Massey products and formality for real projective groups

CIRM

Motivic homotopy in interaction

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This is joint work with Ambrus Pál

Norm Residue Theorem:

for simplicity

k field with $\mathrm{char}(k) \neq p$ containing primitive pth root of unity

Voevodsky, Rost, ...

Milnor K-theory

$T(k^\times)/(u \otimes (1-u), u \neq 0,1)$

continuous cohomology of absolute Galois group

$$K_\bullet^M(k)/_p \xrightarrow{\cong} H^\bullet(k, \mathbb{F}_p)$$

quadratic algebra

- generators in degree 1
- relations in degree 2

strong restriction on which $\mathbb{F}_p$-algebras can occur as the Galois cohomology of a field

Which quadratic algebras occur as $H^\bullet(k, \mathbb{F}_p)$?

Additional properties? $H^\bullet(k, \mathbb{F}_p)$ quadratic algebra

Is $K^M_\bullet(k)/p$ Koszul?

- A is Koszul if coh. $\mathrm{Ext}^\bullet_A(\mathbb{F}_p, \mathbb{F}_p) = A^!$ is the quadratic dual

- Conjecture of Positselski-Vishik-Voevodsky: If k contains a primitive pth root of unity, then $H^\bullet(k, \mathbb{F}_p)$ is Koszul

"The algebra $H^\bullet(k, \mathbb{F}_p)$ has a very nice and simple structure."

- Positselski: local and global fields ✓

- Mináč-Panini-Quadrelli-Tân: finite fields, pseudo algebraically closed fields, elementary type pro p-groups, ... ✓

Additional properties? $H^\bullet(k,\mathbb{F}_p)$ quadratic algebra

- Is $H^\bullet(k,\mathbb{F}_p)$ a Koszul algebra?

Can $H^\bullet(k,\mathbb{F}_p)$ be described in "elementary terms"?

$C^\bullet$ is quasi-isom as a dga to $(H^\bullet(C^\bullet), \delta = 0)$

- Is $\mathscr{C}^\bullet(k,\mathbb{F}_p)$ a formal dg-algebra?

- Can the dga $\mathscr{C}^\bullet(k,\mathbb{F}_p)$ be described in "elementary terms" as well?

Massey products provide an obstruction to formality

- many non-vanishing Massey products in arithm. & alg. geometry: Ekedahl, Morishita, Sharifi, Gärtner, Bleher-Chinburg-Gillibert, ...

- Hopkins and Wickelgren: $0 \in \langle a,b,c\rangle \iff \langle a,b,c\rangle \neq \varnothing$

  k a local or global field of $\mathrm{char}(k) \neq 2$

  elements in $H^1(k,\mathbb{F}_2)$

Massey vanishing conjecture of Mináč-Tân:

for every field k, all $n \geq 3$, all primes p

Conjecture: For $a_1, \ldots, a_n \in H^1(k, \mathbb{F}_p)$,

$$0 \in \langle a_1, \ldots, a_n \rangle \iff \langle a_1, \ldots, a_n \rangle \neq \varnothing.$$

- known in many cases by the work of Efrat-Matzri, Mináč-Tân, Harpaz-Wittenberg, Merkurjev-Scavia, Pál-Szabó, Quadrelli,... ✓

Hopkins-Wickelgren: For a field k and a prime p, is $\mathscr{C}^\bullet(k, \mathbb{F}_p)$ formal?

- The answer is no in general.
- Counterexamples by Positselski, Harpaz-Wittenberg, Merkurjev-Scavia

However, there are also positive cases...

Our main results:

every dga $C^\bullet$ over $\mathbb{F}_p$ with $H^\bullet \cong H^\bullet(G, \mathbb{F}_p)$ is formal

- Theorem (Pál-Q.): If G is a real projective profinite group, then $H^\bullet(G, \mathbb{F}_p)$ is intrinsically formal.

absolute Galois group $\Gamma(k)$ is real projective

G is real projective

k has virtual cohomological dimension ≤ 1

Haran-Jarden

k is pseudo real closed

there is a PRC field k with $\Gamma(k) \cong G$

- Theorem (Pál-Q.): If k has virtual cohomological dimension ≤ 1, then $H^\bullet(k, \mathbb{F}_p)$ is intrinsically formal and Koszul.

Hochschild vanishing theorem:

Scheiderer

connected sum of quadratic algebras

G a real projective group $\Longrightarrow$ $H^\bullet(G, \mathbb{F}_2) = A = B^\bullet \sqcap V^\bullet$

quadratic algebra with generators in deg 1 are orthogonal

$V^i = 0$ for $i \geq 2$

- Theorem (Pál-Q.): Such an algebra A is Koszul.

for p odd: $H^i(G, \mathbb{F}_p) = 0$ for $i \geq 2$ and intrinsic formality and Koszulity are easy

graded Hochschild cohomology

- Theorem (Pál-Q.): $\mathrm{HH}^{n,2-n}(A, A) = 0$ for all $n \geq 3$.

contain the obstructions to define an A_∞-map $A \to \mathscr{C}^\bullet$ which lifts $A \xrightarrow{\cong} H^\bullet(\mathscr{C}^\bullet)$

Kadeishvili: $\Longrightarrow$ A is intrinsically formal

Thank you!