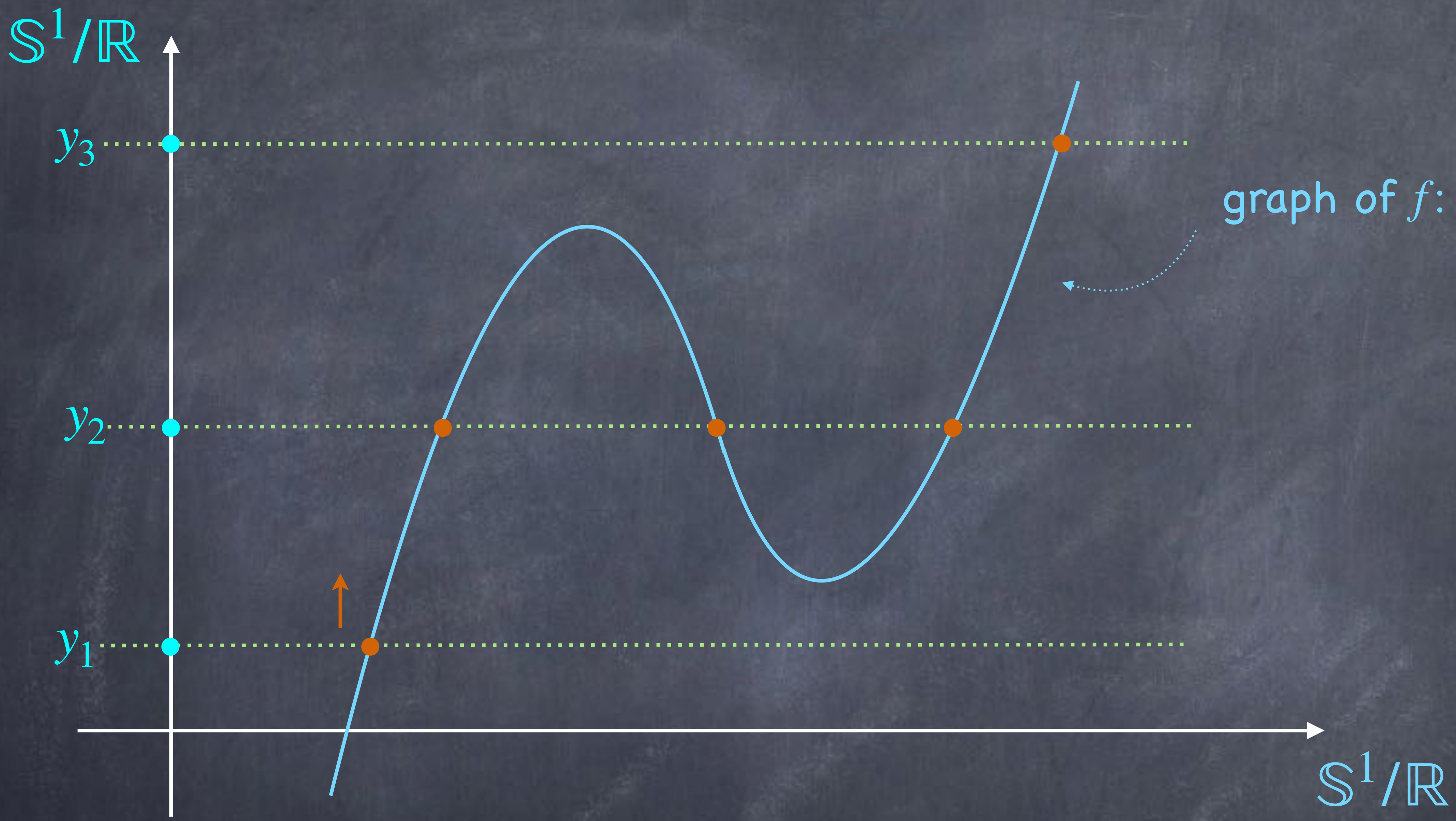

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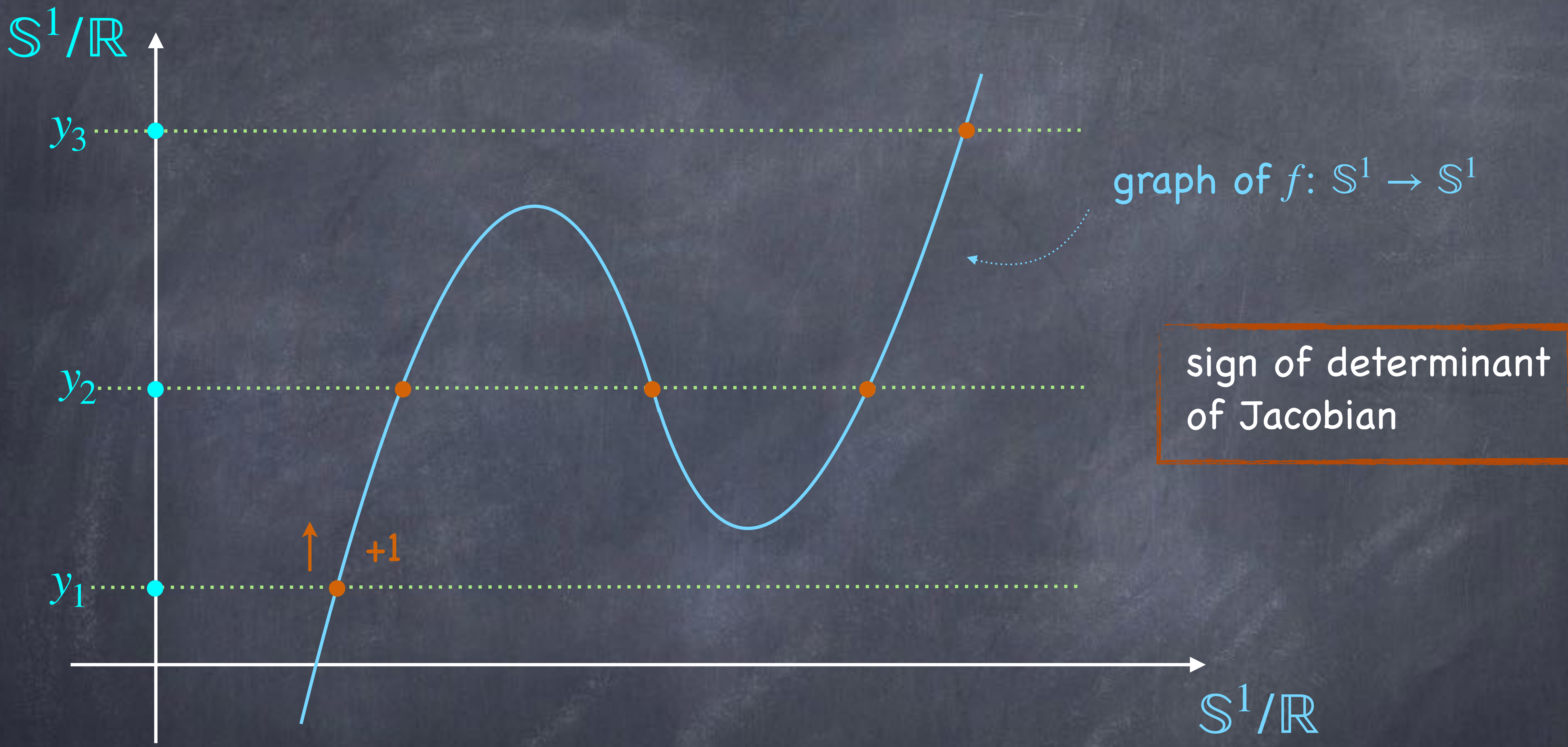

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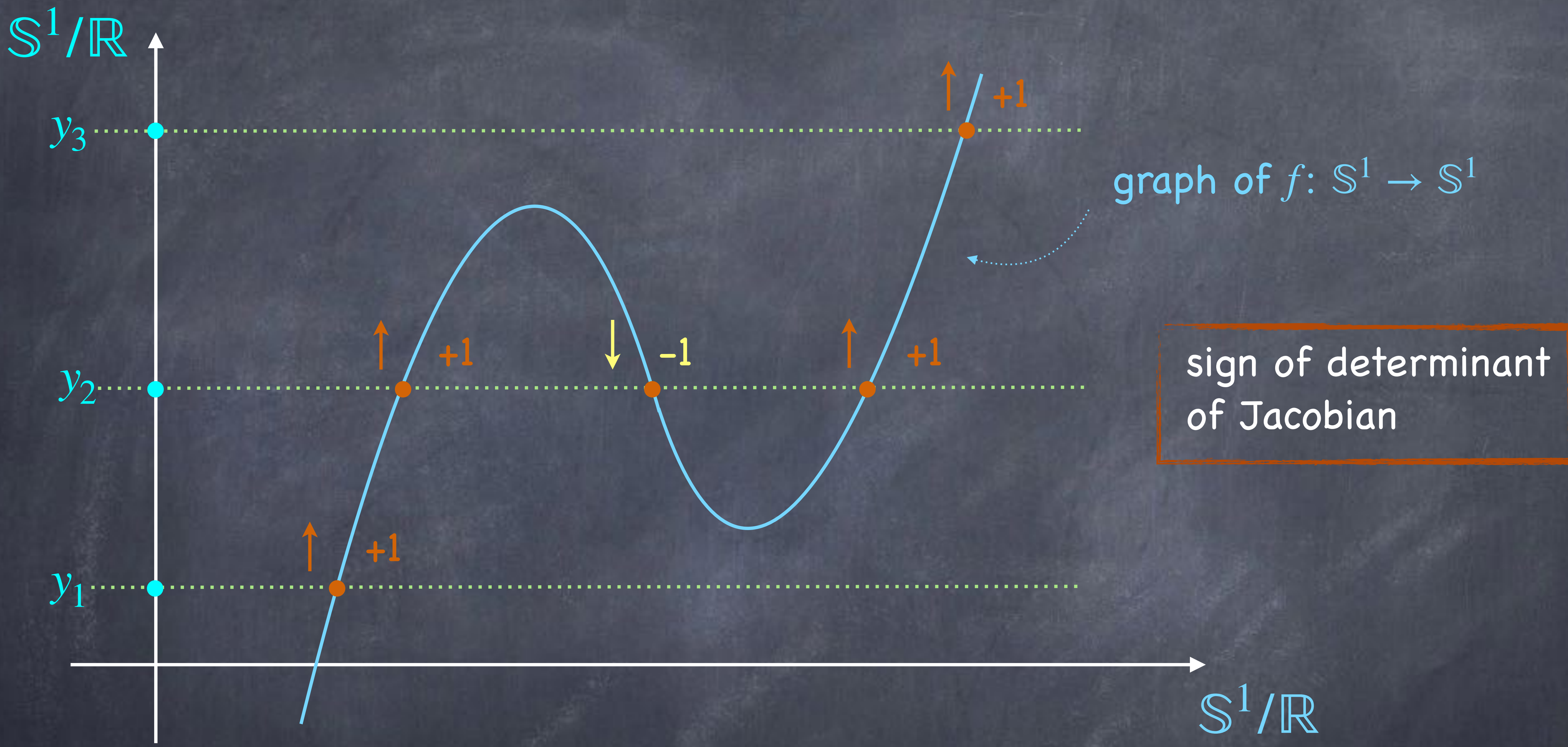

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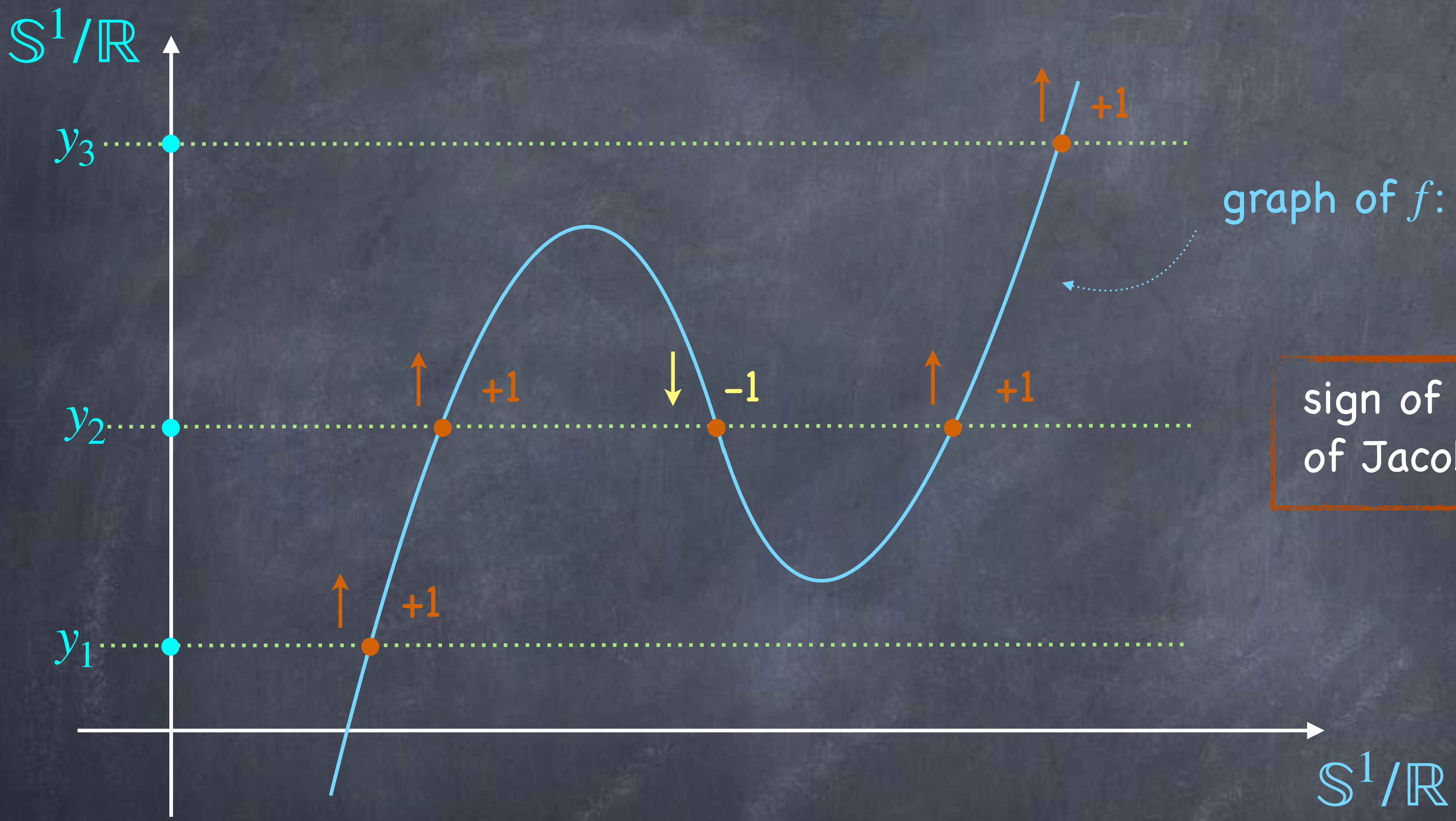

sign of determinant of Jacobian

degree of f = $\# f^{-1}(y)$

counted with signs

any regular value

The Hopf's Theorem:

n-dimensional sphere

counted with signs

homotopy classes of pointed maps

$$
\begin{array}{ccc}
[\mathbb{S}^n, \mathbb{S}^n] & \longrightarrow & \mathbb{Z} \\
[f] & \longmapsto & \deg(f) = \# f^{-1}(y)
\end{array}
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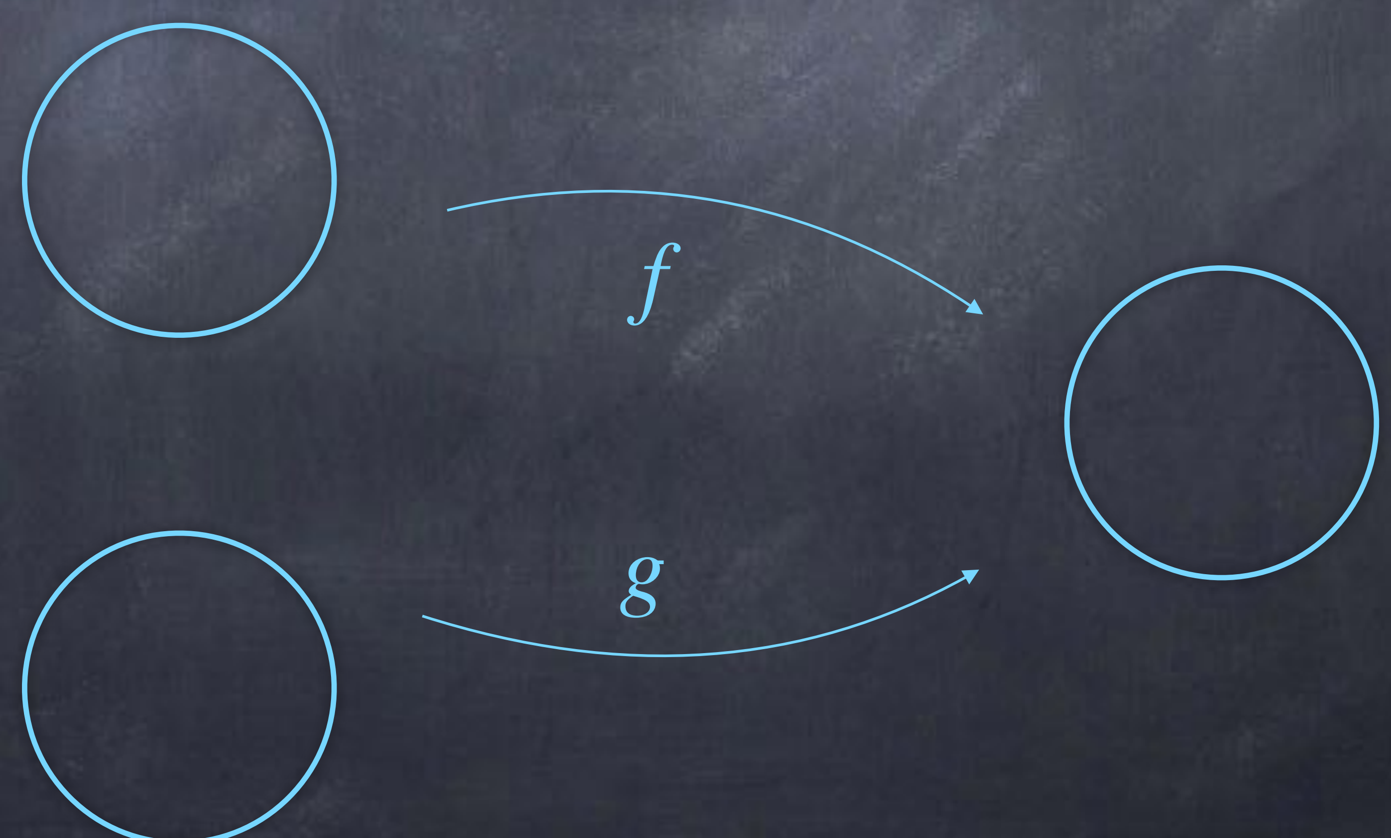

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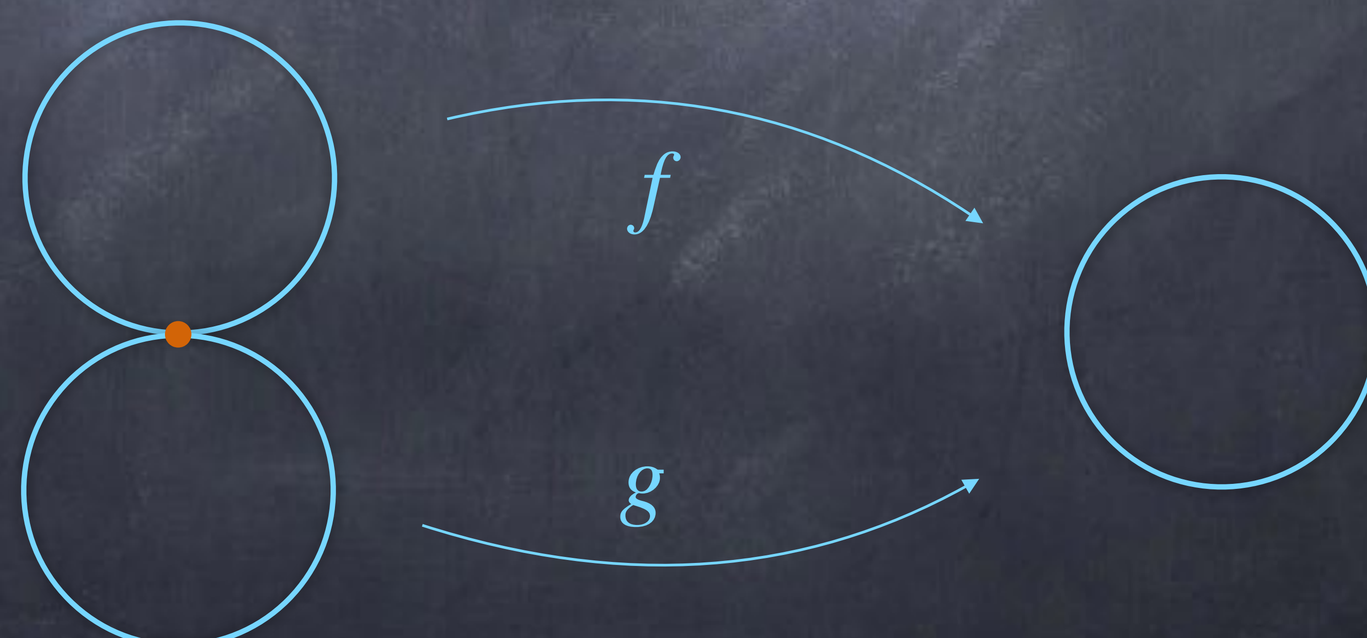

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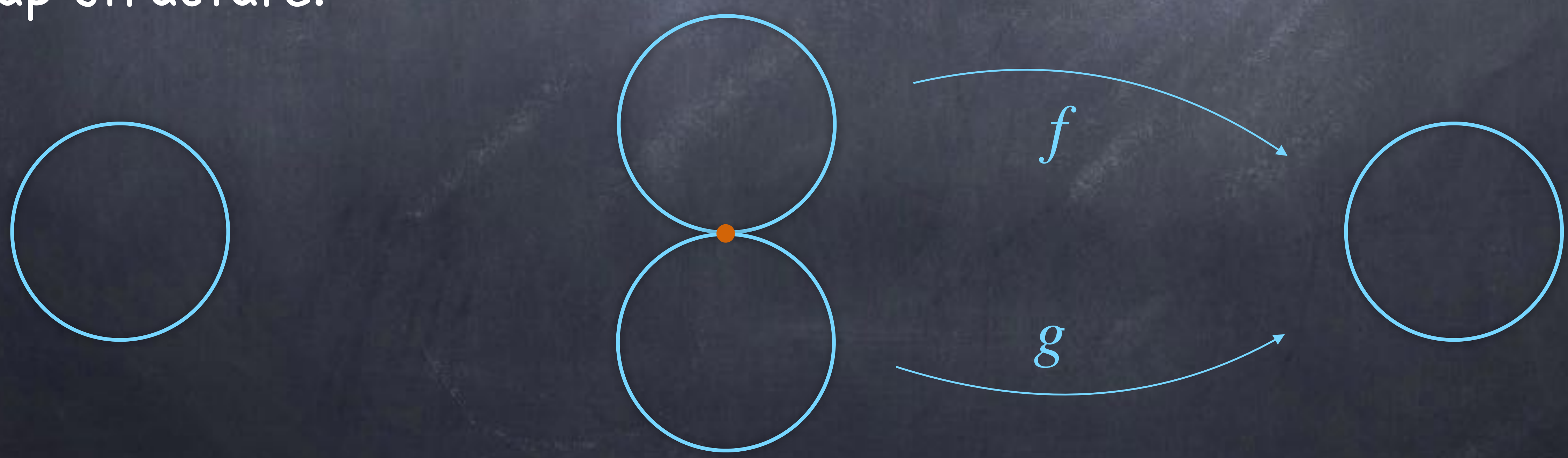

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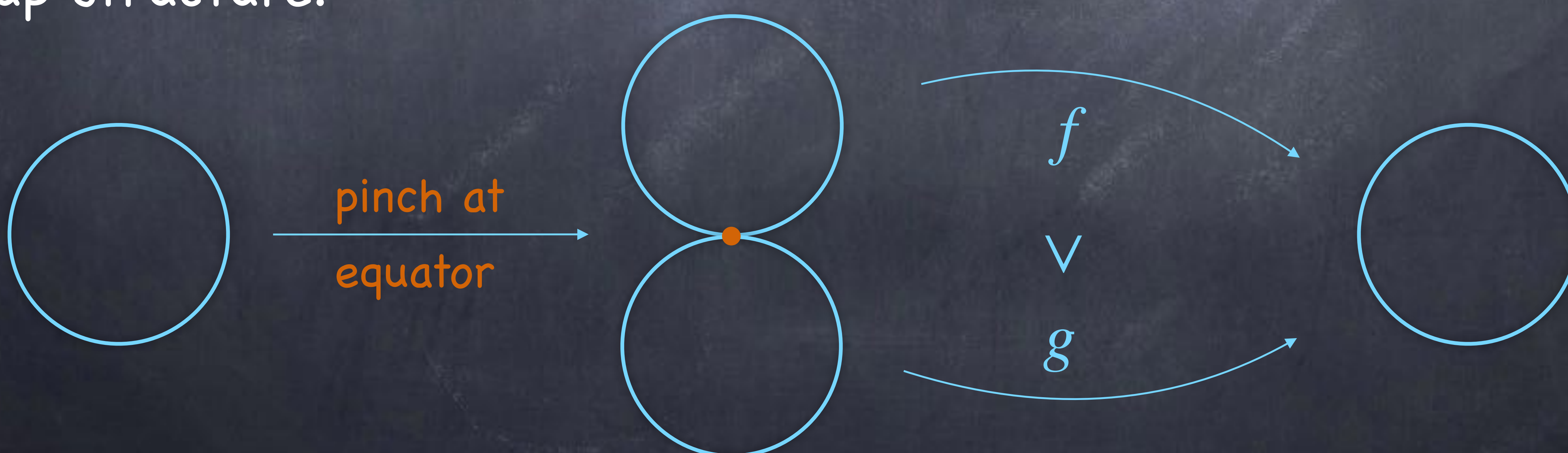

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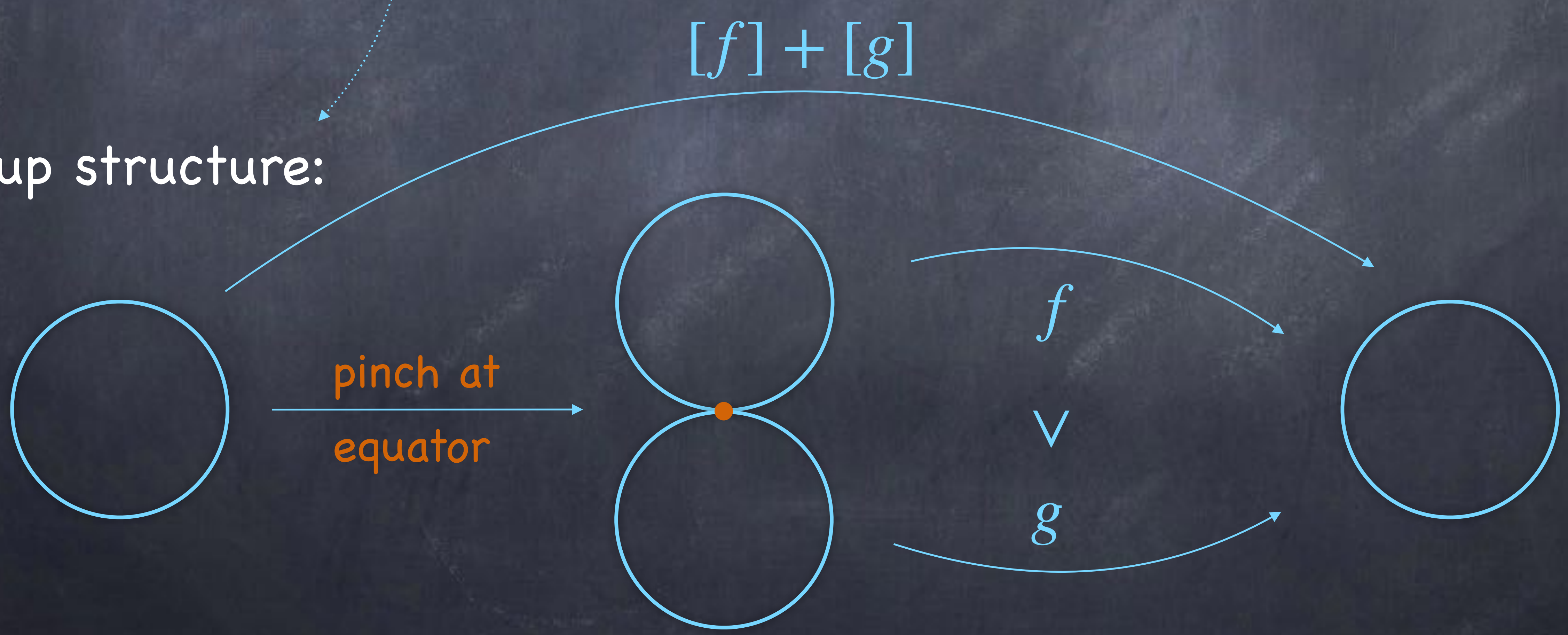

for simplicity

Degree in algebraic geometry:

- k a field of char $\neq 2$
- $\mathbb{P}^1$ projective line over k

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

$$[x_0 : x_1]$$

Degree in algebraic geometry:

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- k a field of char $\neq 2$
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- group $k^\times/(k^\times)^2$ replaces orientations $\{\pm 1\}$ in $\mathbb{R}$

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$\mathrm{Map}(\mathbb{P}^1, \mathbb{P}^1)$ $GW(k)$

Grothendieck-Witt group of quadratic forms over k

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modulo some homotopy relation such that
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$$\begin{array}{ccc} f_u & & \langle u \rangle \\ \mathrm{Map}(\mathbb{P}^1, \mathbb{P}^1) & \rightleftarrows & GW(k) \\ f & & \deg_{\mathbb{A}^1}(f) \end{array}$$

Grothendieck-Witt group of quadratic forms over k

$$= \sum_x \mathrm{Tr}\langle \det(\partial f_i / \partial x_j) \rangle$$

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algebraic Brouwer degree

Morel's Theorem: • k a field • $\mathbb{P}^1$ projective line over k

$$[f_u] \longleftrightarrow \langle u \rangle$$

$$[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \xrightarrow{\deg_{\mathbb{A}^1}} GW(k)$$

$\mathbb{A}^1$-homotopy classes of pointed morphisms

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• $\deg_{\mathbb{A}^1}$ is an $\boxed{\text{epimorphism}}$ with kernel $(k^\times)^2$

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$\cong GW(k) \times_{k^\times/(k^\times)^2} k^\times$

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{[f_u]} & & \langle u \rangle \arrow[ll, bend right, dotted] \\
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- $\deg_{\mathbb{A}^1}$ stabilises to an $\boxed{\text{isomorphism}}$

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- $\deg_{\mathbb{A}^1}$ is an epimorphism with kernel $(k^\times)^2$
- $\deg_{\mathbb{A}^1}$ stabilises to an isomorphism

Why is this an important computation?

Dold–Puppe: $\mathscr{C}$ a nice category $X \in \mathscr{C}$ a nice object

dual object unit object

$$\begin{tikzcd}[column sep=large] 1_{\mathscr{C}} \arrow[r, "\text{coevaluation}"] \arrow[rrr, bend right, dotted] & X \otimes X^* \arrow[r] & X^* \otimes X \arrow[r, "\text{evaluation}"] & 1_{\mathscr{C}} \end{tikzcd}$$

$\in \mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, 1_{\mathscr{C}})$

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1_{\mathscr{C}} \xrightarrow{\text{coevaluation}} X \otimes X^* \longrightarrow X^* \otimes X \xrightarrow{\text{evaluation}} 1_{\mathscr{C}}
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$\boxed{\text{Euler characteristic } \chi(X)} \in \mathrm{Hom}_{\mathscr{C}}(1_{\mathscr{C}}, 1_{\mathscr{C}})$

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- X a topological space $\mathscr{C}$ = stable homotopy category

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“The degree tells us where invariants live.”

$\mathbb{A}^1$- homotopy category of Morel-Voevodsky:

Sm_k

smooth algebraic varieties over k

$\mathbb{A}^1$- homotopy category of Morel-Voevodsky:

Sm_k $\longrightarrow$ universal homotopy category $\mathscr{H}_{\mathbb{A}^1}(k)$

smooth algebraic varieties over k

$\mathbb{A}^1$- homotopy category of Morel-Voevodsky:

Dugger

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Dugger

Sm_k $\boxed{\text{universal}}$ homotopy category $\mathscr{H}_{\mathbb{A}^1}(k)$

smooth algebraic varieties over k

such that

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- limits and colimits exist

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- $X \times \mathbb{A}^1 \to X$ is an equivalence for all X

Milnor conjecture: k field with $\mathrm{char}(k) \neq 2$

Milnor K-theory
$T(k^\times)/(a \otimes (1-a), a \neq 0,1)$

continuous Galois cohomology

Voevodsky

$$K_*^M(k)/2 \xrightarrow{\cong} H^*(k, \mathbb{F}_2)$$

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quadratic algebra

- generators in degree 1
- relations in degree 2

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quadratic algebra

- generators in degree 1
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strong restriction on which $\mathbb{F}_2$-algebras can occur as the Galois cohomology of a field

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X Y

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$$\begin{tikzcd} & Z_1 & & \\ X \arrow[ur] & & Z_2 \arrow[ul, "\sim"'] & & Y \end{tikzcd}$$

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$$\begin{tikzcd}[column sep=small]
 & Z_1 & & Z_3 & & \cdots & & Z_n & \\
X \arrow[ur] & & Z_2 \arrow[ul, "\sim"'] \arrow[ur] & & Z_4 \arrow[ul, "\sim"'] \arrow[ur] & & Z_{n-1} \arrow[ul, "\sim"'] \arrow[ur] & & Y \arrow[ul, "\sim"']
\end{tikzcd}$$

$\cdots$

smooth algebraic varieties

Naive $\mathbb{A}^1$-homotopy:

morphisms of varieties

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

Naive $\mathbb{A}^1$-homotopy:

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$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \qquad \simeq ?$$

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$$X \times \mathbb{A}^1 \xrightarrow{H} Y \quad \text{with} \quad \begin{cases} H(0) = f \\ H(1) = g \end{cases}$$

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f and g are $\boxed{\text{naively}}$ homotopic

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\end{tikzcd}
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$$[X, Y]_{\mathrm{N}} := \mathrm{Hom}_{\mathrm{Sm}_k}(X, Y)$$

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Naive homotopy may produce too few equivalence classes

Group completion:

group

Morel

$[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \cong GW(k) \times_{k^\times/(k^\times)^2} k^\times$

gen. by $(\pm\langle u\rangle, v)$, $u, v \in k^\times$

genuine

$\mathbb{A}^1$-homotopy classes

Group completion:

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Group completion:

Cazanave

monoid

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Group completion:

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$$\underset{k^\times/(k^\times)^2}{MW(k) \times k^\times} \cong [\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}} \xrightarrow[\text{canonical map}]{} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \cong GW(k) \times_{k^\times/(k^\times)^2} k^\times$$

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$$\underset{k^\times/(k^\times)^2}{MW(k) \times k^\times} \cong [\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}} \xrightarrow[\text{canonical map}]{\text{group completion}} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \cong GW(k) \times_{k^\times/(k^\times)^2} k^\times$$

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Group completion:

Asok-
Hoyois-
Wendt

Cazanave

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$$\underset{k^\times/(k^\times)^2}{MW(k) \times k^\times} \cong [\mathbb{P}^1, \mathbb{P}^1]_N \xrightarrow[\text{canonical map}]{\text{group completion}} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \cong GW(k) \times_{k^\times/(k^\times)^2} k^\times$$

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Group completion:

Jouanolou device

$[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}$

Asok-Hoyois-Wendt

Cazanave

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$$MW(k) \underset{k^\times/(k^\times)^2}{\times} k^\times \cong [\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}} \xrightarrow[\text{canonical map}]{\text{group completion}} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \cong GW(k) \times_{k^\times/(k^\times)^2} k^\times$$

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naive homotopy classes

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Group completion:

Jouanolou device

Asok-Hoyois-Wendt

$$\begin{tikzcd}
 & {[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}} \arrow[d] \\
{[\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}}} \arrow[r, "\text{group completion}", "\text{canonical map}"'] & {[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}}
\end{tikzcd}$$

bijection of sets

Cazanave

monoid

group   Morel

$MW(k) \times_{k^\times/(k^\times)^2} k^\times \cong [\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}}$

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Group completion:

Jouanolou device

naive homotopy classes

Asok-Hoyois-Wendt

$$\begin{array}{c} [\mathcal{J}, \mathbb{P}^1]_{\mathrm{N}} \\ \downarrow \\ [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \end{array}$$

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Group completion:

Question: Is there a group structure on $[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}$?

Jouanolou device

naive homotopy classes

Asok-Hoyois-Wendt

$[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}$

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Jouanolou device

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Question: Is there a group structure on $[\mathscr{J}, \mathbb{P}^1]_N$?

Jouanolou device

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Asok-Hoyois-Wendt

$$[\mathscr{J}, \mathbb{P}^1]_N$$

group completion ?

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Jouanolou device:

we define the polynomial ring

$$R = \frac{k[x, y, z, w]}{(x + w - 1,\ xw - yz)}$$

Jouanolou device:

we define the polynomial ring

imagine as 2x2-matrices $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with

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trace = 1 det = 0

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set $\mathcal{J} := \mathrm{Spec}(R)$

$$\begin{array}{c} \mathcal{J} \\ \big\downarrow \pi \\ \mathbb{P}^1 \end{array}$$

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set $\mathscr{J} := \mathrm{Spec}(R)$ $\boxed{\text{affine space}}$

$$\begin{array}{c} \mathscr{J} \\ \Big\downarrow \pi \\ \mathbb{P}^1 \end{array}$$

$\boxed{\mathbb{A}^1\text{-weak equivalence}}$

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$$\mathcal{J} \xrightarrow{\pi} \mathbb{P}^1$$

$\mathbb{A}^1$-weak equivalence

Asok-Hoyois-Wendt

bijections of sets

$$[\mathcal{J}, \mathbb{P}^1]_{\mathrm{N}} \cong [\mathcal{J}, \mathbb{P}^1]_{\mathbb{A}^1}$$

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Asok-Hoyois-Wendt

$\boxed{\text{bijections of sets}}$

$$[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}} \cong [\mathscr{J}, \mathbb{P}^1]_{\mathbb{A}^1}$$

$$\cong [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}$$

Morphisms to projective line:

$$\mathscr{J} \xrightarrow{\text{morphism}} \mathbb{P}^1$$

Morphisms to projective line:

1-1 correspondence

$$\mathscr{J} \xrightarrow{\text{morphism}} \mathbb{P}^1$$

Morphisms to projective line:

1-1 correspondence

$$\mathscr{J} \xrightarrow{\text{morphism}} \mathbb{P}^1$$

Morphisms to projective line:

$$\begin{tikzcd} \mathscr{L} \arrow[d] & \text{line bundle} + \text{two generating sections} \\ \mathscr{J} \arrow[r, "\text{morphism}"] & \mathbb{P}^1 \end{tikzcd}$$

1-1 correspondence

Exact sequences:

$$\begin{tikzcd}
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\mathscr{I} \arrow[r, "\text{morphism}"] & \mathbb{P}^1
\end{tikzcd}$$

1-1 correspondence

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1-1 correspondence

$$1 \longrightarrow [\mathbb{P}^1, \mathbb{A}^2 \backslash 0]_{\mathbb{A}^1} \longrightarrow [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \longrightarrow \mathrm{Pic}(\mathbb{P}^1) \longrightarrow 1$$

exact sequence of groups

Exact sequences:

$$\begin{array}{ccccc} \mathscr{L} & \text{line bundle} & + & \text{two generating sections} & \\ \downarrow & & & & \\ \mathscr{J} & & \xrightarrow{\text{morphism}} & & \mathbb{P}^1 \end{array}$$

1-1 correspondence

exact sequence of sets

$$1 \longrightarrow [\mathscr{J}, \mathbb{A}^2 \backslash 0]_N \longrightarrow [\mathscr{J}, \mathbb{P}^1]_N \longrightarrow \mathrm{Pic}(\mathscr{J}) \longrightarrow 1$$

$$1 \longrightarrow [\mathbb{P}^1, \mathbb{A}^2 \backslash 0]_{\mathbb{A}^1} \longrightarrow [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \longrightarrow \mathrm{Pic}(\mathbb{P}^1) \longrightarrow 1$$

exact sequence of groups

Exact sequences:

$$\begin{tikzcd}
\mathscr{L} \arrow[d] & \text{line bundle} + \text{two generating sections} \arrow[d, leftrightarrow, dotted, "\text{1-1 correspondence}"] & \\
\mathscr{J} \arrow[rr, "\text{morphism}"] & & \mathbb{P}^1
\end{tikzcd}$$

exact sequence of sets

$$\begin{tikzcd}
1 \arrow[r] & {[\mathscr{J}, \mathbb{A}^2 \backslash 0]_{\mathrm{N}}} \arrow[r] \arrow[d] & {[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}} \arrow[r] \arrow[d, "{[\pi] \mapsto [\mathrm{id}]}"] & \mathrm{Pic}(\mathscr{J}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & {[\mathbb{P}^1, \mathbb{A}^2 \backslash 0]_{\mathbb{A}^1}} \arrow[r] & {[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}} \arrow[r] & \mathrm{Pic}(\mathbb{P}^1) \arrow[r] & 1
\end{tikzcd}$$

exact sequence of groups

Exact sequences:

$$\begin{tikzcd}
\mathscr{L} \arrow[d] & \text{line bundle} + \text{two generating sections} \arrow[d, bend right, leftrightarrow, dotted, "\text{1-1 correspondence}"] & \\
\mathscr{J} \arrow[rr, "\text{morphism}"] & & \mathbb{P}^1
\end{tikzcd}$$

exact sequence of sets

$$[f] \longmapsto [\mathscr{L}]$$

$$\begin{tikzcd}
1 \arrow[r] & {[\mathscr{J}, \mathbb{A}^2\backslash 0]_{\mathrm{N}}} \arrow[r] \arrow[d] & {[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}} \arrow[r] \arrow[d] & \mathrm{Pic}(\mathscr{J}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & {[\mathbb{P}^1, \mathbb{A}^2\backslash 0]_{\mathbb{A}^1}} \arrow[r] & {[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}} \arrow[r] & \mathrm{Pic}(\mathbb{P}^1) \arrow[r] & 1
\end{tikzcd}$$

$$[\pi] \longmapsto [\mathrm{id}]$$

exact sequence of groups

Exact sequences:

$$\begin{tikzcd}
\mathscr{L} \arrow[d] & \text{line bundle} + \text{two generating sections} \arrow[d, bend left, dotted] \\
\mathscr{J} \arrow[rr, "\text{morphism}"] & \text{1-1 correspondence} \arrow[u, bend left, dotted] & \mathbb{P}^1
\end{tikzcd}$$

exact sequence of sets

$$[f] \longmapsto [\mathscr{L}]$$

$$\begin{tikzcd}
1 \arrow[r] & {[\mathscr{J}, \mathbb{A}^2\backslash 0]_{\mathrm{N}}} \arrow[r] \arrow[d] & {[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}} \arrow[r] \arrow[d, "{[\pi] \mapsto [\mathrm{id}]}"] & \mathrm{Pic}(\mathscr{J}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & {[\mathbb{P}^1, \mathbb{A}^2\backslash 0]_{\mathbb{A}^1}} \arrow[r] & {[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}} \arrow[r] & \mathrm{Pic}(\mathbb{P}^1) \arrow[r] & 1
\end{tikzcd}$$

exact sequence of groups

Exact sequences:

$\mathscr{L}$ line bundle + two generating sections

1-1 correspondence

$$\begin{tikzcd} \mathscr{L} \arrow[d] & \\ \mathscr{J} \arrow[rr, "\text{morphism}"] & & \mathbb{P}^1 \end{tikzcd}$$

exact sequence of sets

$[f] \longmapsto [\mathscr{L}]$ group ✓

$$\begin{tikzcd} 1 \arrow[r] & {[\mathscr{J}, \mathbb{A}^2\backslash 0]_{\mathrm{N}}} \arrow[r] \arrow[d] & {[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}} \arrow[r] \arrow[d] & \mathrm{Pic}(\mathscr{J}) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & {[\mathbb{P}^1, \mathbb{A}^2\backslash 0]_{\mathbb{A}^1}} \arrow[r] & {[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}} \arrow[r] & \mathrm{Pic}(\mathbb{P}^1) \arrow[r] & 1 \end{tikzcd}$$

$[\pi] \longmapsto [\mathrm{id}]$

exact sequence of groups

Exact sequences:

$$\begin{tikzcd}
\mathscr{L} \arrow[d] & \text{line bundle} + \text{two generating sections} \\
\mathscr{J} \arrow[r, "\text{morphism}"] & \mathbb{P}^1
\end{tikzcd}$$

1-1 correspondence

exact sequence of sets

group ?

$[f] \longmapsto [\mathscr{L}]$

group ✓

$$\begin{tikzcd}
1 \arrow[r] & {[\mathscr{J}, \mathbb{A}^2 \backslash 0]_{\mathrm{N}}} \arrow[r] \arrow[d] & {[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}} \arrow[r] \arrow[d, "{[\pi] \mapsto [\mathrm{id}]}"] & \mathrm{Pic}(\mathscr{J}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & {[\mathbb{P}^1, \mathbb{A}^2 \backslash 0]_{\mathbb{A}^1}} \arrow[r] & {[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}} \arrow[r] & \mathrm{Pic}(\mathbb{P}^1) \arrow[r] & 1
\end{tikzcd}$$

exact sequence of groups

The group structure:

group ✓

$$1 \longrightarrow [\mathscr{J}, \mathbb{A}^2\backslash 0]_{\mathrm{N}} \longrightarrow [\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}} \longrightarrow \{[n\pi] \,|\, n \in \mathbb{Z}\} \longrightarrow 1$$

The group structure:

$f_0\colon \mathscr{J} \to \mathbb{A}^2\backslash 0 \simeq \mathrm{SL}_2$

group ✓

$$1 \longrightarrow [\mathscr{J}, \mathbb{A}^2\backslash 0]_{\mathrm{N}} \longrightarrow [\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}} \longrightarrow \{[n\pi] \,|\, n \in \mathbb{Z}\} \longrightarrow 1$$

The group structure:

$M \in \mathrm{SL}_2(R)$

1–1

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$f_0 \colon \mathscr{J} \to \mathbb{A}^2 \backslash 0 \simeq \mathrm{SL}_2$

action

$\begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$ generating sections

group ✓ group ✓

$$1 \longrightarrow [\mathscr{J}, \mathbb{A}^2 \backslash 0]_{\mathrm{N}} \longrightarrow [\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}} \longrightarrow \{[n\pi] \,|\, n \in \mathbb{Z}\} \longrightarrow 1$$

The group structure:

$M \cdot \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$

$M \in \mathrm{SL}_2(R)$

1–1

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$f_0 \colon \mathscr{J} \to \mathbb{A}^2 \backslash 0 \simeq \mathrm{SL}_2$

group ✓ group ✓

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$[f]$ $[g]$

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$$\begin{cases} [f] = [f_0] \oplus [n\pi] \\ [g] = [g_0] \oplus [m\pi] \end{cases}$$

$[f]$ $[g]$

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$$\boxed{[f] \oplus [g] \quad := [f_0 \oplus g_0] \oplus [(n+m)\pi]}$$

The group structure:

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action ✓

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group ✓ group group ✓

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$$\boxed{[f] \oplus [g] \quad := [f_0 \oplus g_0] \oplus [(n+m)\pi]}$$

a lot of compatibilities need to be checked...

Our results: (Balch Barth, Hornslien, Q., Wilson)

$$[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}$$

$$[\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}} \xrightarrow[\text{group completion}]{} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}$$

Morel

Cazanave

Our results: (Balch Barth, Hornslien, Q., Wilson)

group

$\boxed{[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}}$

$$[\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}} \xrightarrow[\text{group completion}]{} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \quad \text{Morel}$$

Cazanave

Our results: (Balch Barth, Hornslien, Q., Wilson)

group

$$\boxed{[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}}$$

π_{N}^{*}

morphism of monoids

requires key computations

$$[\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}} \xrightarrow[\text{group completion}]{} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}$$

Morel

Cazanave

Our results: (Balch Barth, Hornslien, Q., Wilson)

subgroup generated by
"simple" morphisms

group

$$\mathbf{G} \subset \boxed{[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}}$$

π_{N}^*

morphism of
monoids

requires key
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$$[\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}} \xrightarrow[\text{group completion}]{} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}$$

Morel

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Our results: (Balch Barth, Hornslien, Q., Wilson)

subgroup generated by
"simple" morphisms

$\mathbf{G} \subset \boxed{\text{group} \quad [\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}}$

$\boxed{\text{group completion}}$

$\boxed{\pi_{\mathrm{N}}^{*} \quad \text{morphism of monoids}}$

requires key
computations

$$[\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}} \xrightarrow[\text{group completion}]{} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}$$

Morel

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Our results: (Balch Barth, Hornslien, Q., Wilson)

subgroup generated by
"simple" morphisms

$\mathbf{G} \subset$ $[\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}}$

group

group
completion

π_{N}^*
morphism of
monoids

canonical
isomorphism
of groups

requires key
computations

$$[\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}} \xrightarrow[\text{group completion}]{} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}$$

Morel

Cazanave

Our results: (Balch Barth, Hornslien, Q., Wilson)

we conjecture that this is an equality

subgroup generated by "simple" morphisms

$$\begin{tikzcd}
 & \mathbf{G} \subset [\mathscr{J}, \mathbb{P}^1]_{\mathrm{N}} \arrow[dr, "\text{canonical isomorphism of groups}"] & \\
{[\mathbb{P}^1, \mathbb{P}^1]_{\mathrm{N}}} \arrow[ur, bend left, "\text{group completion}"] \arrow[ur, "\pi_{\mathrm{N}}^{*}\ \text{morphism of monoids}"'] \arrow[rr, "\text{group completion}"'] & & {[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}}
\end{tikzcd}$$

group

requires key computations

Cazanave

Morel

Tusen takk!