

Examples of non- algebraic classes in the Brown-Peterson tower

Derived algebraic geometry and
chromatic homotopy theory

Isaac Newton Institute

28th September 2018

Gereon Quick

NTNU

Steenrod's question:

Can *every class* in $H_n(X;Z)$ be realized as the fundamental class of a *smooth n-manifold* $M \rightarrow X$?

*& compact,
oriented*

Steenrod's question:

Can **every class** in $H_n(X;Z)$ be realized as the fundamental class of a **smooth n-manifold** $M \rightarrow X$?

 & compact,
oriented

Thom's answer: In general, **no!**

Modifying the question:

- choose a nice X

Modifying the question:

- choose a nice X

projective

complex projective
space

complex smooth algebraic variety $X \subset \mathbb{C}P^N$

set of zeros of
homogeneous polynomial

Modifying the question:

- choose a nice X

projective

complex projective
space

complex smooth algebraic variety $X \subset \mathbb{C}P^N$

set of zeros of
homogeneous polynomial

- consider an algebraic subset $V \subset X$

satisfy additional
polynomial equations

Modifying the question:

- choose a nice X

projective

complex projective
space

complex smooth algebraic variety $X \subset \mathbb{C}P^N$

set of zeros of
homogeneous polynomial

- consider an algebraic subset $V \subset X$

satisfy additional
polynomial equations

if V smooth
fundamental

class of V $\rightarrow [V] \in H_{2n}(V(\mathbb{C}); \mathbb{Z})$

Modifying the question:

- choose a nice X

projective

complex projective
space

complex smooth algebraic variety $X \subset \mathbb{C}P^N$

set of zeros of
homogeneous polynomial

- consider an algebraic subset $V \subset X$

satisfy additional
polynomial equations

if V smooth
fundamental

class of V \rightarrow $[V] \in H_{2n}(V(\mathbb{C}); \mathbb{Z})$ $j_*[V] \in H_{2n}(X(\mathbb{C}); \mathbb{Z})$

Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

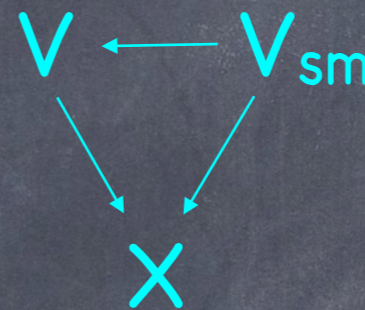
Can *every class* in $H_{2n}(X; \mathbb{Z})$
be realized as the
fundamental class of an
algebraic subset $V \subset X$?

Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Can every class in $H_{2n}(X; \mathbb{Z})$ be realized as the fundamental class of an algebraic subset $V \subset X$?

desingularization



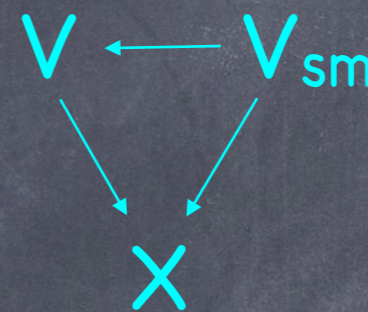
$[V] = [V_{sm}]$
in $H_{2n}(X; \mathbb{Z})$

Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Can every class in $H_{2n}(X; \mathbb{Z})$ of Hodge type
be realized as the
fundamental class of an
algebraic subset $V \subset X$?

desingularization



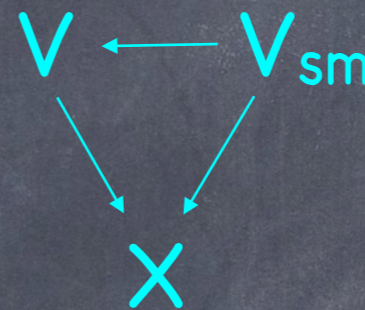
$[V] = [V_{sm}]$
in $H_{2n}(X; \mathbb{Z})$

Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Can every class in $H_{2n}(X; \mathbb{Z})$ of Hodge type
be realized as the
fundamental class of an
algebraic subset $V \subset X$?

desingularization



$[V] = [V_{sm}]$
in $H_{2n}(X; \mathbb{Z})$

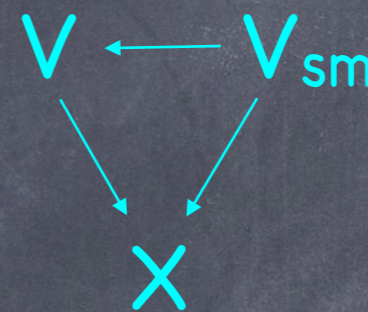
Atiyah-Hirzebruch: In general, **no!**

Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Can every class in $H_{2n}(X; \mathbb{Z})$ of Hodge type be realized as the fundamental class of an algebraic subset $V \subset X$?

desingularization



$[V] = [V_{sm}]$
in $H_{2n}(X; \mathbb{Z})$

Atiyah-Hirzebruch: In general, **no!**

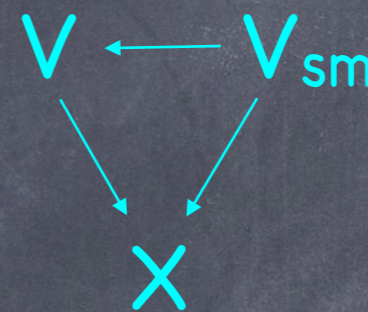
- However, for $n = \dim X - 1$, the answer is **yes**.

Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Can every class in $H_{2n}(X; \mathbb{Z})$ of Hodge type
be realized as the
fundamental class of an
algebraic subset $V \subset X$?

desingularization



$[V] = [V_{sm}]$
in $H_{2n}(X; \mathbb{Z})$

Atiyah-Hirzebruch: In general, **no!**

- However, for $n = \dim X - 1$, the answer is **yes.**

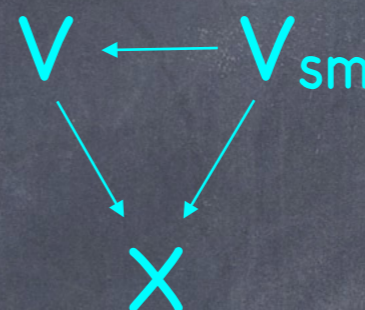
given α in $H_{2d-2}(X; \mathbb{Z})$

Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Can every class in $H_{2n}(X; \mathbb{Z})$ of Hodge type be realized as the fundamental class of an algebraic subset $V \subset X$?

desingularization



$[V] = [V_{sm}]$
in $H_{2n}(X; \mathbb{Z})$

Atiyah-Hirzebruch: In general, **no!**

- However, for $n = \dim X - 1$, the answer is **yes.**

given α in $H_{2d-2}(X; \mathbb{Z})$

Poincaré
dual

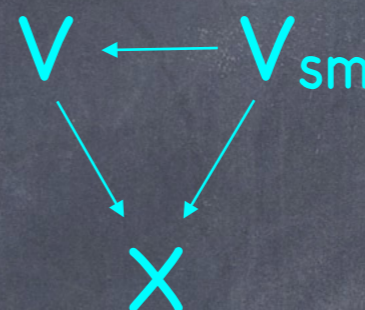
α_{PD} in $H^2(X; \mathbb{Z})$

Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Can every class in $H_{2n}(X; \mathbb{Z})$ of Hodge type be realized as the fundamental class of an algebraic subset $V \subset X$?

desingularization



$[V] = [V_{sm}]$
in $H_{2n}(X; \mathbb{Z})$

Atiyah-Hirzebruch: In general, **no!**

- However, for $n = \dim X - 1$, the answer is **yes.**

given α in $H_{2d-2}(X; \mathbb{Z})$

Poincaré
dual

α_{PD} in $H^2(X; \mathbb{Z})$

$X \xrightarrow{\alpha_{PD}}$

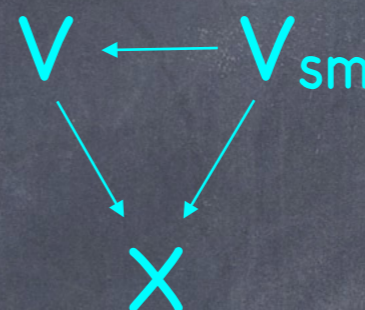
$K(2; \mathbb{Z})$
 $\mathbb{C}P^\infty$

Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Can every class in $H_{2n}(X; \mathbb{Z})$ of Hodge type be realized as the fundamental class of an algebraic subset $V \subset X$?

desingularization



$[V] = [V_{sm}]$
in $H_{2n}(X; \mathbb{Z})$

Atiyah-Hirzebruch: In general, **no!**

- However, for $n = \dim X - 1$, the answer is **yes.**

given α in $H_{2d-2}(X; \mathbb{Z})$

Poincaré
dual

α_{PD} in $H^2(X; \mathbb{Z})$

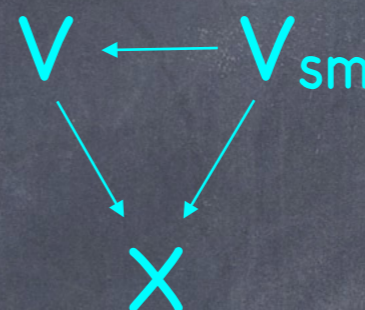


Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Can every class in $H_{2n}(X; \mathbb{Z})$ of Hodge type be realized as the fundamental class of an algebraic subset $V \subset X$?

desingularization



$[V] = [V_{sm}]$
in $H_{2n}(X; \mathbb{Z})$

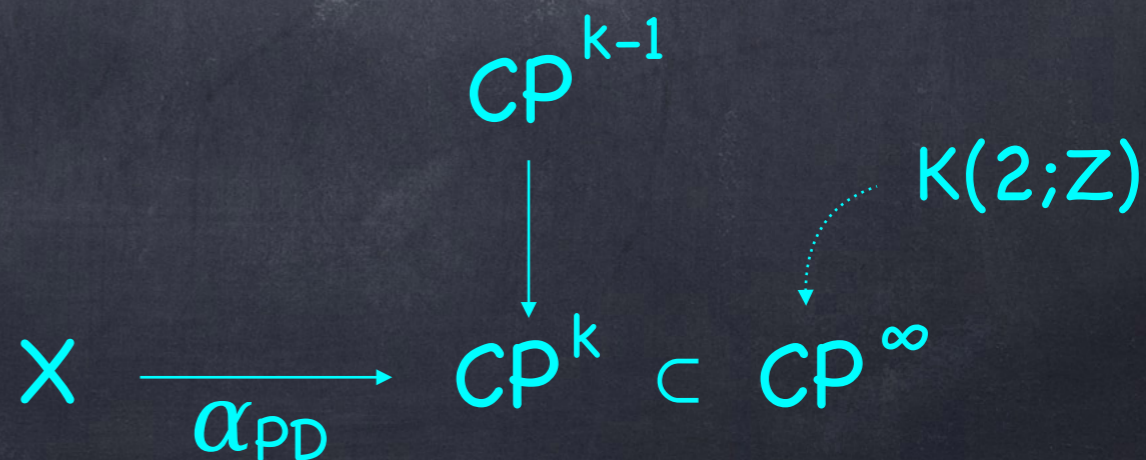
Atiyah-Hirzebruch: In general, **no!**

- However, for $n = \dim X - 1$, the answer is **yes.**

given α in $H_{2d-2}(X; \mathbb{Z})$

Poincaré
dual

α_{PD} in $H^2(X; \mathbb{Z})$

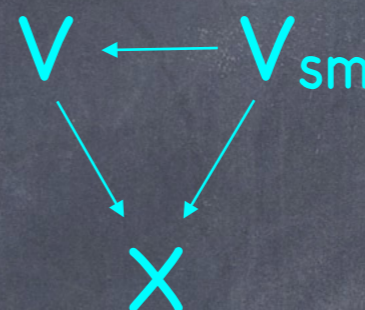


Hodge's question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

Can every class in $H_{2n}(X; \mathbb{Z})$ of Hodge type be realized as the fundamental class of an algebraic subset $V \subset X$?

desingularization



$$[V] = [V_{sm}] \text{ in } H_{2n}(X; \mathbb{Z})$$

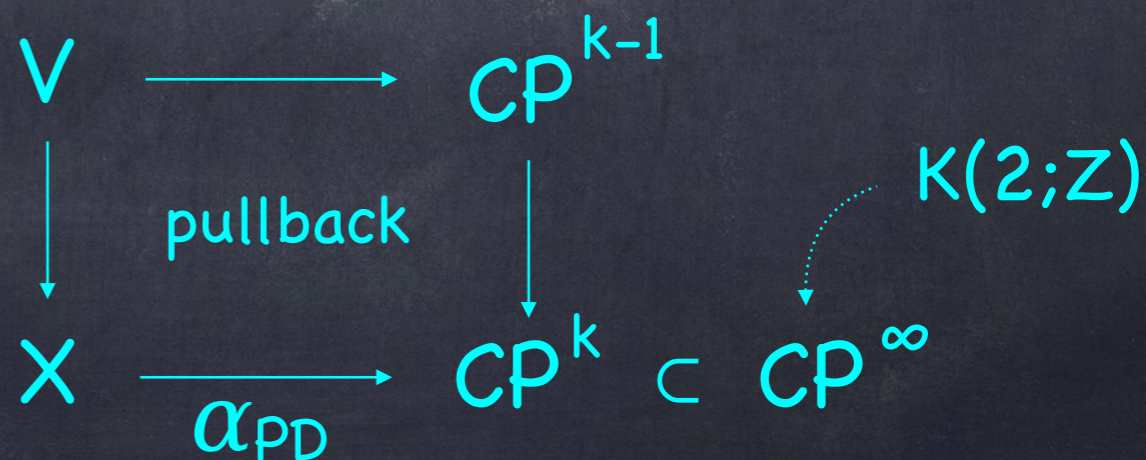
Atiyah-Hirzebruch: In general, **no!**

- However, for $n = \dim X - 1$, the answer is **yes.**

given α in $H_{2d-2}(X; \mathbb{Z})$

Poincaré dual

α_{PD} in $H^2(X; \mathbb{Z})$



How to do homotopy on $\text{Man}_\mathbb{C}$?

category of complex
manifolds



How to do homotopy on $\text{Man}_\mathbb{C}$?

category of complex manifolds

$\text{Man}_\mathbb{C} \longrightarrow \text{Pre}_\Delta$

presheaves of simplicial sets

$M \longmapsto F_M$

$F_M: X \longmapsto$ discrete simplicial set $\text{Hom}_{\text{Man}}(X, M)$

How to do homotopy on Man_C ?

category of complex manifolds

$\text{Man}_C \longrightarrow \text{Pre}_\Delta$

presheaves of simplicial sets

$M \longmapsto F_M$

$F_M: X \longmapsto$ discrete simplicial set $\text{Hom}_{\text{Man}}(X, M)$

Presheaves "remember"

$$\text{Hom}_{\text{Pre}}(F_M, F_{M'}) = \text{Hom}_{\text{Man}_C}(M, M')$$

How to do homotopy on $\text{Man}_\mathbb{C}$?

category of complex manifolds

$$\text{Man}_\mathbb{C} \longrightarrow \text{Pre}_\Delta$$

presheaves of simplicial sets

$$M \longmapsto F_M$$

Presheaves "remember"

$F_M: X \mapsto$ discrete simplicial set $\text{Hom}_{\text{Man}}(X, M)$

$$\text{Hom}_{\text{Pre}}(F_M, F_{M'}) = \text{Hom}_{\text{Man}_\mathbb{C}}(M, M')$$

- Given $n \geq 0$, the n -dimensional stalk of F_\bullet .

$$F_\bullet^{(n)} = \text{colim}_{r \rightarrow 0} F_\bullet(B^n(r)) \text{ in } \text{Set}_\Delta$$

ball of radius r in n -dim. complex affine space

How to do homotopy on $\text{Man}_\mathbb{C}$?

category of complex manifolds

$$\text{Man}_\mathbb{C} \longrightarrow \text{Pre}_\Delta$$

presheaves of simplicial sets

$$M \longmapsto F_M$$

Presheaves "remember"

$F_M: X \mapsto$ discrete simplicial set $\text{Hom}_{\text{Man}}(X, M)$

$$\text{Hom}_{\text{Pre}}(F_M, F_{M'}) = \text{Hom}_{\text{Man}_\mathbb{C}}(M, M')$$

- Given $n \geq 0$, the n -dimensional stalk of F_\bullet .

$$F_\bullet^{(n)} = \text{colim}_{r \rightarrow 0} F_\bullet(B^n(r)) \text{ in } \text{Set}_\Delta$$

ball of radius r in n -dim. complex affine space

- A map $F_\bullet \rightarrow G_\bullet$ is a weak equivalence in Pre_Δ if $F_\bullet^{(n)} \rightarrow G_\bullet^{(n)}$ is a weak equivalence in Set_Δ for all $n \geq 0$.

Homotopy category of \mathbf{Man}_C :

- $\mathbf{Man} \longrightarrow \mathbf{Pre}_\Delta$

Homotopy category of \mathbf{Man}_C : homotopy category of simplicial presheaves on \mathbf{Man}_C

• $\mathbf{Man} \longrightarrow \mathbf{hoPre}_\Delta = \mathbf{Pre}_\Delta[w.e.^{-1}]$

Homotopy category of \mathbf{Man}_C : homotopy category of simplicial presheaves on \mathbf{Man}_C

• $\mathbf{Man} \longrightarrow \mathbf{hoPre}_\Delta = \mathbf{Pre}_\Delta[\mathbf{w.e.}^{-1}]$

• Given M with an open cover $\{U_\alpha\}$:

$\mathbf{F}_{U_\bullet} \rightarrow \mathbf{F}_M$ is a weak equivalence.

$\coprod U_\alpha \rightrightarrows \coprod U_\alpha \times U_\beta \rightrightarrows \dots$

Homotopy category of \mathbf{Man}_C : homotopy category of simplicial presheaves on \mathbf{Man}_C

- $\mathbf{Man} \longrightarrow \mathbf{hoPre}_\Delta = \mathbf{Pre}_\Delta[\mathbf{w.e.}^{-1}]$

- Given M with an open cover $\{U_\alpha\}$:

$\mathbf{F}_{U_\bullet} \rightarrow \mathbf{F}_M$ is a weak equivalence.

$\coprod U_\alpha \rightrightarrows \coprod U_\alpha \times U_\beta \rightrightarrows \dots$

- Can replace \mathbf{Set}_Δ with $\mathbf{Spectra}$ and get a **stable** homotopy category $\mathbf{hoPre}_{\mathbf{Spectra}}$ of \mathbf{Man}_C .

- $S^1 \wedge -$ with S^1 viewed as a simplicial (constant) presheaf is made invertible.

Homotopy category of $\mathbf{Sm}_\mathbb{C}$:

smooth complex varieties

→ $\mathbf{Sm}_\mathbb{C}$

Homotopy category of \mathbf{Sm}_C :

smooth complex varieties

\mathbf{Sm}_C



\mathbf{Pre}_Δ

simplicial presheaves on \mathbf{Sm}_C

Morel

Voevodsky

Jardine

Joyal

Isaksen

Dugger

...

Homotopy category of \mathbf{Sm}_C :

smooth complex varieties

\mathbf{Sm}_C



\mathbf{Pre}_Δ

simplicial presheaves on \mathbf{Sm}_C

- Nisnevich covers (replacing open covers)

Morel

Voevodsky

Jardine

Joyal

Isaksen

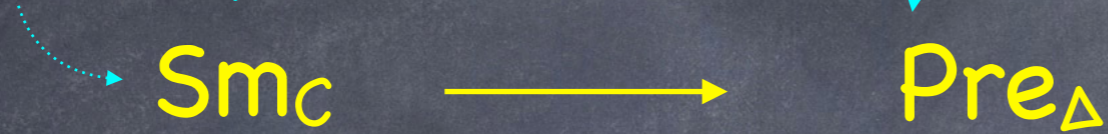
Dugger

...

Homotopy category of \mathbf{Sm}_C :

smooth complex varieties

simplicial presheaves on \mathbf{Sm}_C



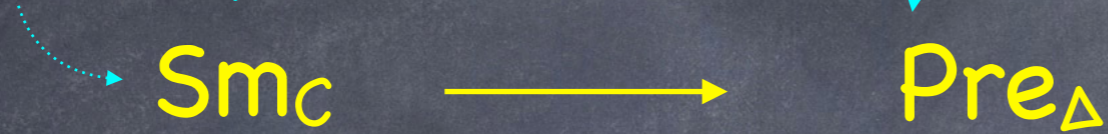
- Nisnevich covers (replacing open covers)
- Localize with respect to

Morel
Voevodsky
Jardine
Joyal
Isaksen
Dugger
...

Homotopy category of \mathbf{Sm}_C :

smooth complex varieties

simplicial presheaves on \mathbf{Sm}_C



- Nisnevich covers (replacing open covers)

- Localize with respect to

$\mathcal{F}_U \rightarrow \mathcal{F}_X$ for any X and any (hyper)cover $U \rightarrow X$

Morel
Voevodsky
Jardine
Joyal
Isaksen
Dugger
...

Homotopy category of \mathbf{Sm}_C :

smooth complex varieties

\mathbf{Sm}_C



\mathbf{hoPre}_Δ

homotopy category of
simplicial presheaves on \mathbf{Sm}_C

Morel
Voevodsky
Jardine
Joyal
Isaksen
Dugger
...

- Nisnevich covers (replacing open covers)
- Localize with respect to

$\mathbf{F}_U \rightarrow \mathbf{F}_X$ for any X and any (hyper)cover $U \rightarrow X$

Homotopy category of \mathbf{Sm}_C : **motivic** homotopy category of simplicial presheaves on \mathbf{Sm}_C

smooth complex varieties

$$\mathbf{Sm}_C \longrightarrow \mathbf{hoPre}_\Delta$$

Morel
Voevodsky
Jardine
Joyal
Isaksen
Dugger
...

- **Nisnevich** covers (replacing open covers)
- **Localize** with respect to

→ $F_U \rightarrow F_X$ for any X and any (hyper)cover $U \rightarrow X$

→ $A^1_C X \rightarrow X$ for any X

↖ affine line over C

Homotopy category of \mathbf{Sm}_C : **motivic** homotopy category of simplicial presheaves on \mathbf{Sm}_C

smooth complex varieties

$$\mathbf{Sm}_C \longrightarrow \mathbf{hoPre}_\Delta$$

Morel
Voevodsky
Jardine
Joyal
Isaksen
Dugger
...

- Nisnevich covers (replacing open covers)
- Localize with respect to

$$F_U \rightarrow F_X \text{ for any } X \text{ and any (hyper)cover } U \rightarrow X$$

$$A^1_C X \rightarrow X \text{ for any } X$$

affine line over C

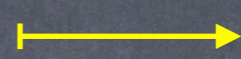
- stable motivic homotopy category of \mathbf{Sm}_C

- P^1_C - the projective line

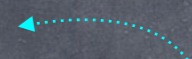
- S^1_C - the "simplicial circle" and $(A^1 - 0)_C$ - the "Tate circle"

Topological realization: $\text{Sm}_C \xrightarrow{\rho} \text{Man}_C$

X



$X(C)$



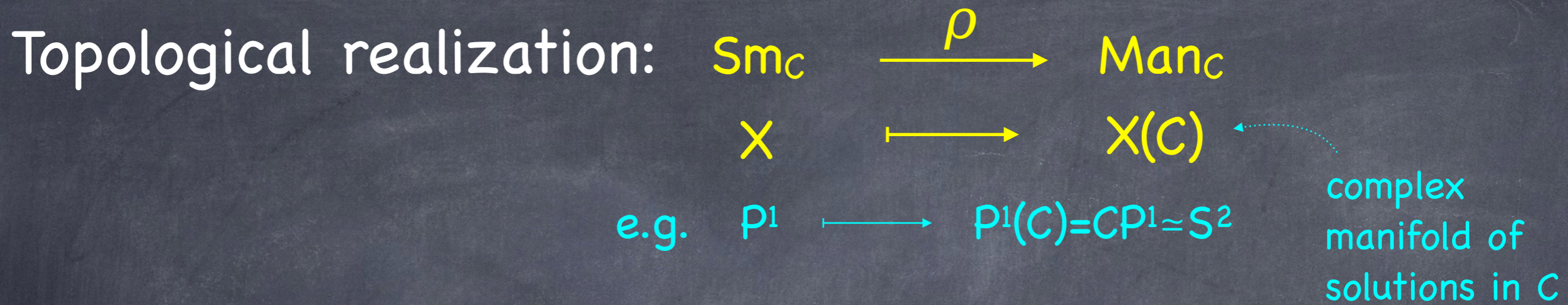
e.g.

P^1



$P^1(C) = CP^1 \simeq S^2$

complex
manifold of
solutions in C



motivic spectrum

$$E_{\text{mot}}^{a,b}(X)$$

"algebraic"

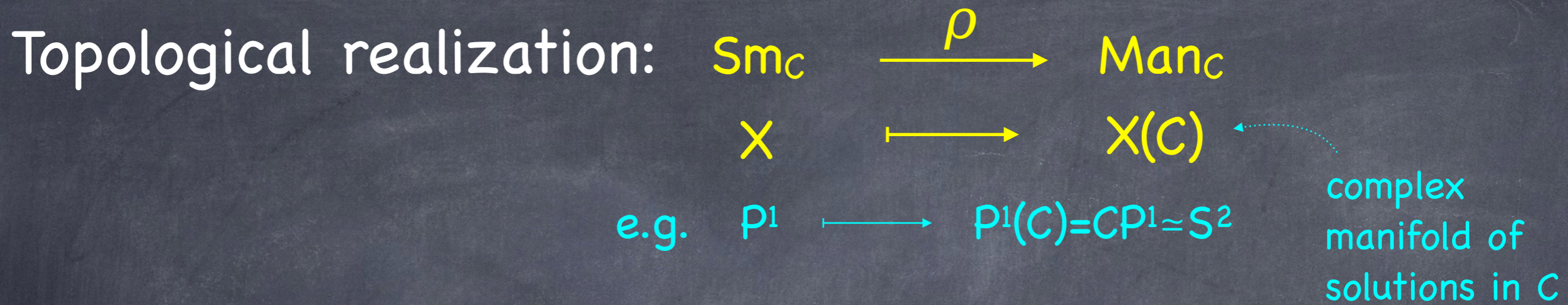
induced map

$$\xrightarrow{\rho}$$

$$E_{\text{top}}^a(X(C))$$

"topological"

top. realization of E



motivic spectrum

$$E_{\text{mot}}^{a,b}(X)$$

"algebraic"

induced map

$$\xrightarrow{\rho}$$

$$E_{\text{top}}^a(X(C))$$

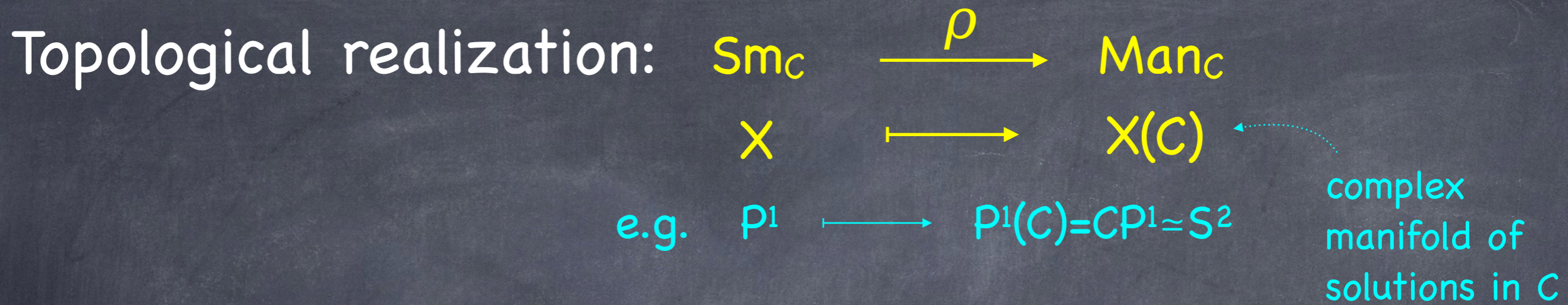
"topological"

top. realization of E

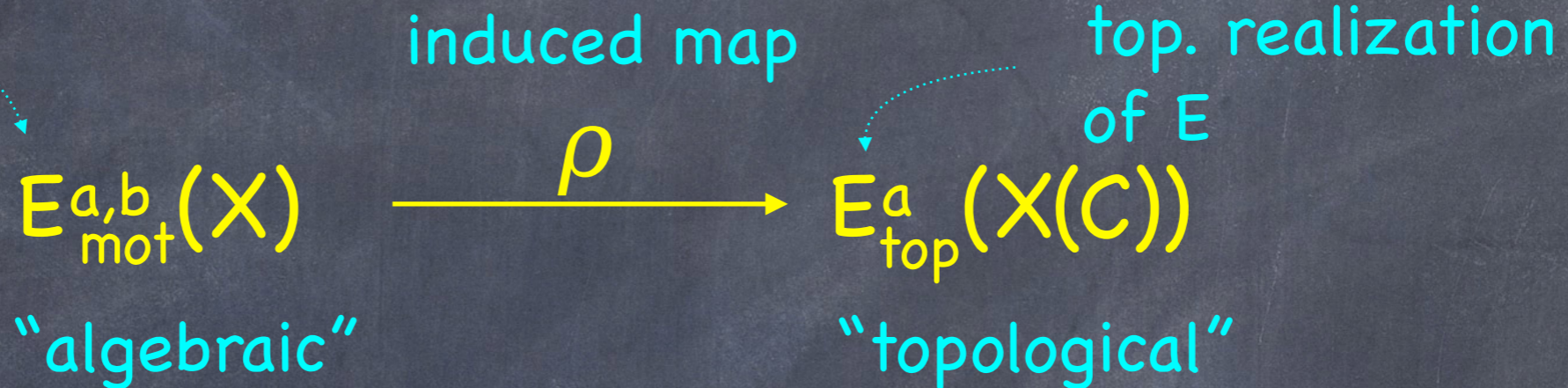
Example:

$$[V_C X] \in \text{HZ}_{\text{mot}}^{2n,n}(X) \longrightarrow \text{HZ}_{\text{top}}^{2n}(X(C)) = H^{2n}(X; \mathbb{Z})$$

[V_{sm}]



motivic spectrum



Example:

$$[V_C X] \in \mathrm{HZ}_{\mathrm{mot}}^{2n,n}(X) \longrightarrow \mathrm{HZ}_{\mathrm{top}}^{2n}(X(C)) = H^{2n}(X; \mathbb{Z})$$

[V_{sm}]

Question: How can we produce classes in $E_{\mathrm{top}}^{2*}(X(C))$ which are **not algebraic**, i.e., are **not** in the image of ρ ?

Atiyah-Hirzebruch obstruction:

given


algebraic $V \subset X$

Atiyah-Hirzebruch obstruction:

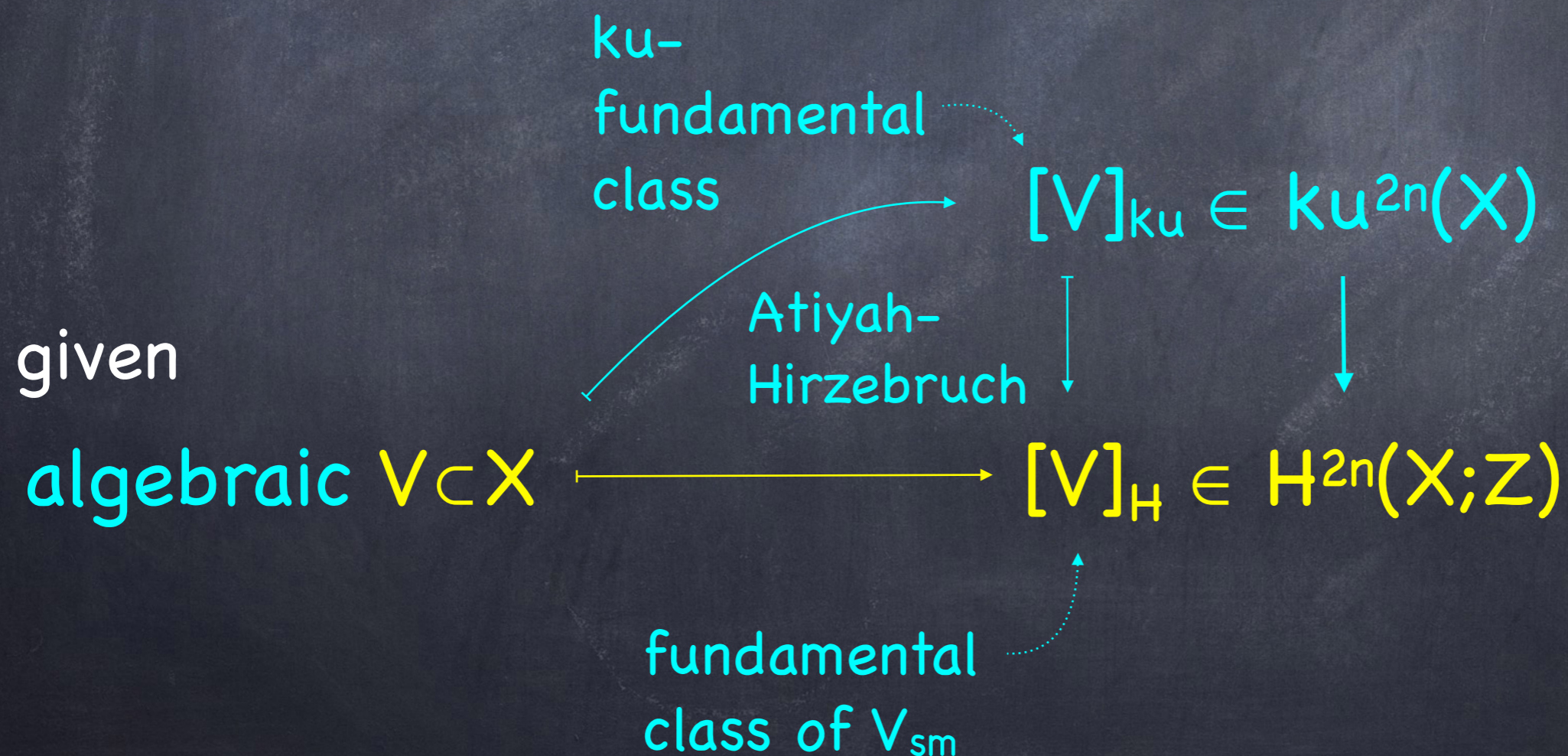
given

$$\text{algebraic } V \subset X \longrightarrow [V]_H \in H^{2n}(X; \mathbb{Z})$$

fundamental
class of V_{sm}

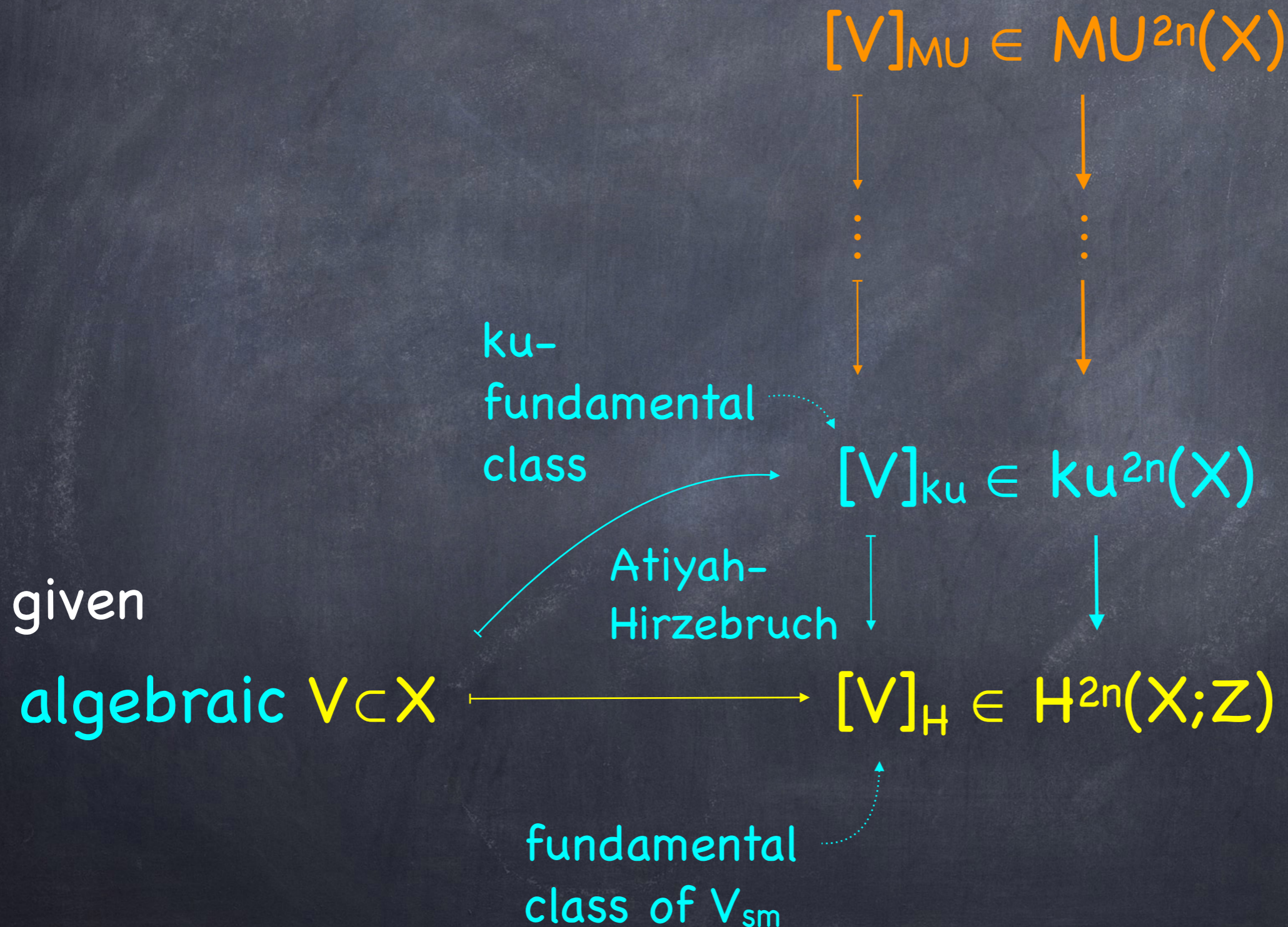


Atiyah-Hirzebruch obstruction:



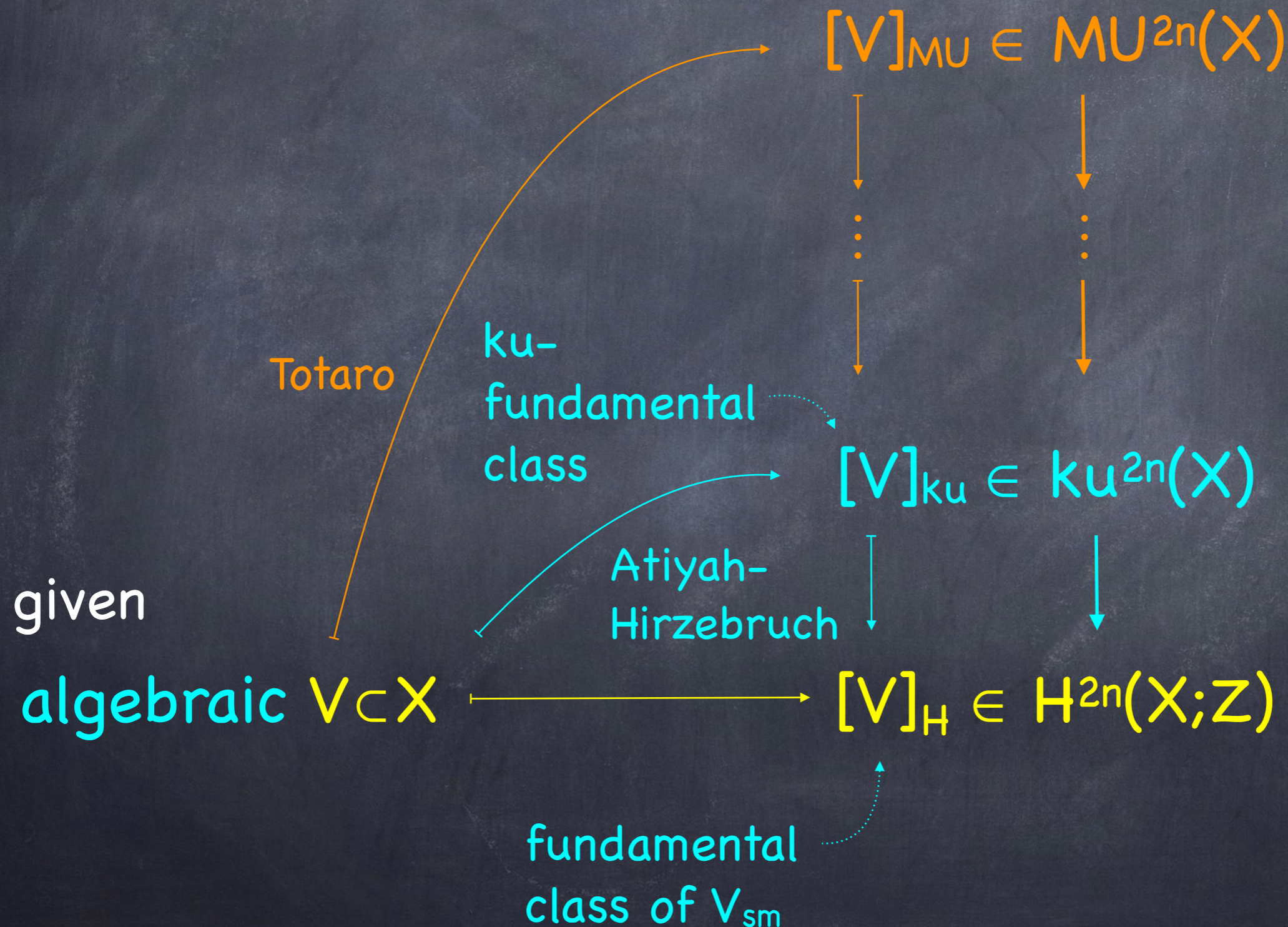
Atiyah-Hirzebruch obstruction:

Thom/Quillen:
universal
fundamental class



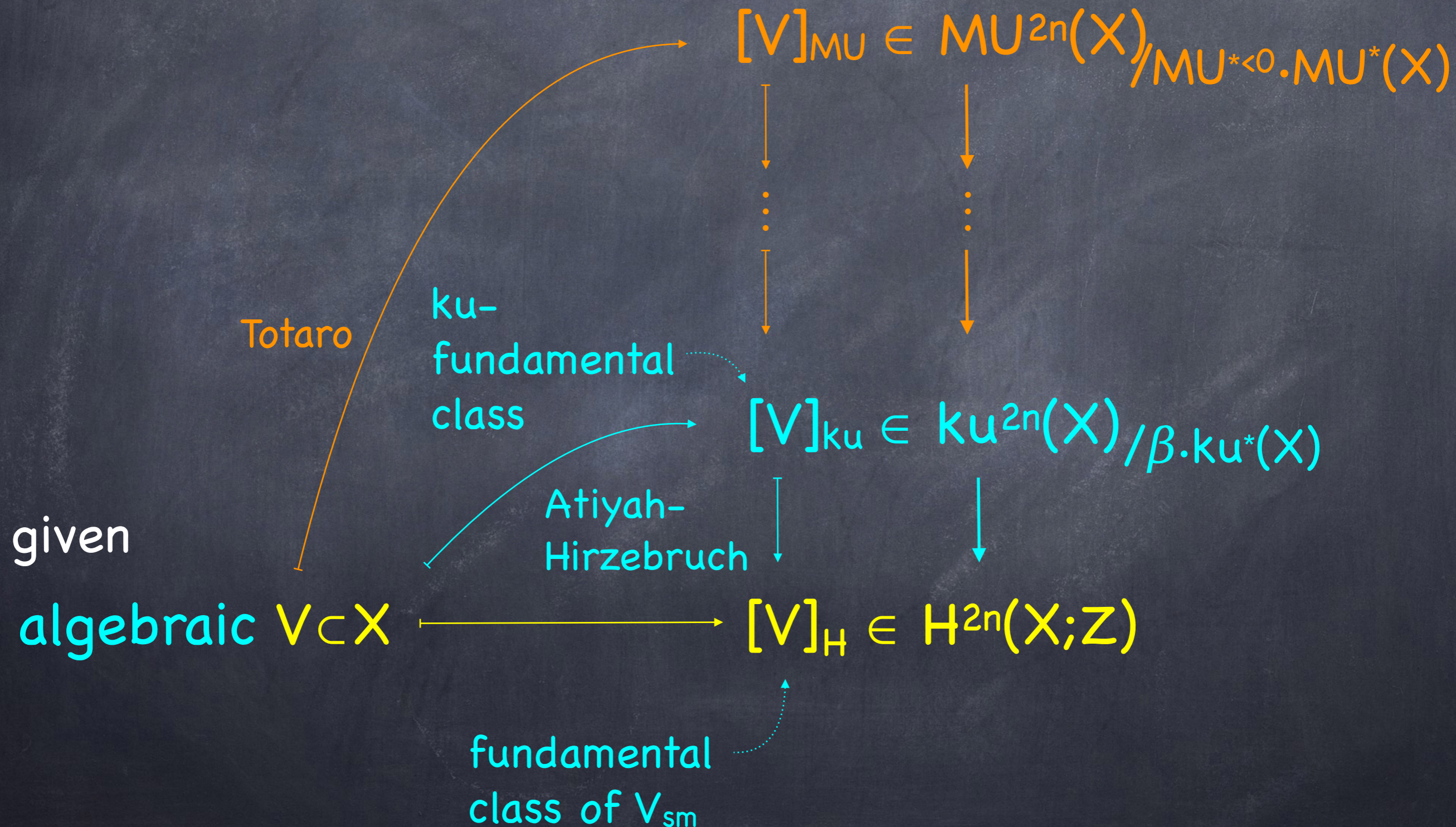
Atiyah-Hirzebruch obstruction:

Thom/Quillen:
universal
fundamental class



Atiyah-Hirzebruch obstruction:

Thom/Quillen:
universal
fundamental class



Atiyah-Hirzebruch obstruction:

Thom/Quillen:

alg. Thom spectrum

universal

fundamental class

$$\text{MGL}^{2n,n}(X) / \text{MU}^{* < 0} \cdot \text{MGL}^{2*,*}(X)$$

$$[V]_{\text{MU}} \in \text{MU}^{2n}(X) / \text{MU}^{* < 0} \cdot \text{MU}^*(X)$$

\approx Levine-Morel

$$\text{H}_{\text{mot}}^{2n,n}(X; Z)$$



Totaro

ku-
fundamental
class

$$[V]_{\text{ku}} \in \text{ku}^{2n}(X) / \beta \cdot \text{ku}^*(X)$$

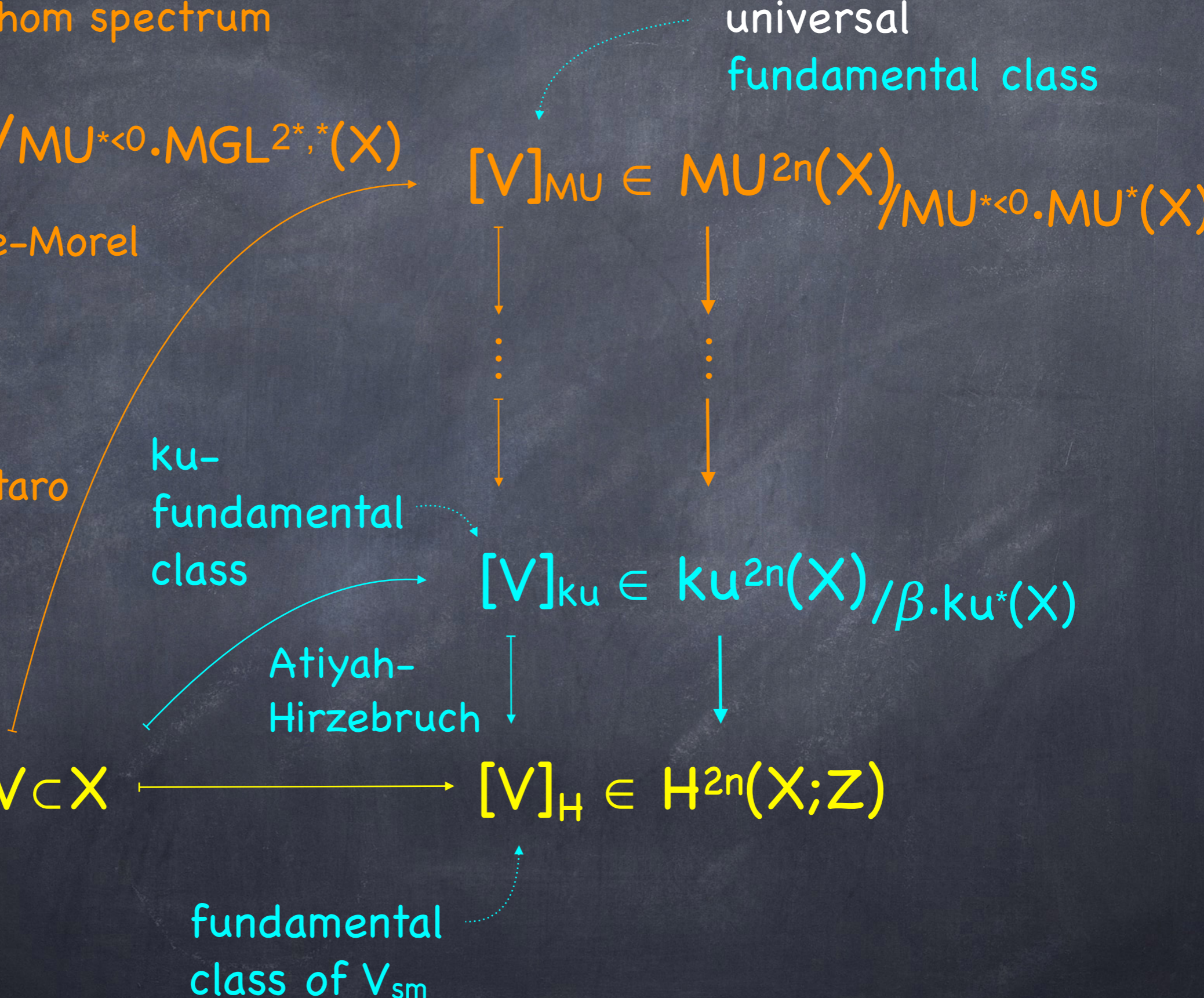
given

Atiyah-
Hirzebruch

$$\text{algebraic } V \subset X$$

$$[V]_{\text{H}} \in \text{H}^{2n}(X; Z)$$

fundamental
class of V_{sm}



Atiyah-Hirzebruch obstruction:

Thom/Quillen:
universal
fundamental class

alg. Thom spectrum

$$\text{MGL}^{2n,n}(X) / \text{MU}^{* < 0} \cdot \text{MGL}^{2*,*}(X)$$

$$[V]_{\text{MU}} \in \text{MU}^{2n}(X) / \text{MU}^{* < 0} \cdot \text{MU}^*(X)$$

\approx Levine-Morel

$$\text{H}_{\text{mot}}^{2n,n}(X; Z)$$

Totaro

ku-
fundamental
class

$$[V]_{\text{ku}} \in \text{ku}^{2n}(X) / \beta \cdot \text{ku}^*(X)$$

given

Atiyah-
Hirzebruch

\neq in
general

$$\text{algebraic } V \subset X \longrightarrow [V]_{\text{H}} \in \text{H}^{2n}(X; Z)$$

fundamental
class of V_{sm}

The Brown-Peterson tower:

fix a prime p

p -local universal theory

Brown-Peterson spectra BP with

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]. \quad |v_i| = 2(p^i - 1)$$

Brown-Peterson
Quillen
Wilson
⋮

The Brown-Peterson tower:

fix a prime p

p -local universal theory

Brown-Peterson spectra BP with

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots].$$

$|v_i| = 2(p^i - 1)$

Brown-Peterson
Quillen
Wilson
⋮

quotient map

For every n :

$$\mathbb{Z}_{(p)}[v_1, v_2, \dots] \longrightarrow \mathbb{Z}_{(p)}[v_1, \dots, v_n]$$

The Brown-Peterson tower:

fix a prime p

p -local universal theory

Brown-Peterson spectra BP with

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots].$$

$|v_i| = 2(p^i - 1)$

Brown-Peterson
Quillen
Wilson
⋮

quotient map

For every n : $BP \longrightarrow BP/(v_{n+1}, \dots) =: BP\langle n \rangle$

$$\mathbb{Z}_{(p)}[v_1, v_2, \dots] \longrightarrow \mathbb{Z}_{(p)}[v_1, \dots, v_n] = BP\langle n \rangle_*$$

The Brown-Peterson tower:

fix a prime p

p -local universal theory

Brown-Peterson spectra BP with

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

$$|v_i| = 2(p^i - 1)$$

Brown-Peterson
Quillen
Wilson
⋮

quotient map

$$\text{For every } n: \quad BP \longrightarrow BP/(v_{n+1}, \dots) =: BP\langle n \rangle$$

$$\mathbb{Z}_{(p)}[v_1, v_2, \dots] \longrightarrow \mathbb{Z}_{(p)}[v_1, \dots, v_n] = BP\langle n \rangle_*$$

The Brown-Peterson tower:

$$BP \longrightarrow \dots \longrightarrow BP\langle n \rangle \longrightarrow \dots \longrightarrow BP\langle 1 \rangle \longrightarrow BP\langle 0 \rangle \longrightarrow BP\langle -1 \rangle$$

$$HZ_{(p)} \longrightarrow HF_p$$

Milnor operations:

Milnor operations:

For every n :

stable cofibre sequence

$$\sum |v_n| \mathbf{BP}\langle n \rangle \xrightarrow{v_n} \mathbf{BP}\langle n \rangle \longrightarrow \mathbf{BP}\langle n-1 \rangle \longrightarrow \sum |v_n|+1 \mathbf{BP}\langle n \rangle$$

Milnor operations:

For every n :

stable cofibre sequence

$$\sum |v_n| \mathbf{BP}\langle n \rangle \xrightarrow{v_n} \mathbf{BP}\langle n \rangle \longrightarrow \mathbf{BP}\langle n-1 \rangle \longrightarrow \sum |v_{n+1}| \mathbf{BP}\langle n \rangle$$

with an induced exact sequence (for any space Y)

$$\begin{array}{ccc} \mathbf{BP}\langle n \rangle^{*+|v_n|}(Y) & \longrightarrow & \mathbf{BP}\langle n \rangle^*(Y) \\ & \searrow & \uparrow \\ & & \mathbf{BP}\langle n-1 \rangle^*(Y) \xrightarrow{q_n} \mathbf{BP}\langle n \rangle^{*+|v_{n+1}|}(Y) \end{array}$$

Milnor operations:

For every n :

stable cofibre sequence

$$\Sigma^{|v_n|} \mathbf{BP}\langle n \rangle \xrightarrow{v_n} \mathbf{BP}\langle n \rangle \longrightarrow \mathbf{BP}\langle n-1 \rangle \longrightarrow \Sigma^{|v_n|+1} \mathbf{BP}\langle n \rangle$$

with an induced exact sequence (for any space Y)

$$\begin{array}{ccccc} \mathbf{BP}\langle n \rangle^{*+|v_n|}(Y) & \longrightarrow & \mathbf{BP}\langle n \rangle^*(Y) & & \\ & \searrow & & \searrow & \\ \mathbf{BP}\langle n-1 \rangle^* & \xrightarrow{q_n} & \mathbf{BP}\langle n \rangle^{*+|v_n|+1} & & \\ \downarrow \text{Thom map} & & \downarrow & & \downarrow \\ \mathbf{HF}_p & \longrightarrow & H^*(Y; \mathbb{F}_p) & \longrightarrow & H^{*+|v_n|+1}(Y; \mathbb{F}_p) \end{array}$$

Milnor operations:

For every n :

stable cofibre sequence

$$\Sigma^{|v_n|} \mathbf{BP}\langle n \rangle \xrightarrow{v_n} \mathbf{BP}\langle n \rangle \longrightarrow \mathbf{BP}\langle n-1 \rangle \longrightarrow \Sigma^{|v_{n+1}|} \mathbf{BP}\langle n \rangle$$

with an induced exact sequence (for any space Y)

$$\mathbf{BP}\langle n \rangle^{*+|v_n|}(Y) \longrightarrow \mathbf{BP}\langle n \rangle^*(Y)$$

$$\mathbf{BP}\langle n-1 \rangle \longrightarrow \mathbf{BP}\langle n-1 \rangle^*(Y) \xrightarrow{q_n} \mathbf{BP}\langle n \rangle^{*+|v_{n+1}|}(Y)$$

Thom
map
 \downarrow
 \mathbf{HF}_p

\downarrow
 $H^*(Y; \mathbf{F}_p)$

\downarrow
 $\xrightarrow{Q_n} H^{*+|v_{n+1}|}(Y; \mathbf{F}_p)$

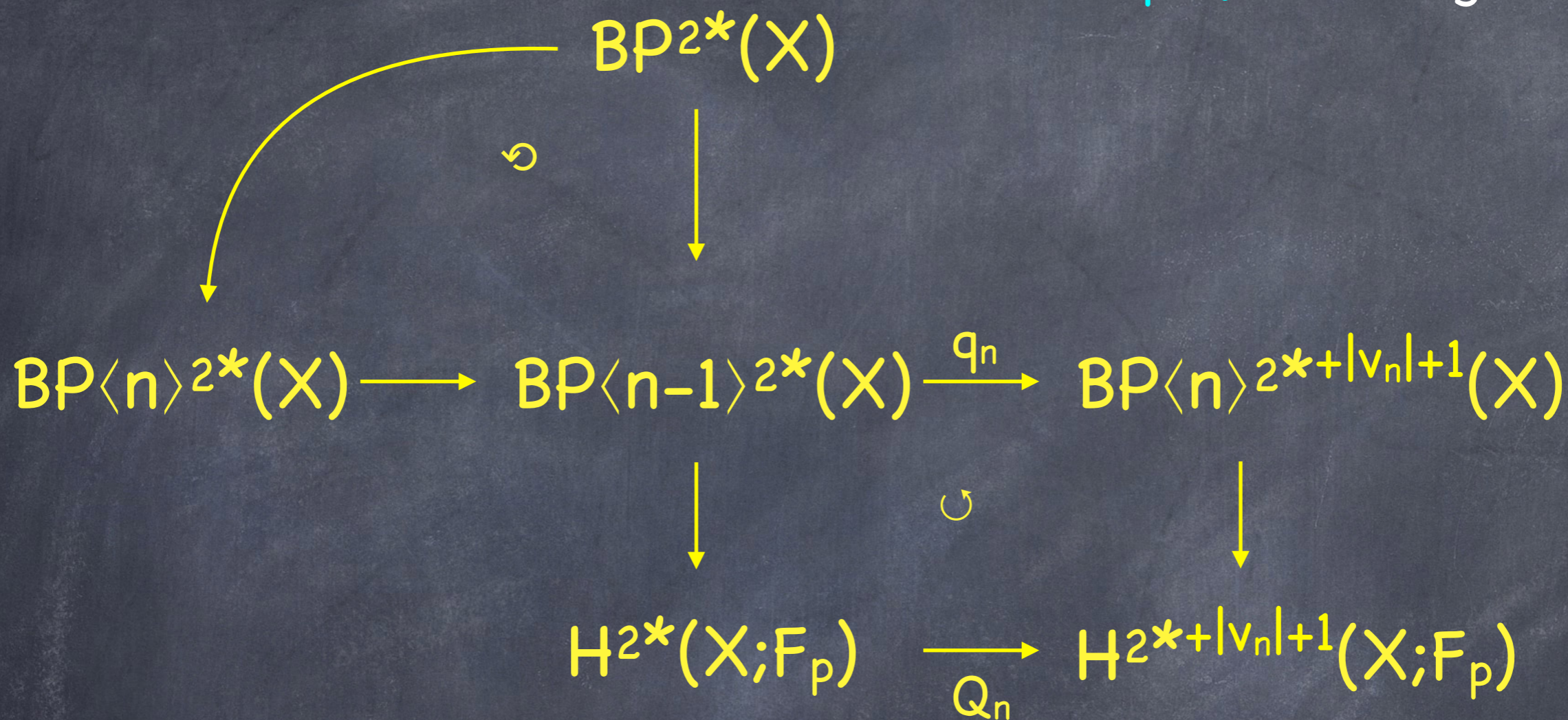
n th Milnor
operation:

$Q_0 = \text{Bockstein}$

$Q_n = P^{p^{n-1}} Q_{n-1} - Q_{n-1} P^{p^{n-1}}$

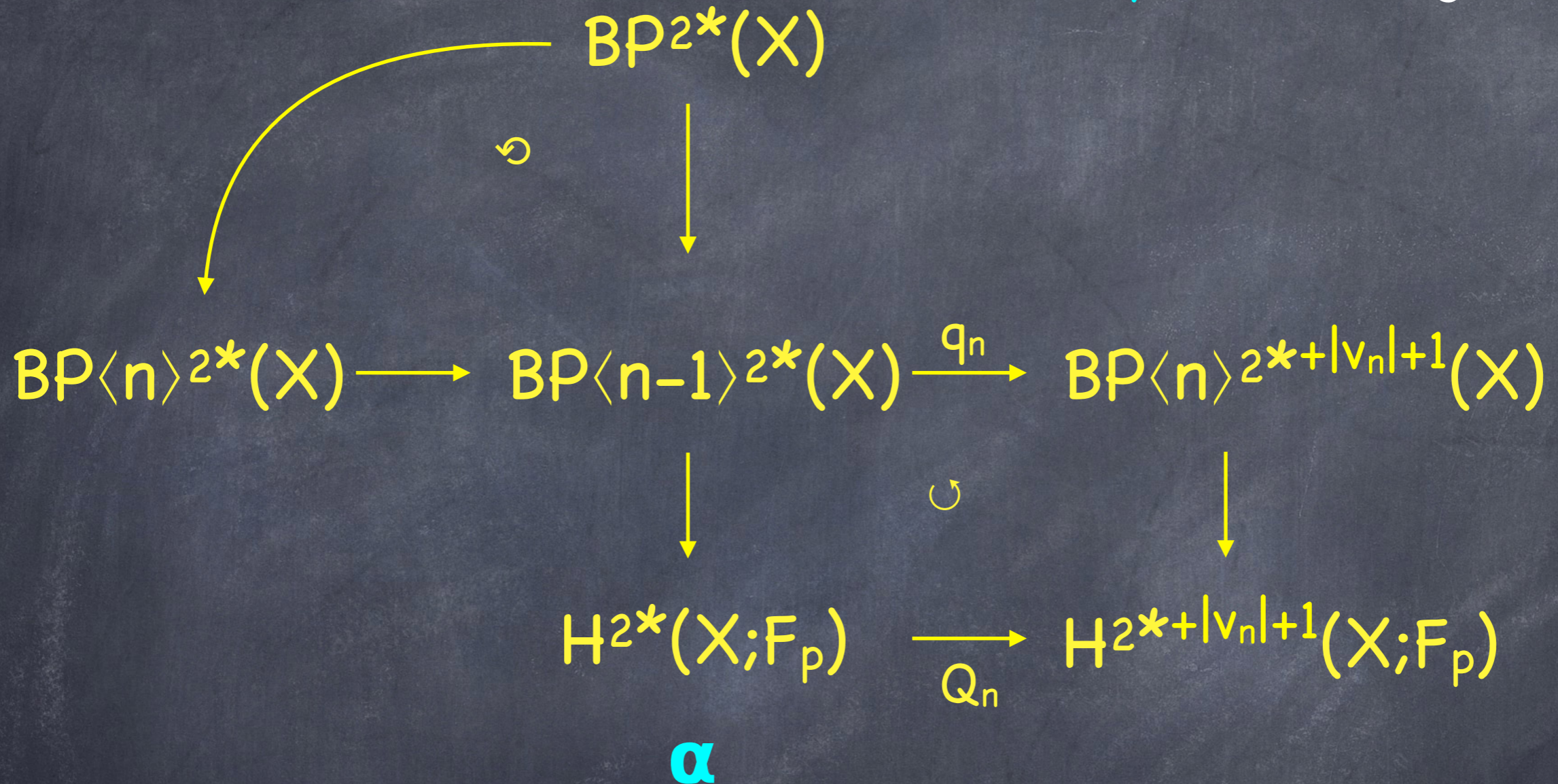
The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth
projective algebraic



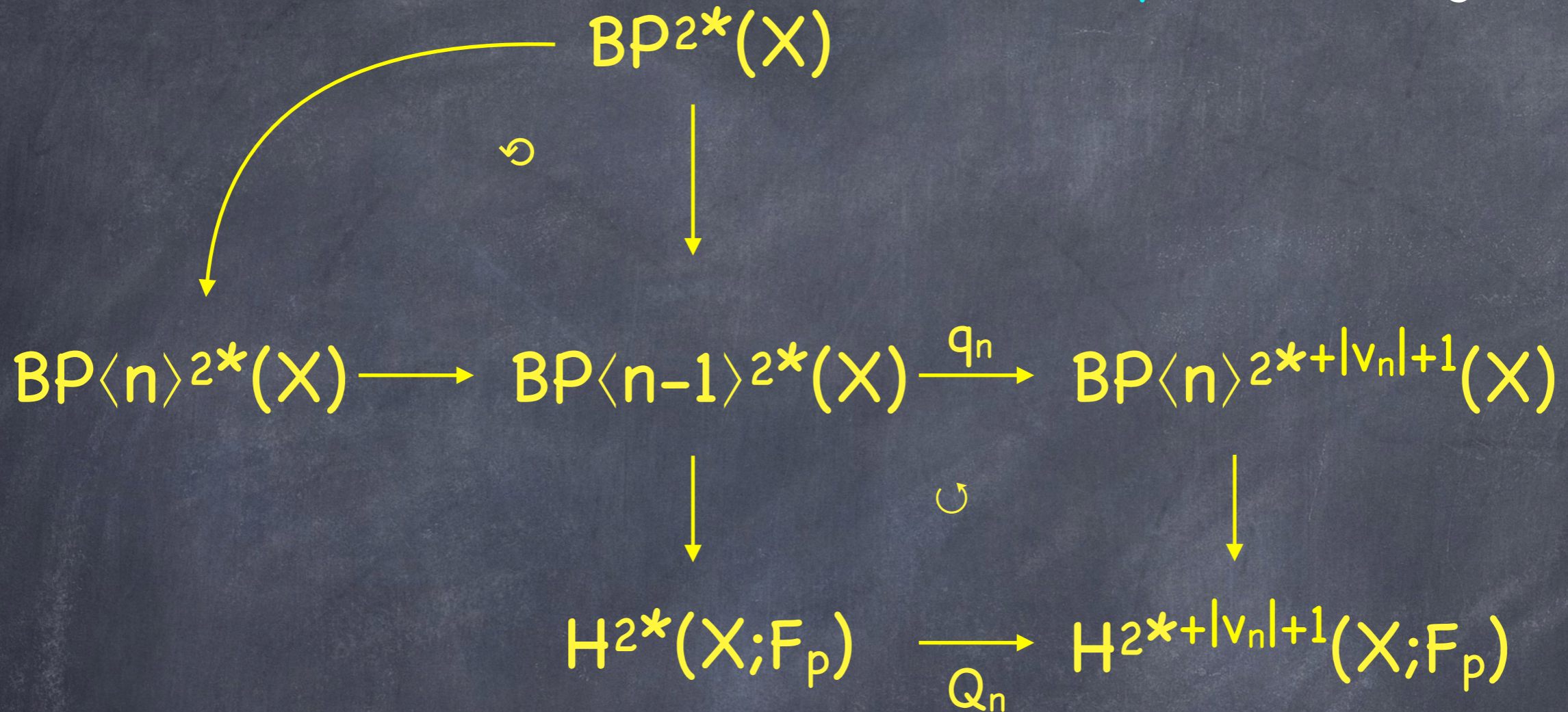
The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth
projective algebraic



The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth
projective algebraic

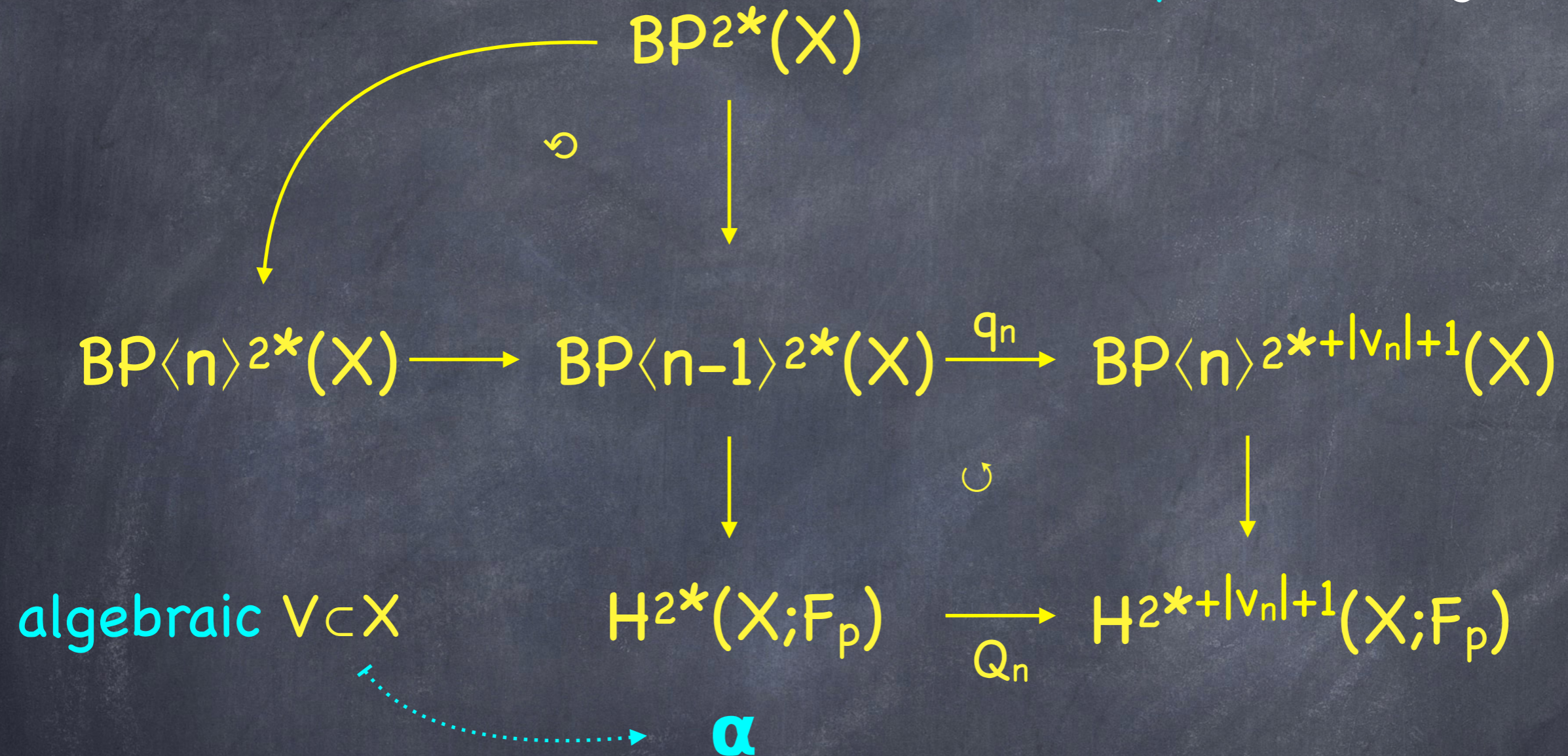


α

Question: Is α algebraic?

The LMT obstruction in action:

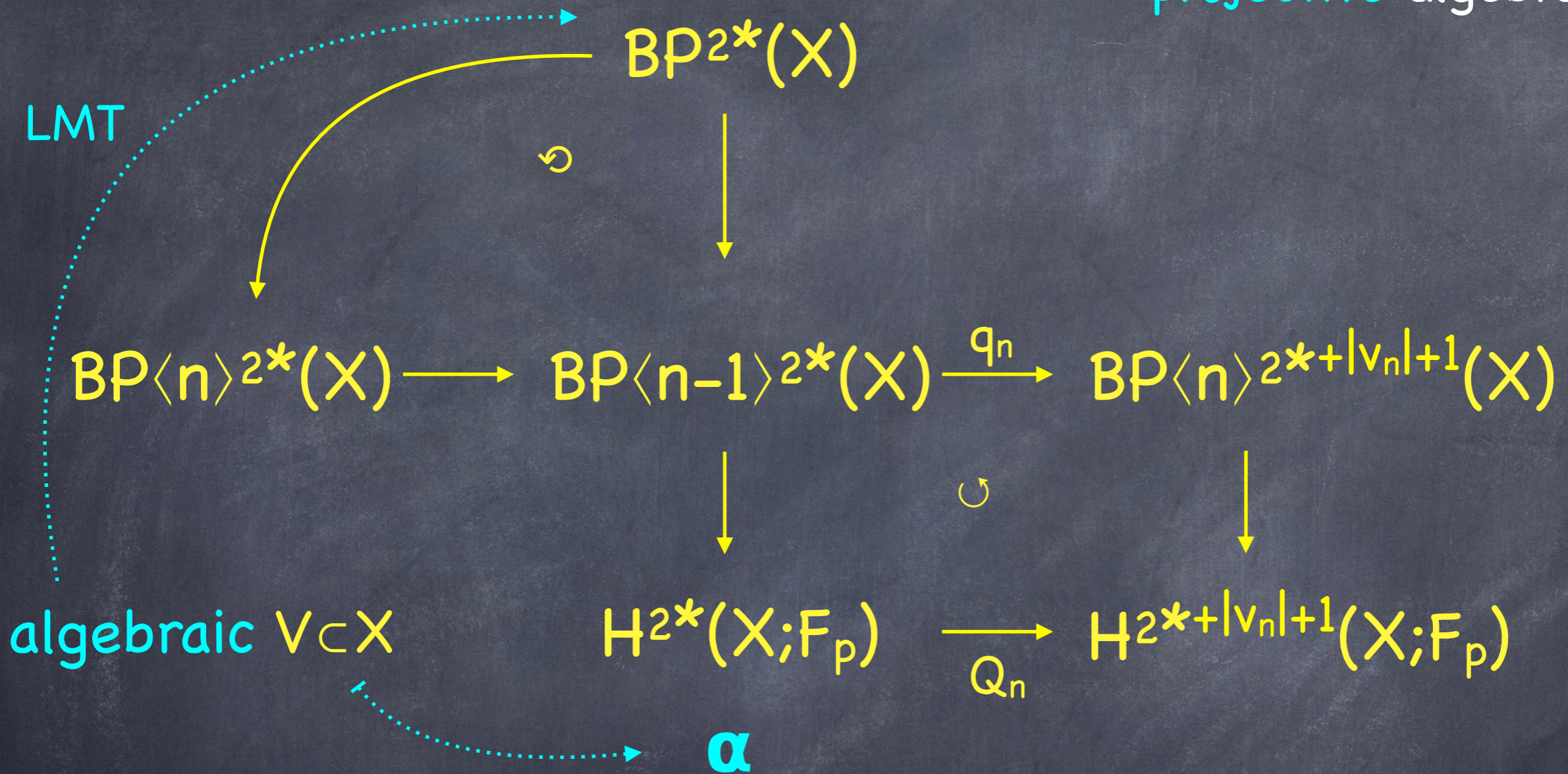
$X \subset \mathbb{C}P^N$ smooth
projective algebraic



Question: Is α algebraic?

The LMT obstruction in action:

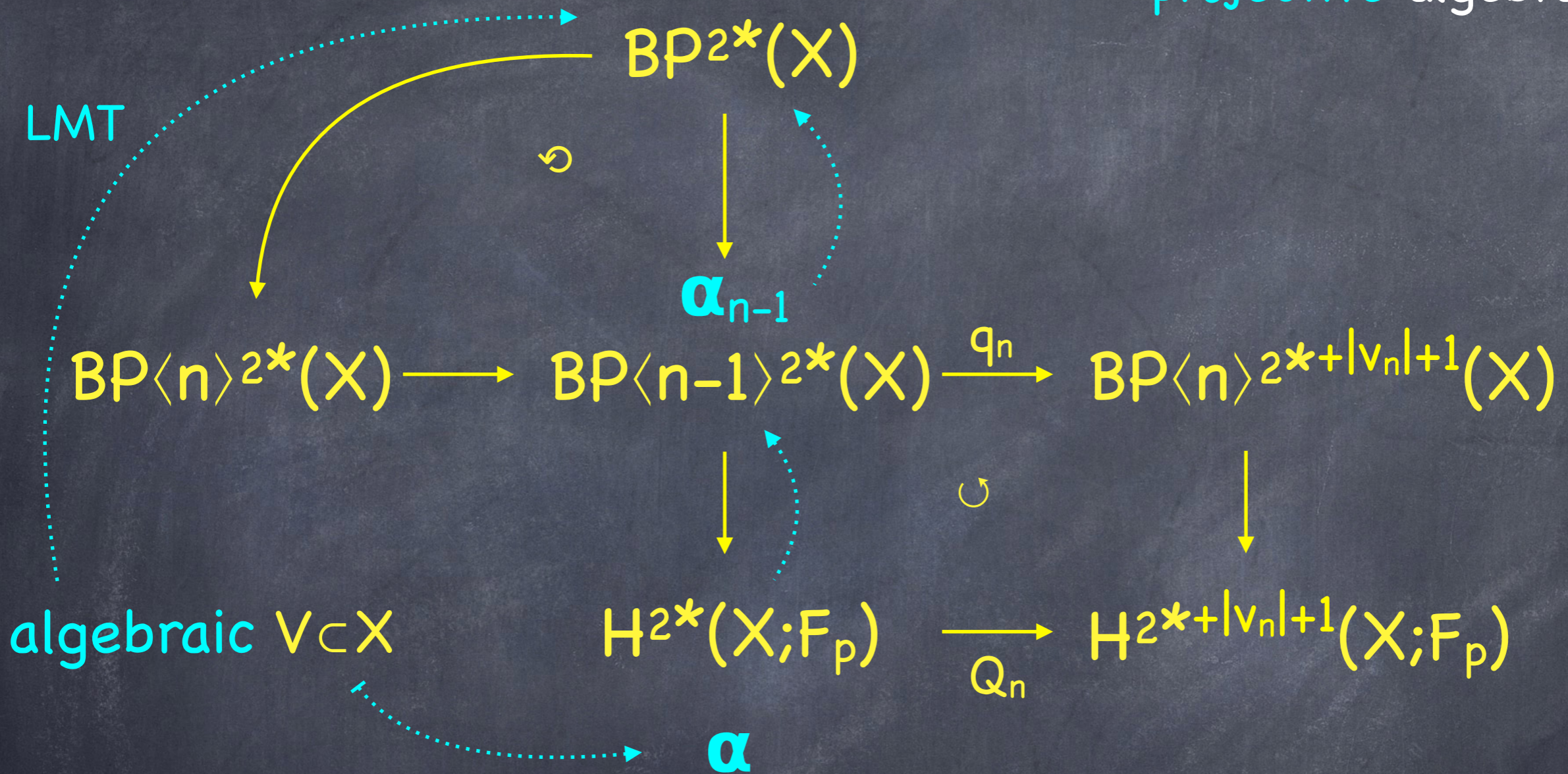
$X \subset \mathbb{C}P^N$ smooth
projective algebraic



Question: Is α algebraic?

The LMT obstruction in action:

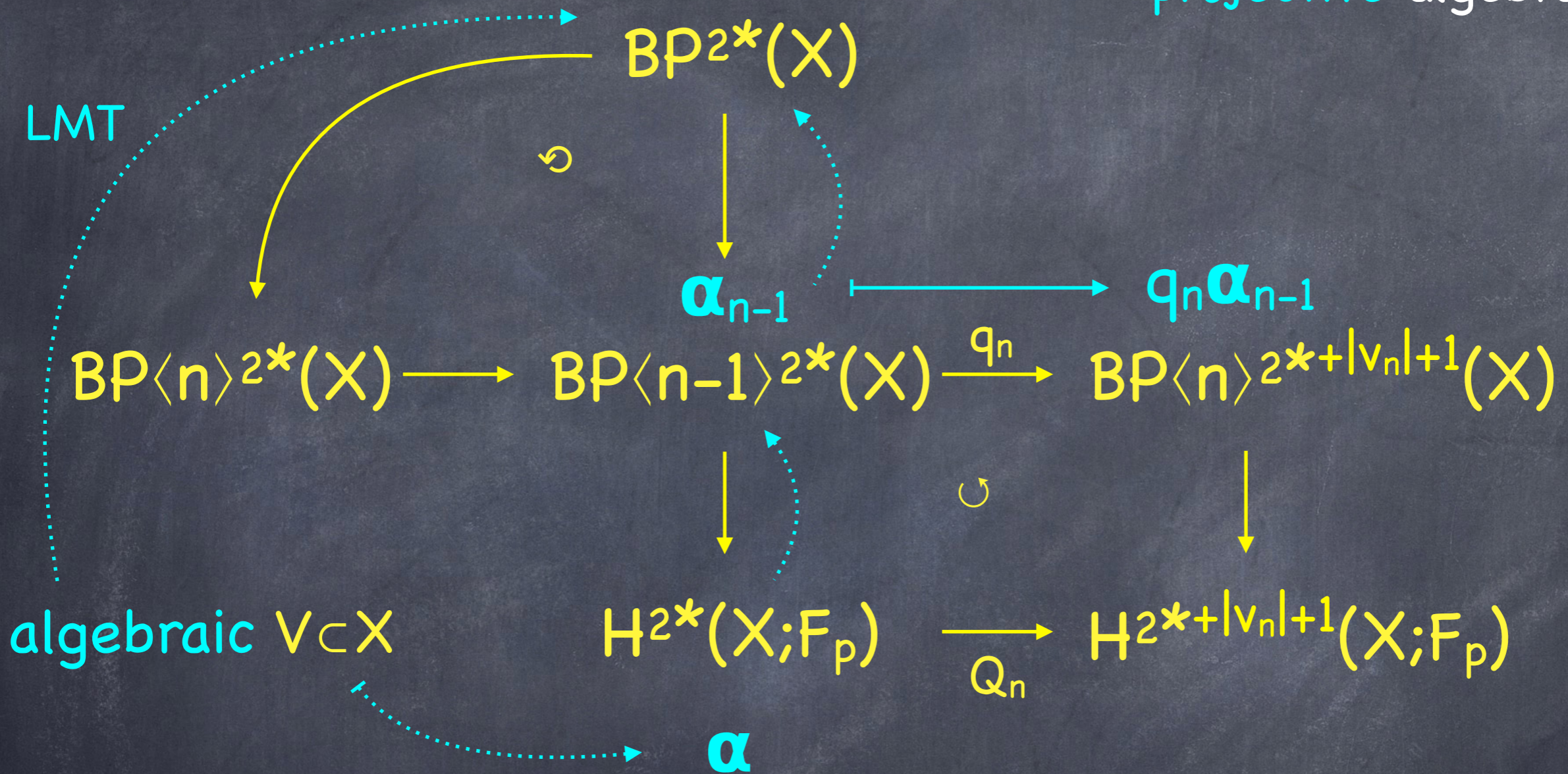
$X \subset \mathbb{C}P^N$ smooth
projective algebraic



Question: Is α algebraic?

The LMT obstruction in action:

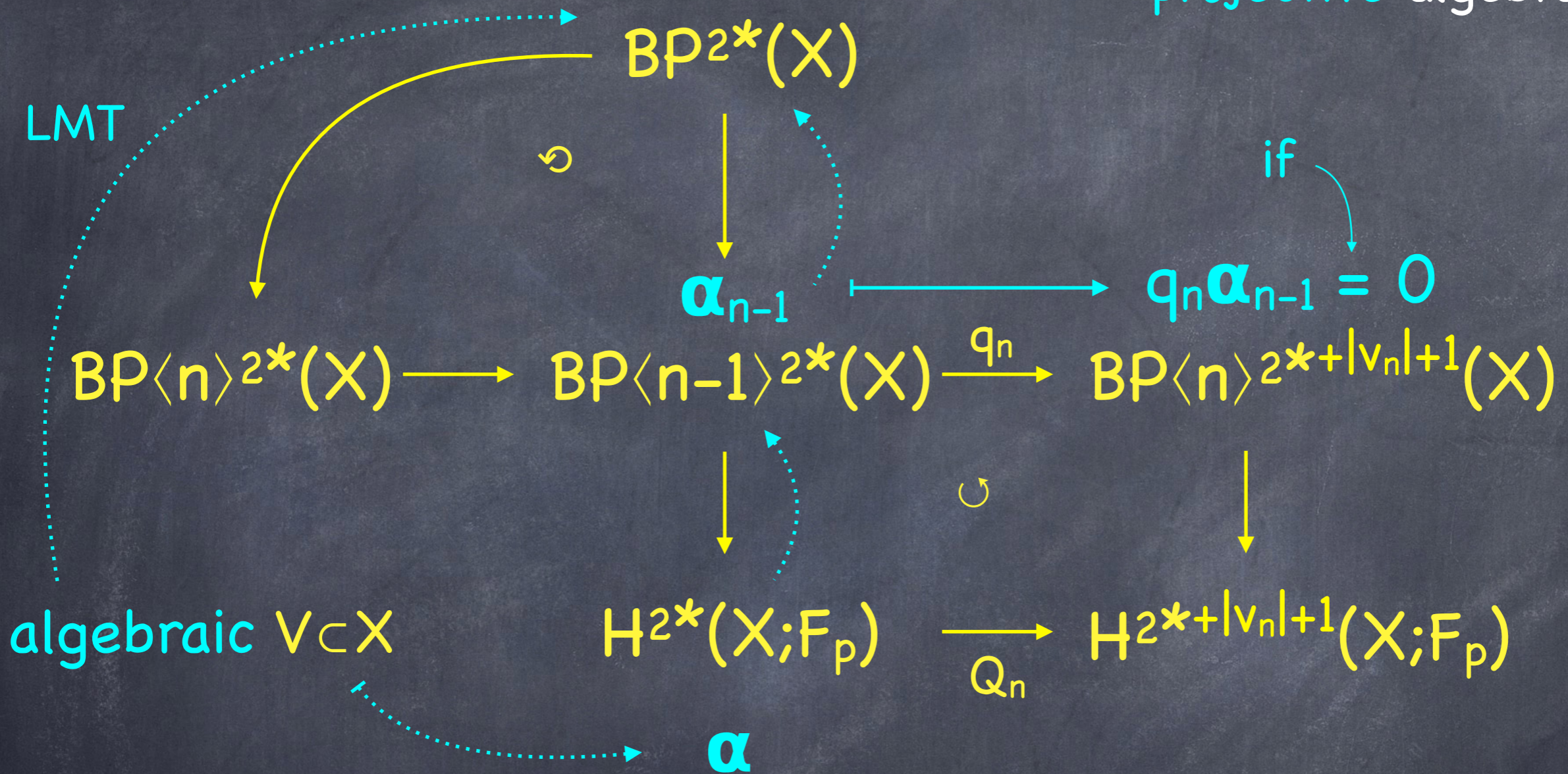
$X \subset \mathbb{C}P^N$ smooth
projective algebraic



Question: Is α algebraic?

The LMT obstruction in action:

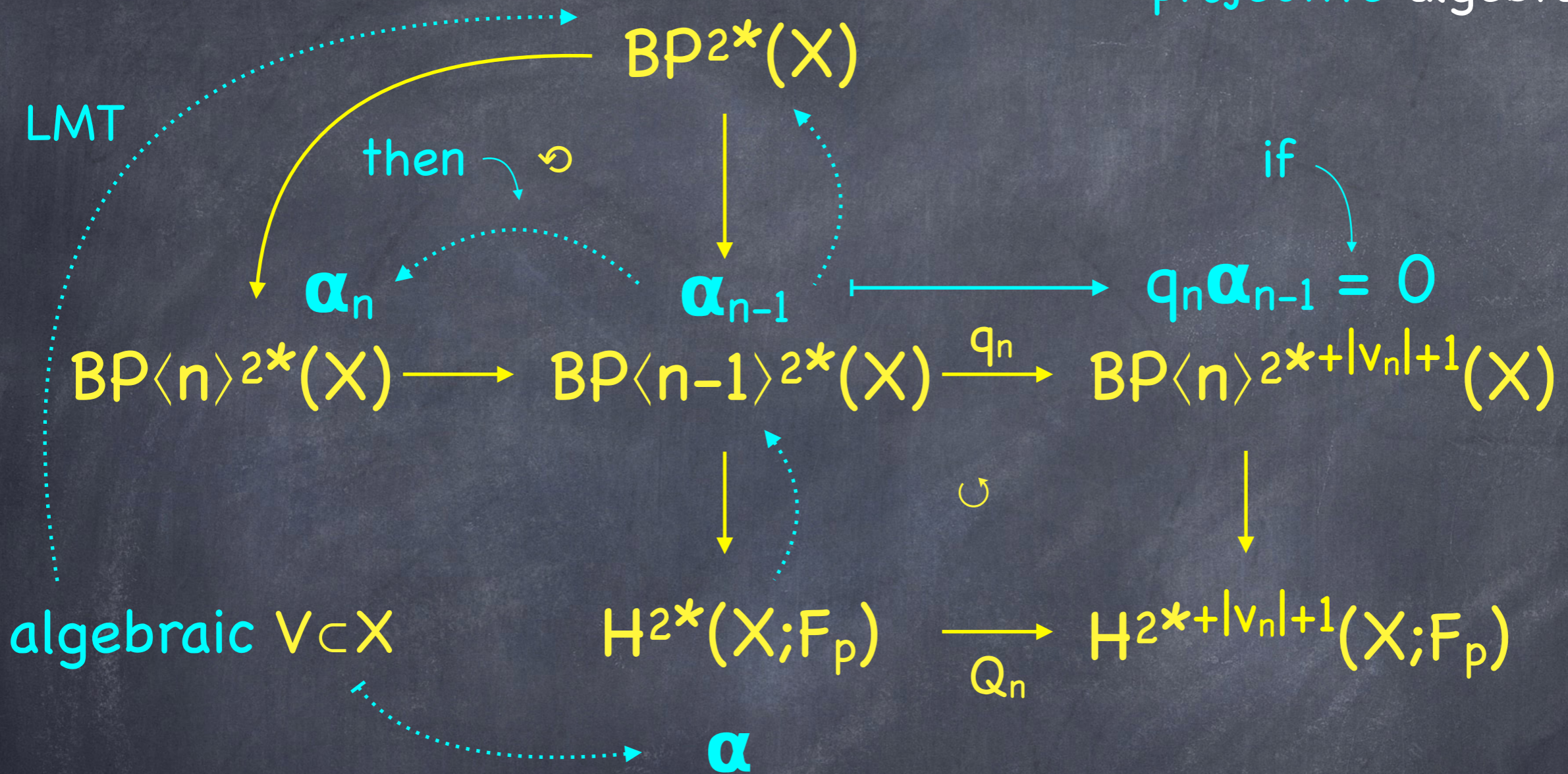
$X \subset \mathbb{C}P^N$ smooth
projective algebraic



Question: Is α algebraic?

The LMT obstruction in action:

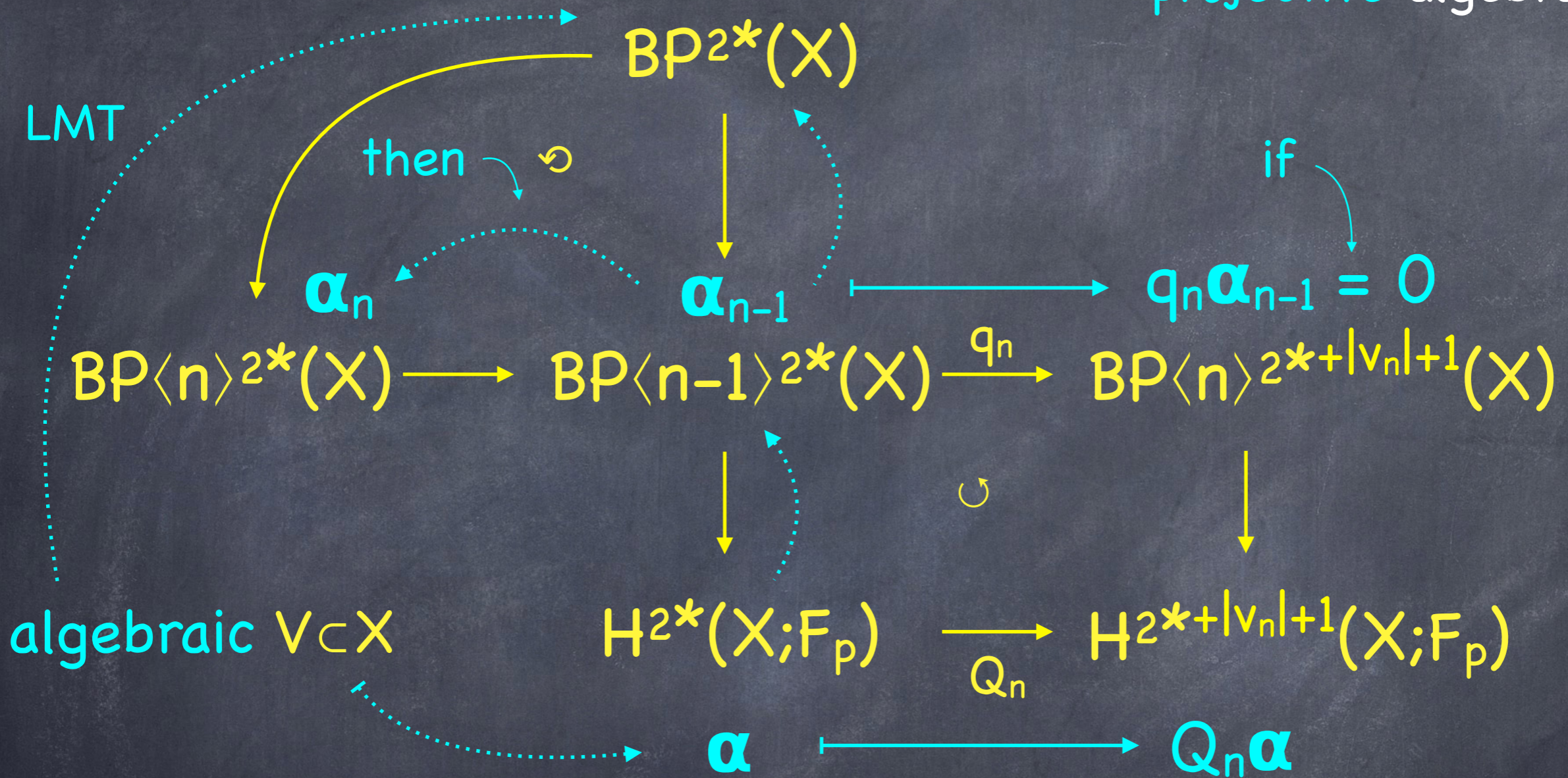
$X \subset \mathbb{C}P^N$ smooth
projective algebraic



Question: Is α algebraic?

The LMT obstruction in action:

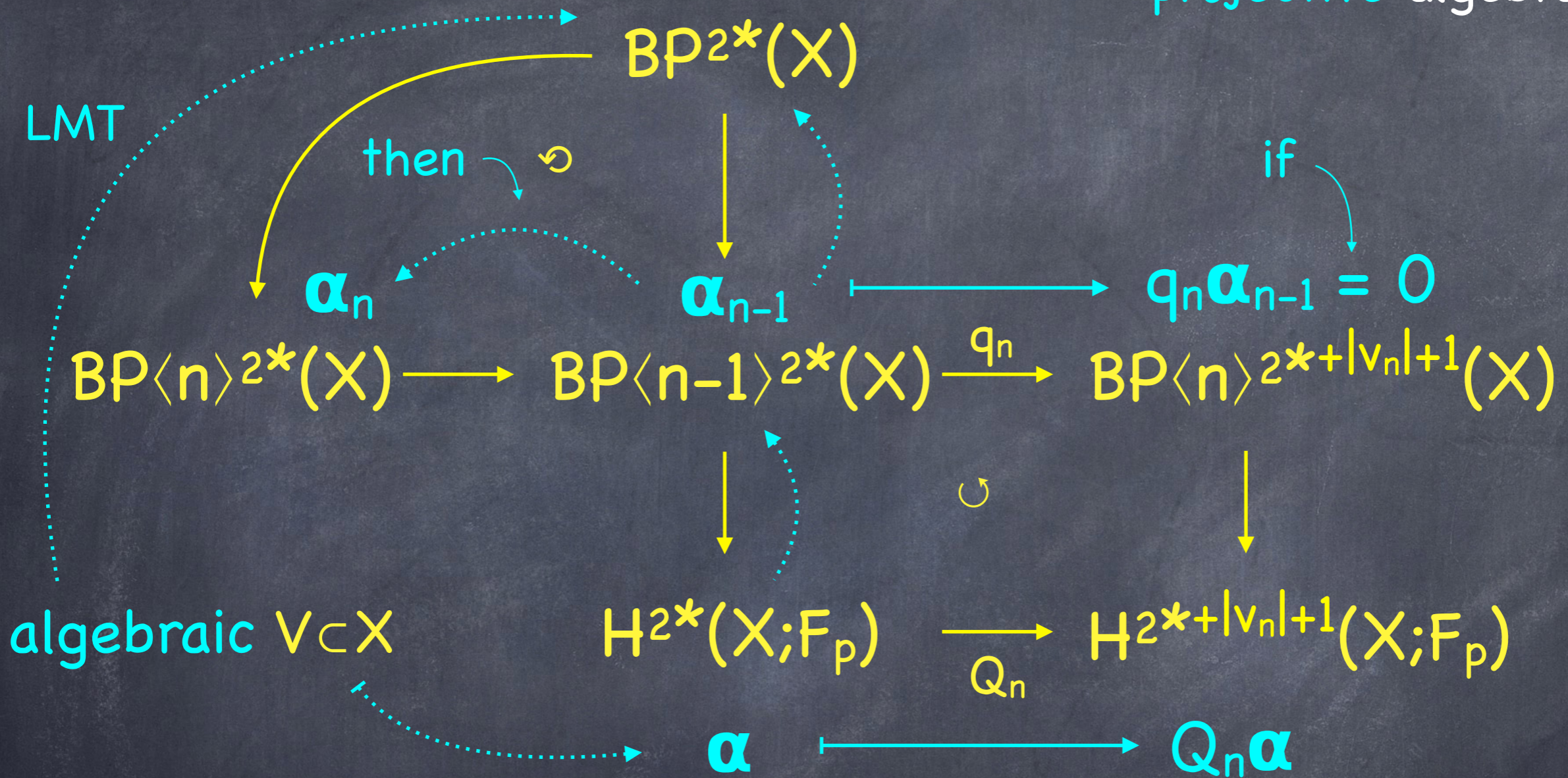
$X \subset \mathbb{C}P^N$ smooth
projective algebraic



Question: Is α algebraic?

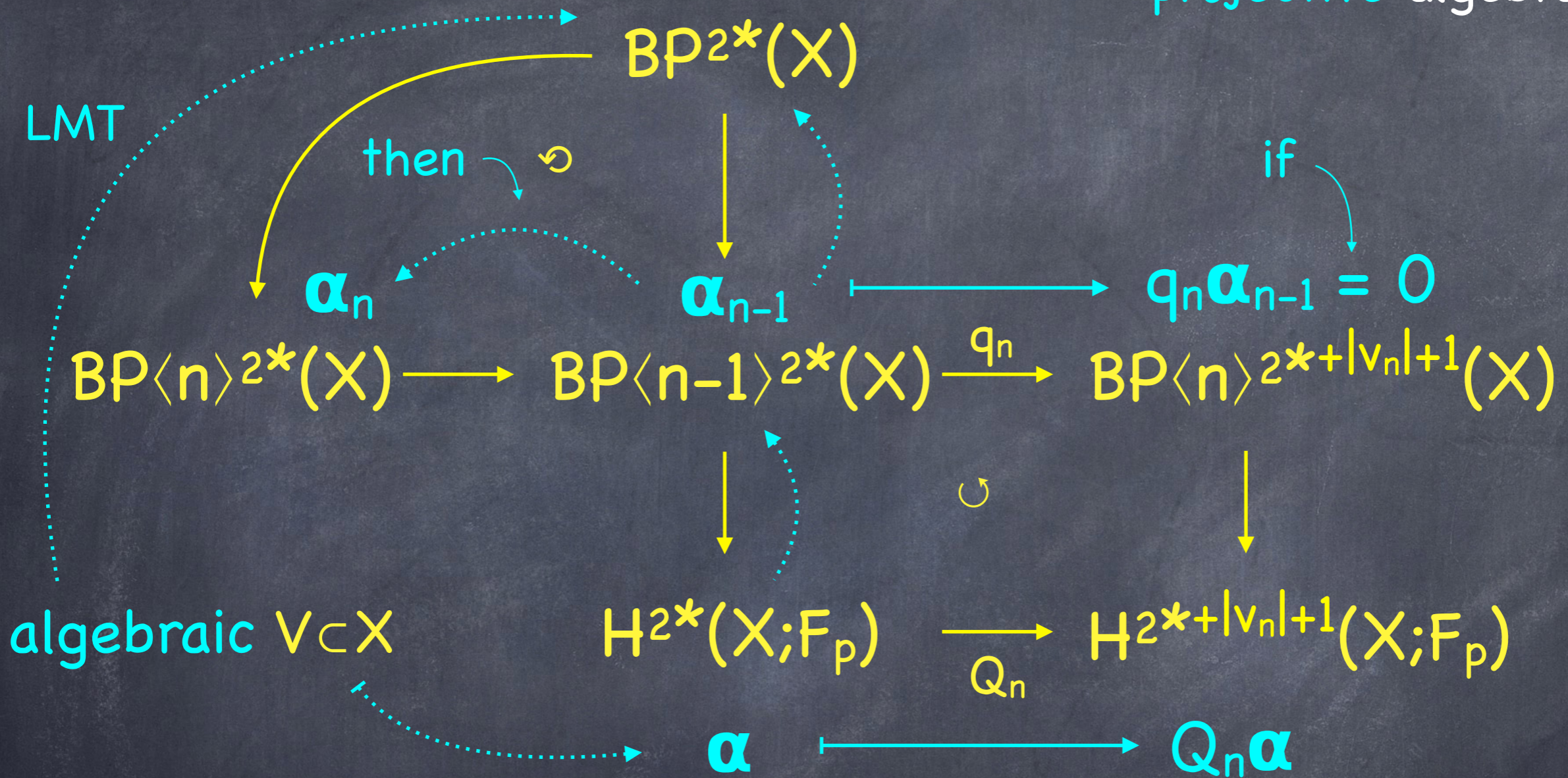
The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth
projective algebraic



The LMT obstruction in action:

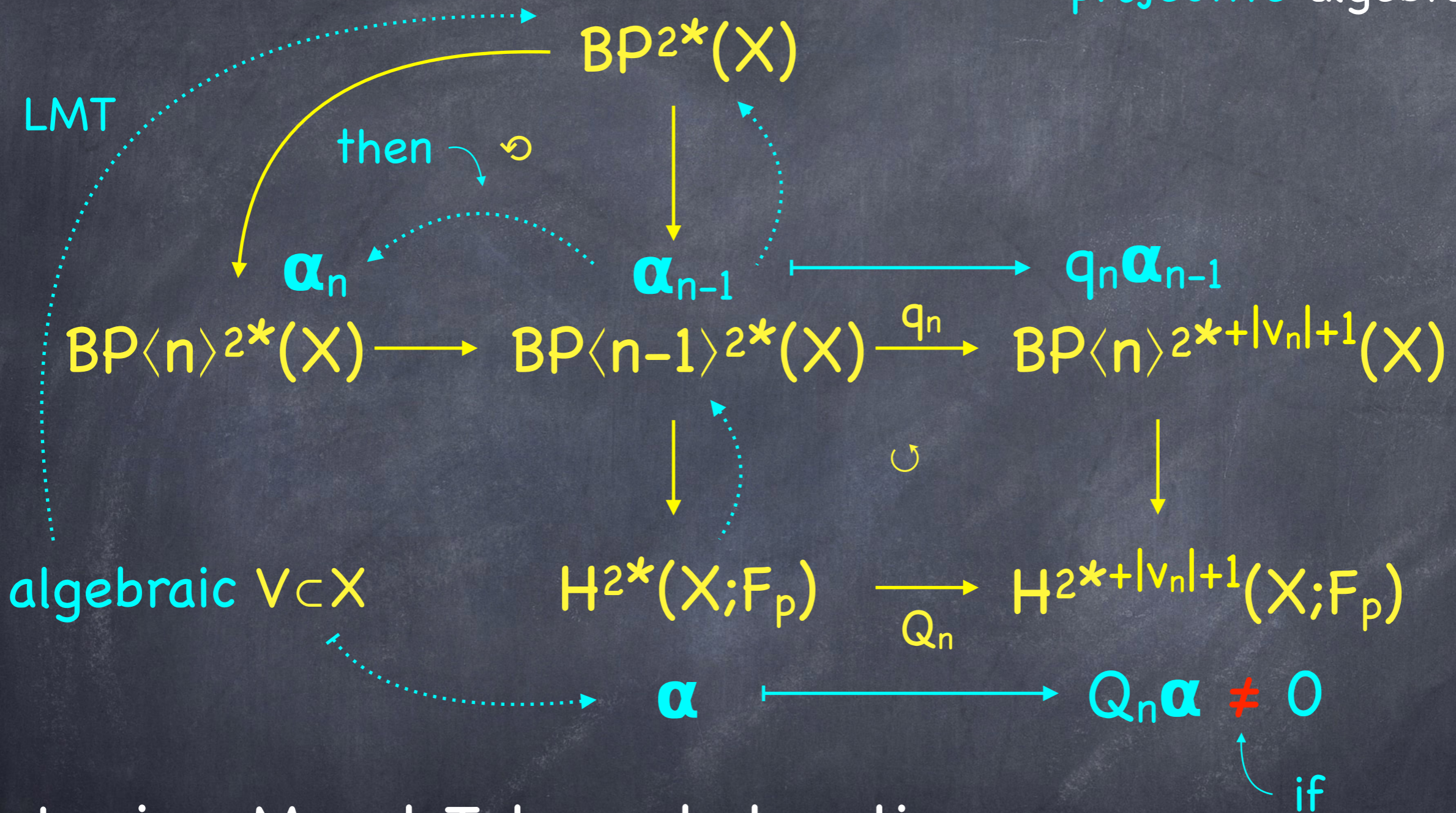
$X \subset \mathbb{C}P^N$ smooth
projective algebraic



Levine-Morel-Totaro obstruction:

The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth
projective algebraic

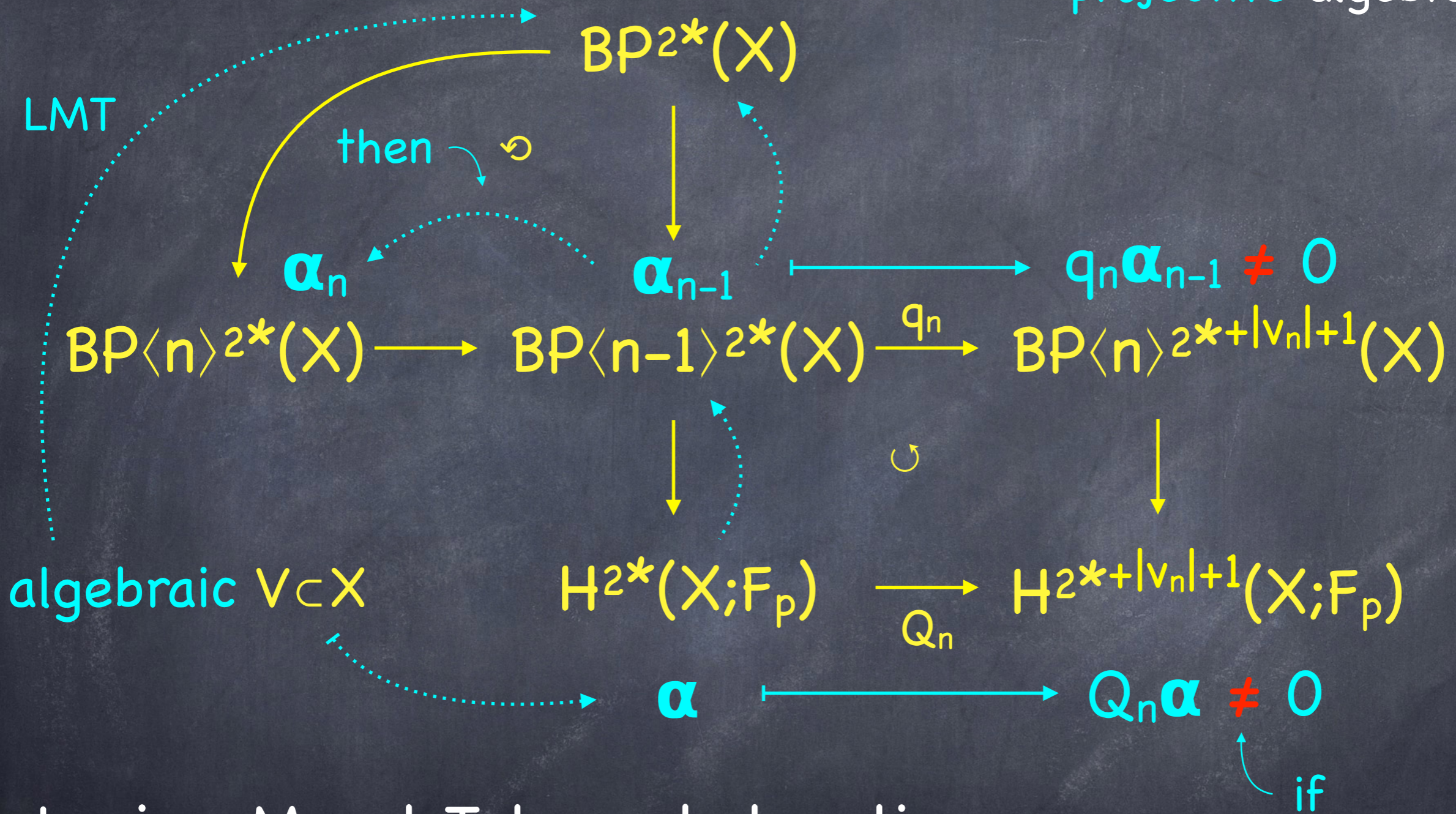


Levine–Morel–Totaro obstruction:

If $Q_n \alpha \neq 0$,

The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth
projective algebraic

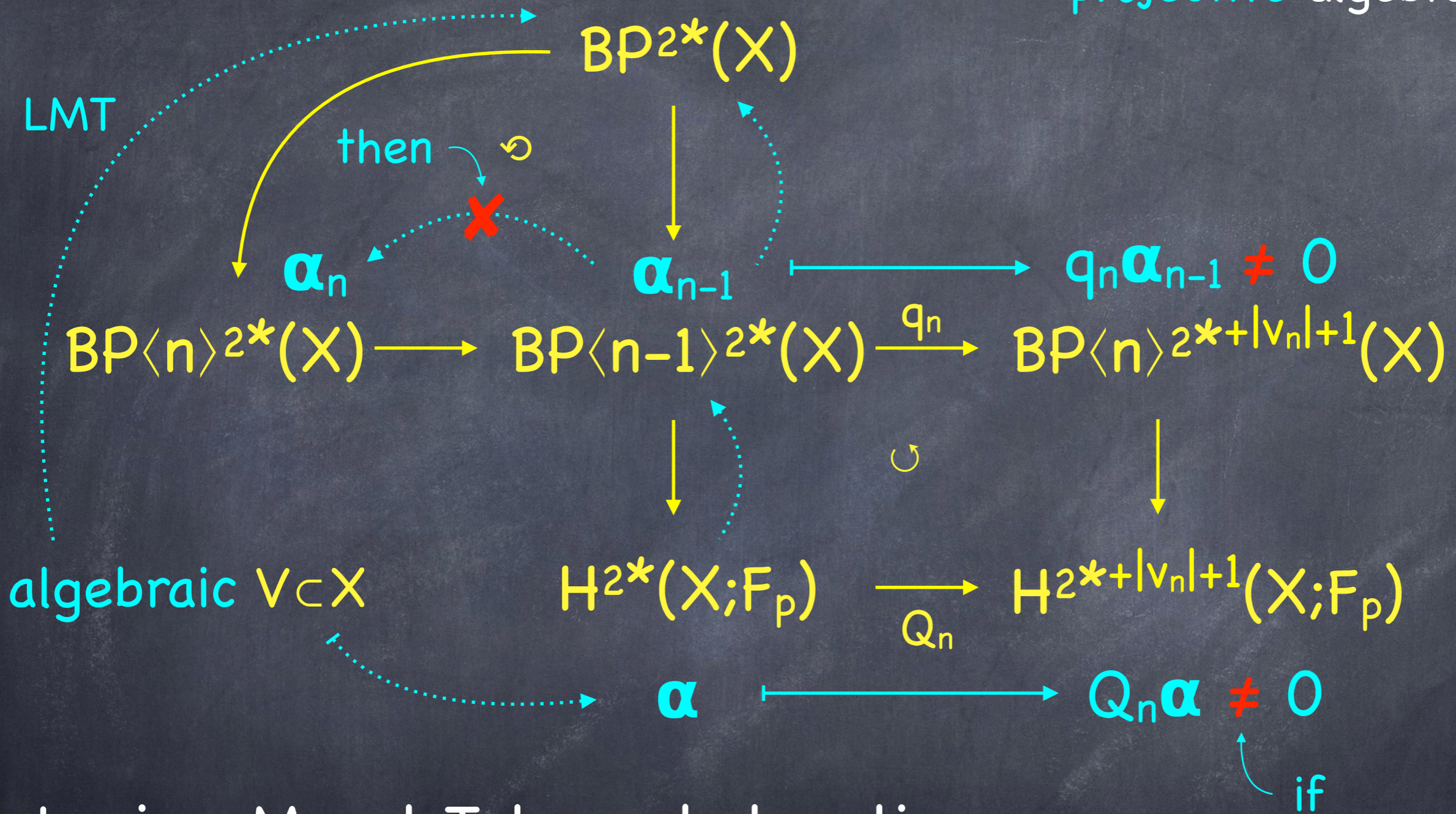


Levine–Morel–Totaro obstruction:

If $Q_n \alpha \neq 0$,

The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth
projective algebraic



Levine–Morel–Totaro obstruction:

If $Q_n \alpha \neq 0$,

Voevodsky's motivic Milnor operations:

Voevodsky's motivic Milnor operations:

There are motivic operations

$$Q_n^{\text{mot}} \in \mathcal{A}^{2p^n-1, p^n-1}$$

mod p -motivic
Steenrod algebra

Voevodsky's motivic Milnor operations:

There are motivic operations

$$Q_n^{\text{mot}} \in \mathcal{A}^{2p^n-1, p^n-1}$$

mod p -motivic
Steenrod algebra

For a smooth complex variety X :

$$H_{\text{mot}}^{i,j}(X; \mathbb{F}_p) \xrightarrow{Q_n^{\text{mot}}} H_{\text{mot}}^{i+2p^n-1, j+p^n-1}(X; \mathbb{F}_p)$$

mod p -motivic
cohomology

Voevodsky's motivic Milnor operations:

There are motivic operations

$$Q_n^{\text{mot}} \in \mathcal{A}^{2p^n-1, p^n-1}$$

mod p -motivic
Steenrod algebra

For a smooth complex variety X :

$$H_{\text{mot}}^{i,j}(X; \mathbb{F}_p) \xrightarrow{Q_n^{\text{mot}}} H_{\text{mot}}^{i+2p^n-1, j+p^n-1}(X; \mathbb{F}_p)$$

mod p -motivic
cohomology

Recall: $H_{\text{mot}}^{i,j}(X; \mathbb{F}_p) = 0$ if $i > 2j$.

Obstructions revisited:

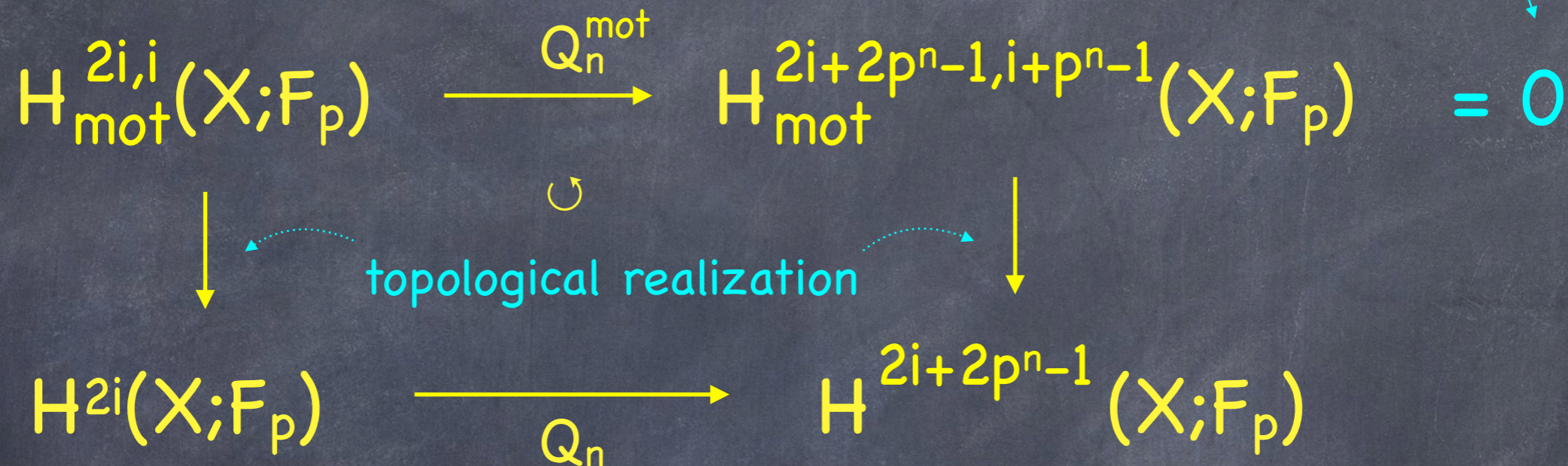
X smooth complex variety

$$\begin{array}{ccc} H_{\text{mot}}^{2i,i}(X; \mathbb{F}_p) & \xrightarrow{Q_n^{\text{mot}}} & H_{\text{mot}}^{2i+2p^{n-1}, i+p^{n-1}}(X; \mathbb{F}_p) \\ \downarrow & \curvearrowright & \downarrow \\ H^{2i}(X; \mathbb{F}_p) & \xrightarrow{Q_n} & H^{2i+2p^{n-1}}(X; \mathbb{F}_p) \end{array}$$

topological realization

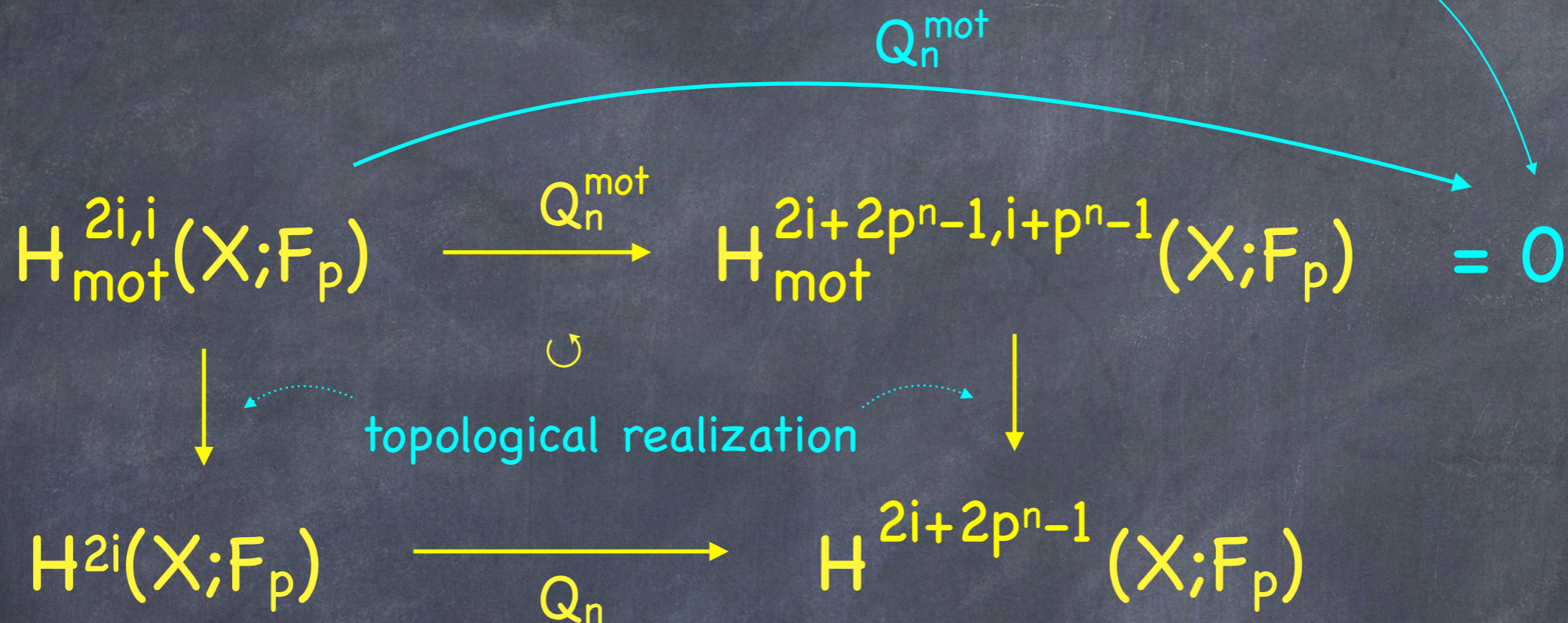
Obstructions revisited:

X smooth complex variety



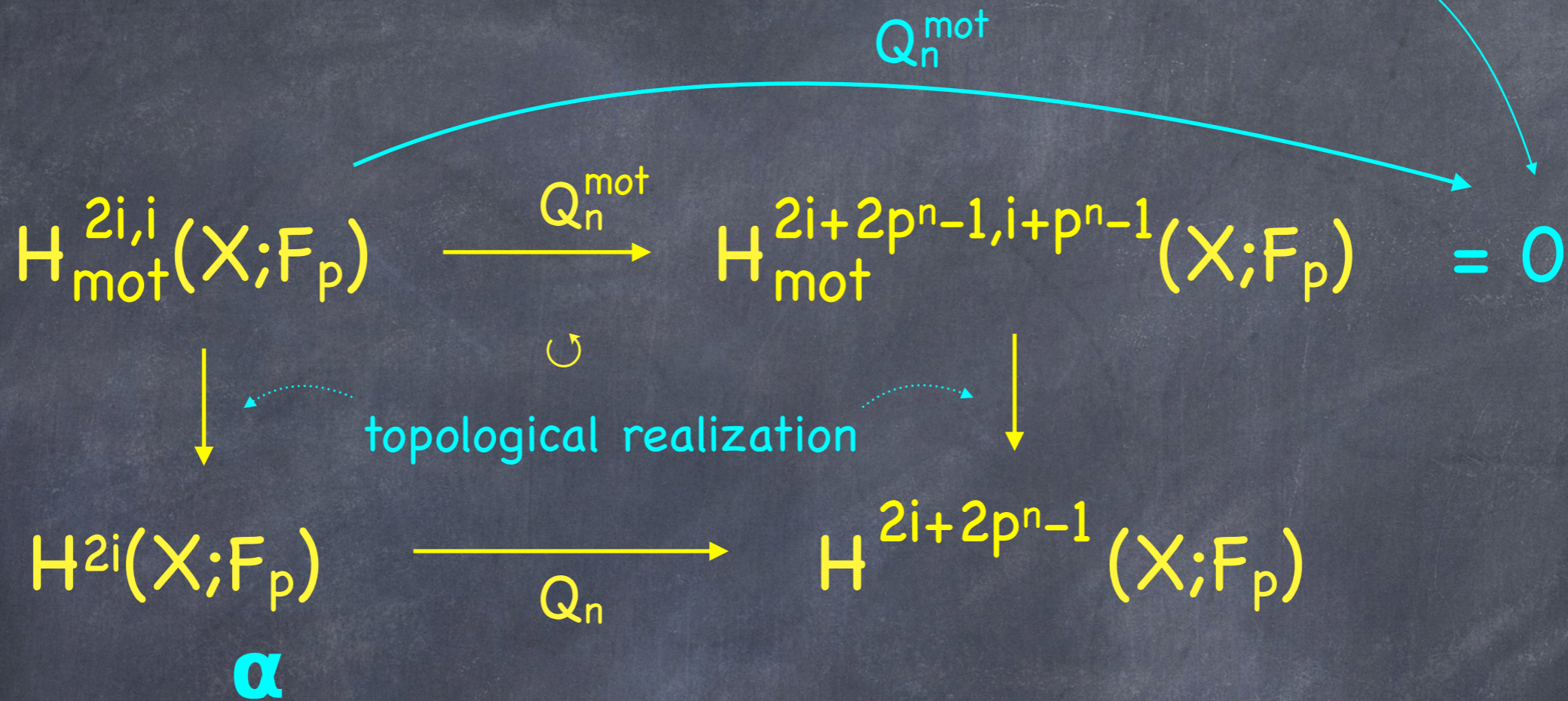
Obstructions revisited:

X smooth complex variety



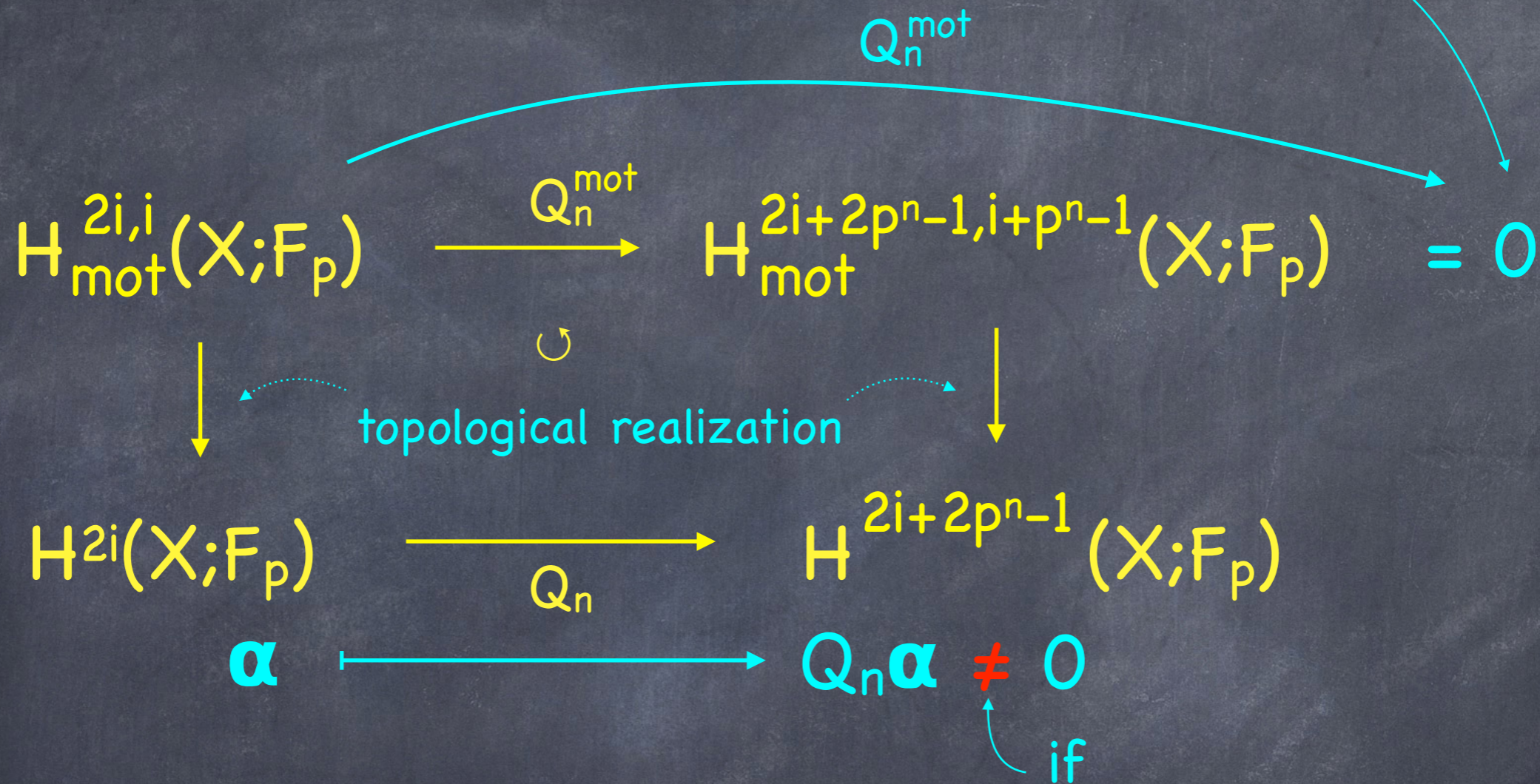
Obstructions revisited:

X smooth complex variety



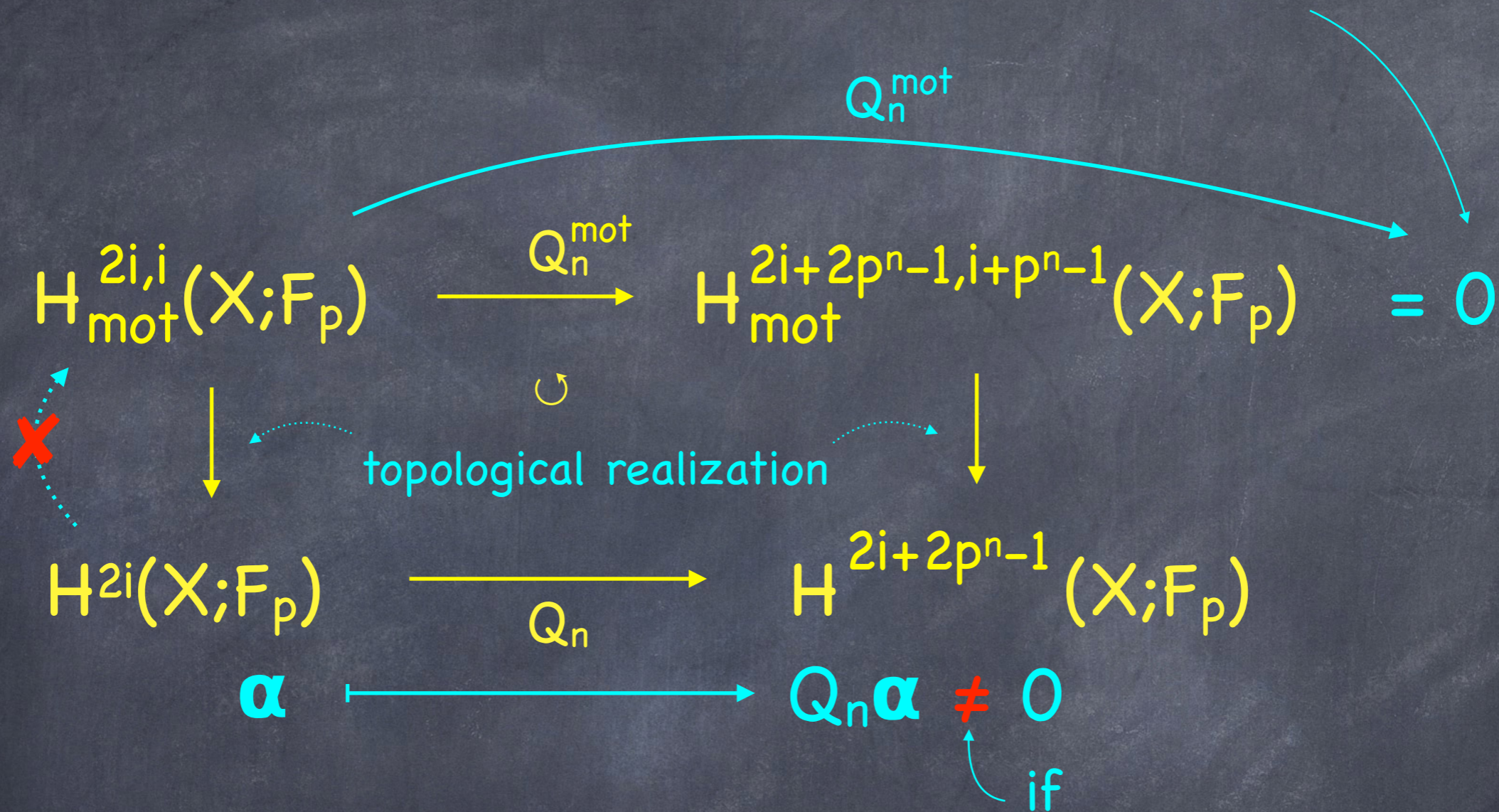
Obstructions revisited:

X smooth complex variety



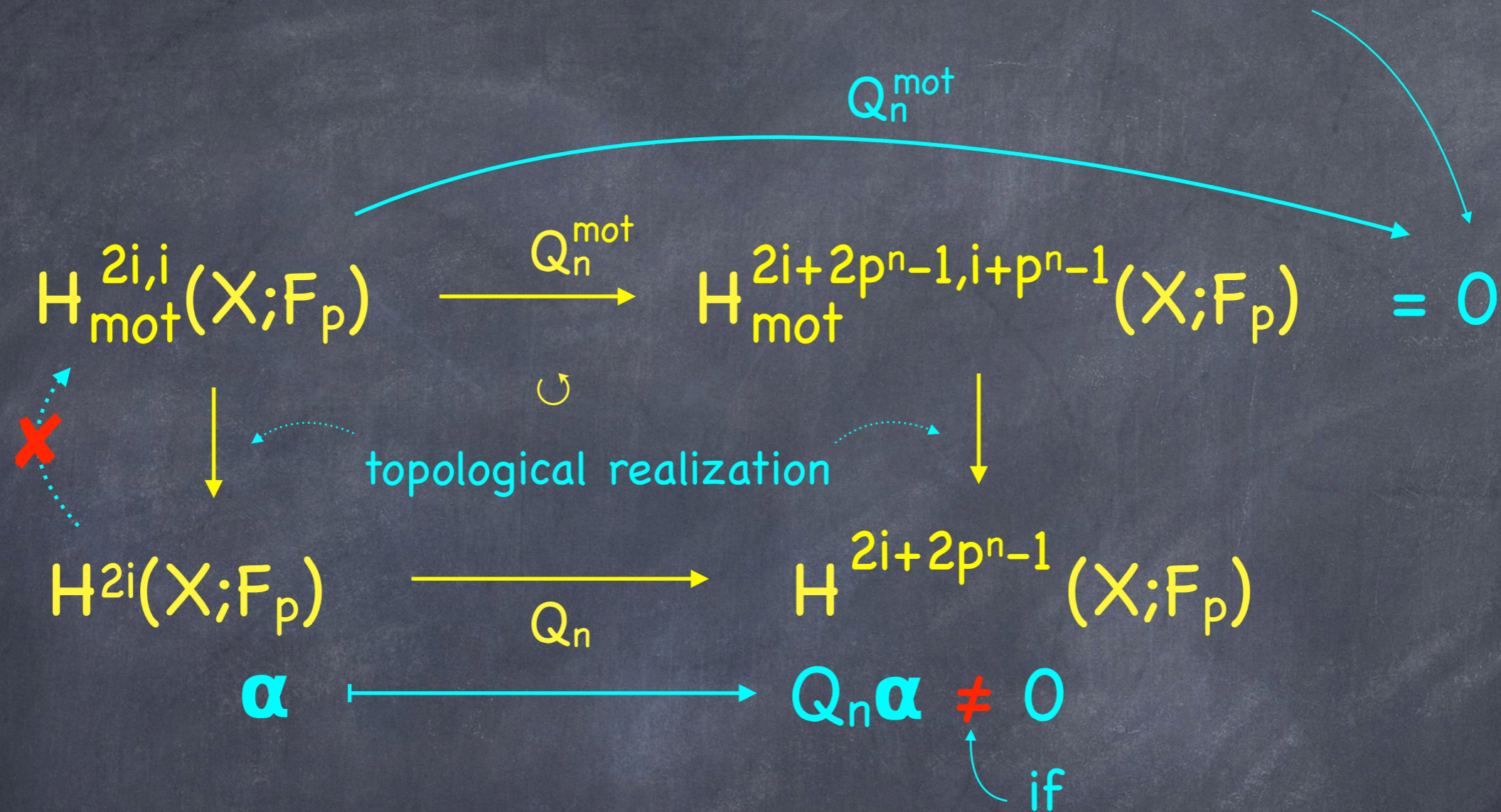
Obstructions revisited:

X smooth complex variety



Obstructions revisited:

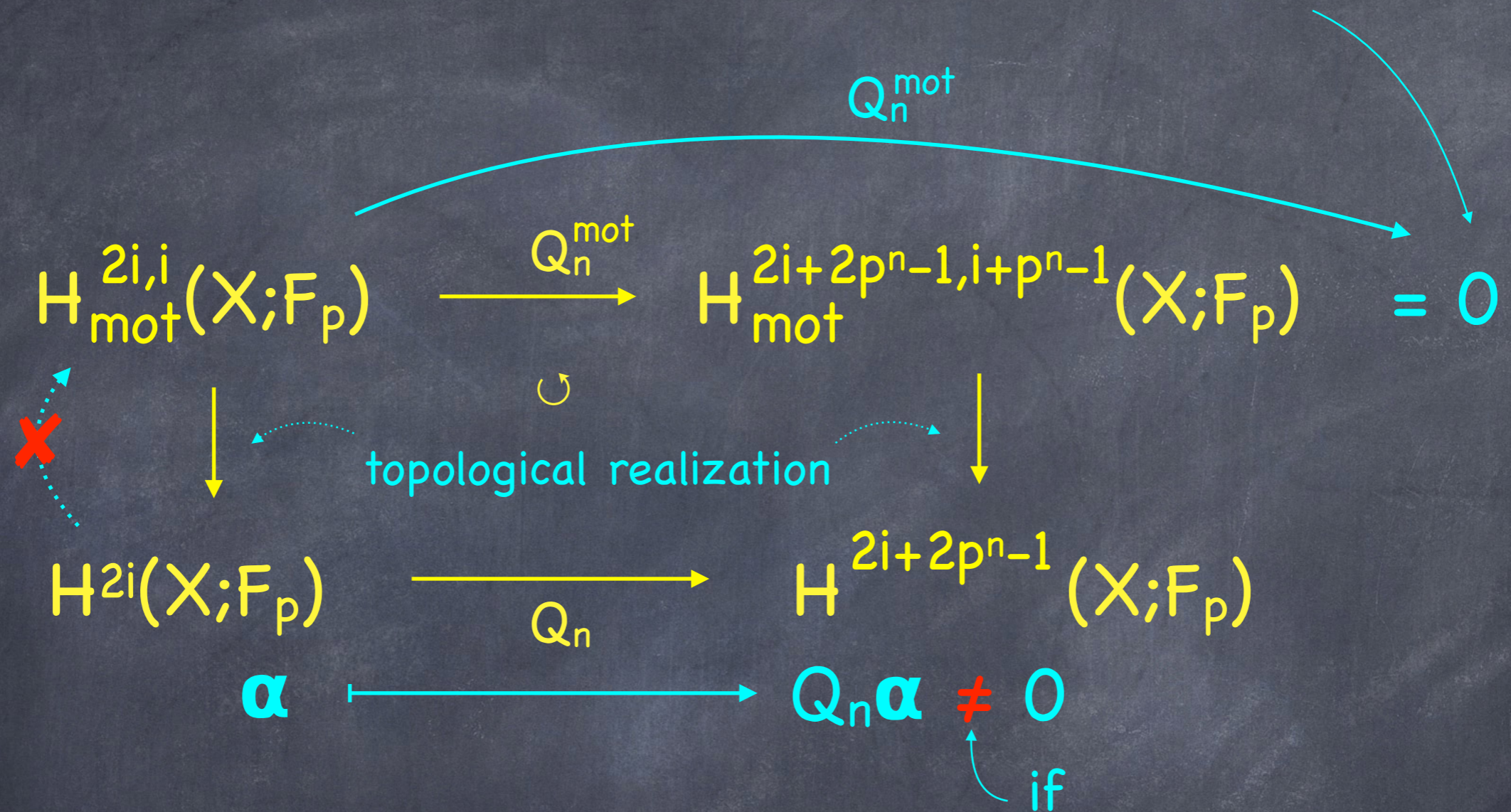
X smooth complex variety



Observation: The LMT-obstruction is particular to smooth varieties and bidegrees $(2i, i)$.

Obstructions revisited:

X smooth complex variety



Observation: The LMT-obstruction is particular to smooth varieties and bidegrees $(2i, i)$.

Example: $Q_n \mathbf{1} \neq 0$ for $\mathbf{1}$ the fundamental class of a suitable Eilenberg-MacLane space, though $\mathbf{1}$ is algebraic.

Back to our task:

Back to our task:

Find non-algebraic classes in $E_{\text{top}}^{2*}(X)$.

Back to our task:

Find non-algebraic classes in $E_{\text{top}}^{2*}(X)$.

For example: $E=BP\langle n\rangle$?

Back to our task:

Find non-algebraic classes in $E_{\text{top}}^{2*}(X)$.

For example: $E=BP\langle n\rangle$?

Recall: BP and $BP\langle n\rangle$ exist in the motivic world (e.g. Hopkins, Vezzosi, Hu-Kriz, Ormsby, Hoyois, Ormsby-Østvær).

Back to our task:

Find non-algebraic classes in $E_{\text{top}}^{2*}(X)$.

For example: $E=BP\langle n\rangle$?

Recall: BP and $BP\langle n\rangle$ exist in the motivic world (e.g. Hopkins, Vezzosi, Hu-Kriz, Ormsby, Hoyois, Ormsby-Østvær).

Question: How can we produce non-algebraic elements in $BP\langle n\rangle_{\text{top}}^{2*}(X)$?

We can lift...

$$H^k(X; \mathbb{F}_p)$$

We can lift...

$$H^{k+1}(X; \mathbb{Z}_{(p)})$$

$$q_0 \uparrow$$

$$H^k(X; \mathbb{F}_p)$$

We can lift...

$$\mathrm{BP}\langle 1 \rangle^{k+1+2p-1}(X)$$

$$q_1 \uparrow$$

$$H^{k+1}(X; \mathbb{Z}_{(p)})$$

$$q_0 \uparrow$$

$$H^k(X; \mathbb{F}_p)$$

We can lift...

$$BP\langle n \rangle^{k+|Q_0|+\dots+|Q_n|}(X)$$

$$q_n \uparrow$$

⋮

$$q_2 \uparrow$$

$$BP\langle 1 \rangle^{k+1+2p-1}(X)$$

$$q_1 \uparrow$$

$$H^{k+1}(X; \mathbb{Z}_{(p)})$$

$$q_0 \uparrow$$

$$H^k(X; \mathbb{F}_p)$$

We can lift...

$$\mathrm{BP}\langle n \rangle_{k+|Q_0|+\dots+|Q_n|}(X) \xrightarrow{q_{n+1}} \mathrm{BP}\langle n+1 \rangle_{k+|Q_0|+\dots+|Q_{n+1}|}(X)$$

$$q_n \uparrow$$

⋮

$$q_2 \uparrow$$

$$\mathrm{BP}\langle 1 \rangle_{k+1+2p-1}(X)$$

$$q_1 \uparrow$$

$$H^{k+1}(X; \mathbb{Z}_{(p)})$$

$$q_0 \uparrow$$

$$H^k(X; \mathbb{F}_p)$$

We can lift...

$$\mathrm{BP}\langle n \rangle_{k+|Q_0|+\dots+|Q_n|}(X) \xrightarrow{q_{n+1}} \mathrm{BP}\langle n+1 \rangle_{k+|Q_0|+\dots+|Q_{n+1}|}(X)$$

$$q_n \uparrow$$

⋮

$$q_2 \uparrow$$

$$\mathrm{BP}\langle 1 \rangle_{k+1+2p-1}(X)$$

$$q_1 \uparrow$$

$$H^{k+1}(X; \mathbb{Z}_{(p)})$$

$$q_0 \uparrow$$

$$H^k(X; \mathbb{F}_p)$$



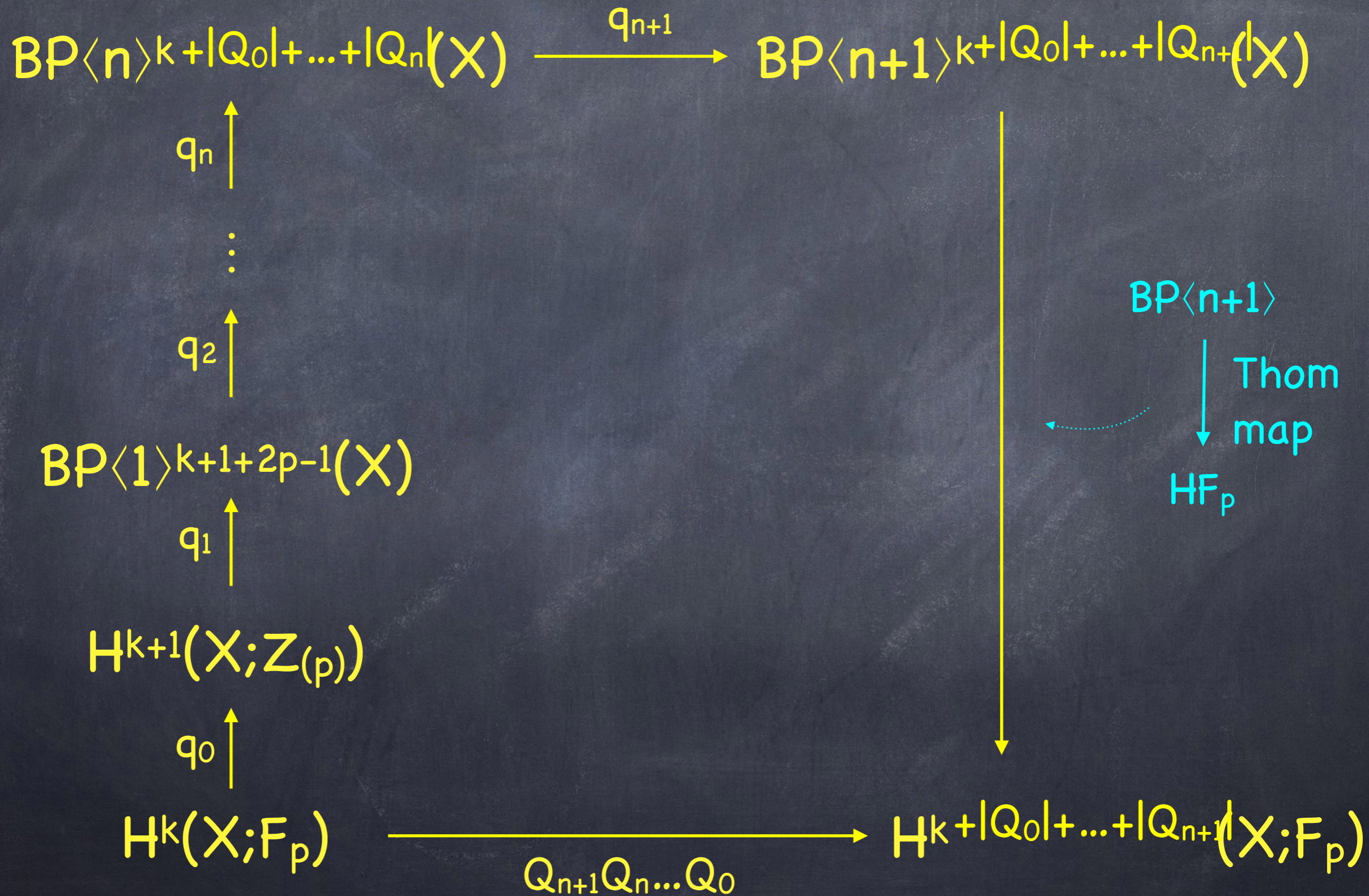
$\mathrm{BP}\langle n+1 \rangle$

Thom
map

$H\mathbb{F}_p$

$$H^{k+|Q_0|+\dots+|Q_{n+1}|}(X; \mathbb{F}_p)$$

We can lift...



Lifting classes:

We produced

Lifting classes:

We produced

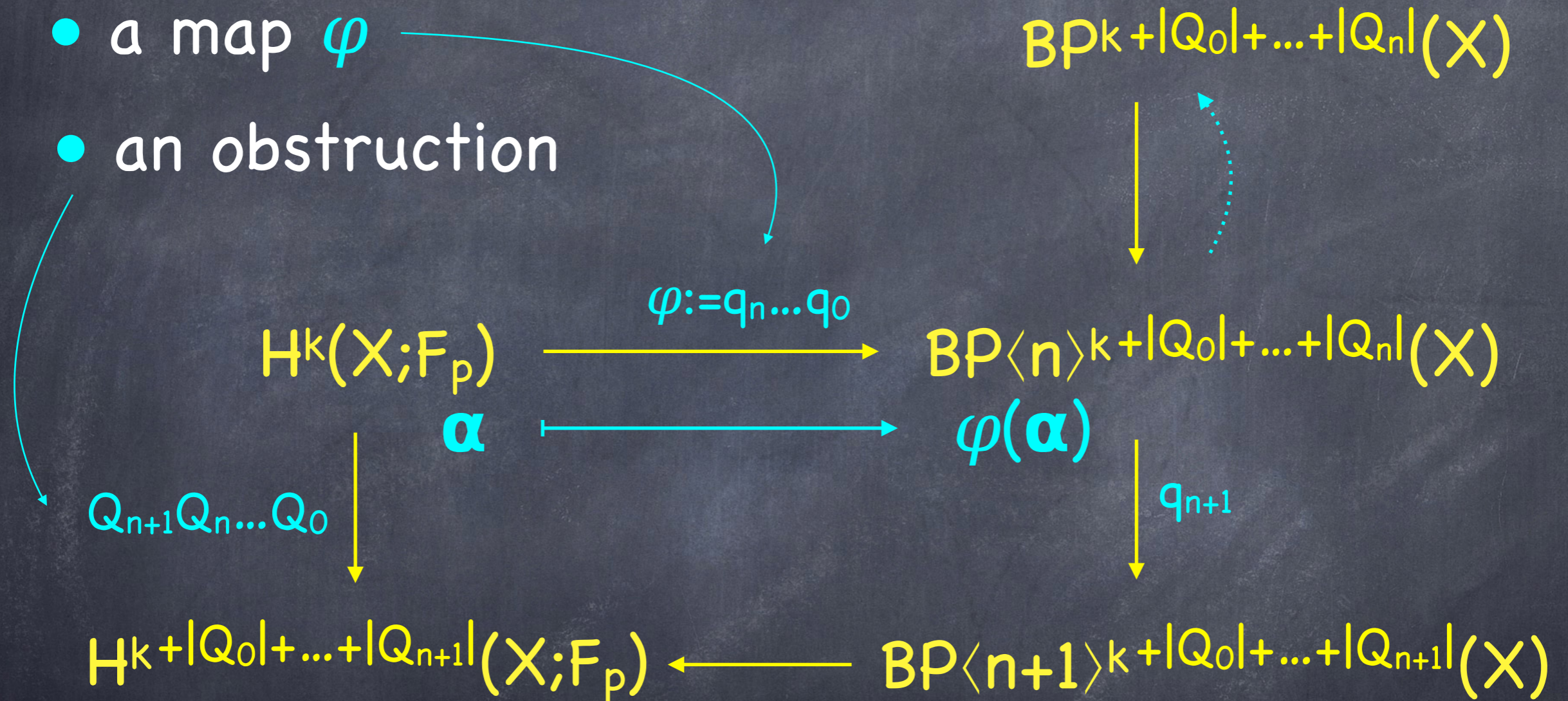
- a map φ

$$\begin{array}{ccc} H^k(X; \mathbb{F}_p) & \xrightarrow{\varphi := q_n \dots q_0} & \text{BP}\langle n \rangle^{k+|Q_0|+\dots+|Q_n|}(X) \\ \alpha & \xrightarrow{\quad} & \varphi(\alpha) \end{array}$$

Lifting classes:

We produced

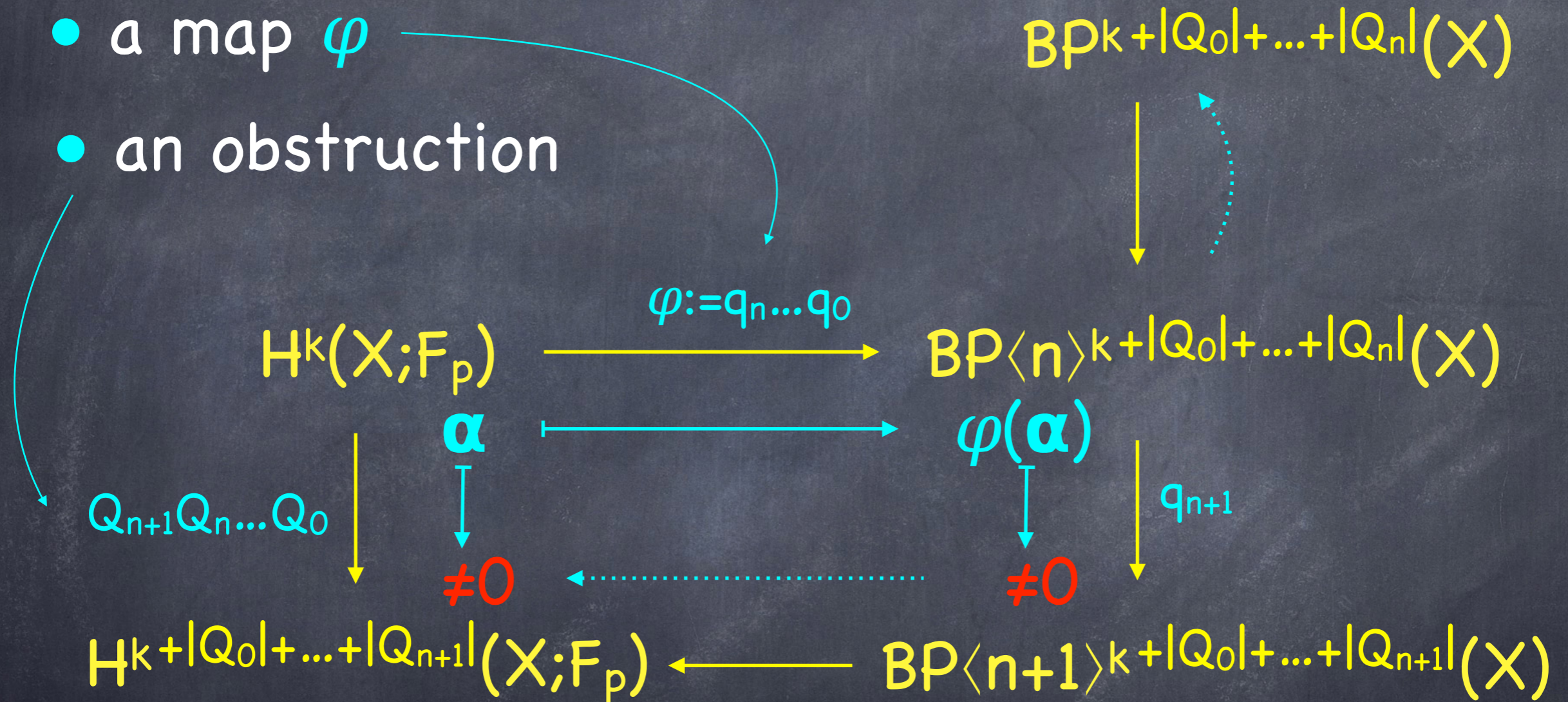
- a map φ
- an obstruction



Lifting classes:

We produced

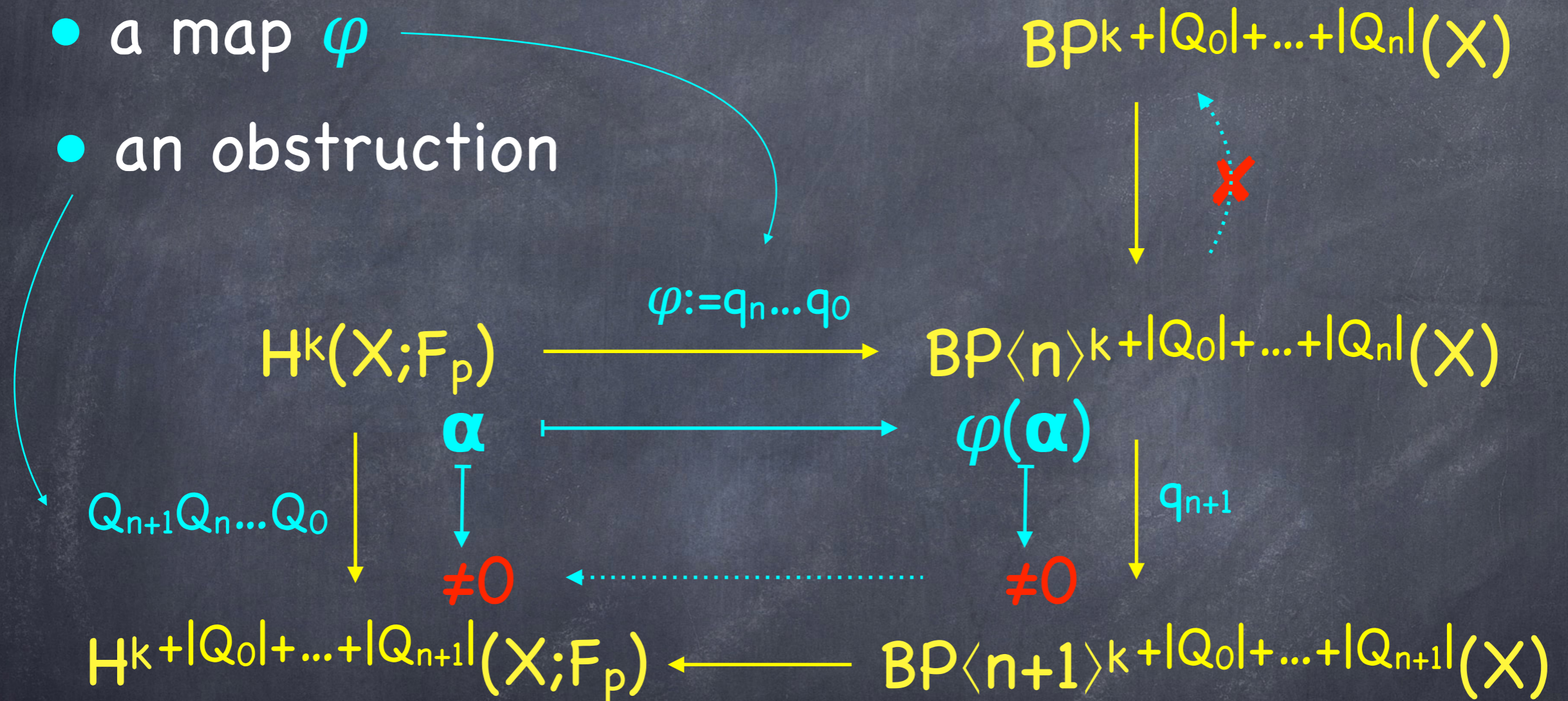
- a map φ
- an obstruction



Lifting classes:

We produced

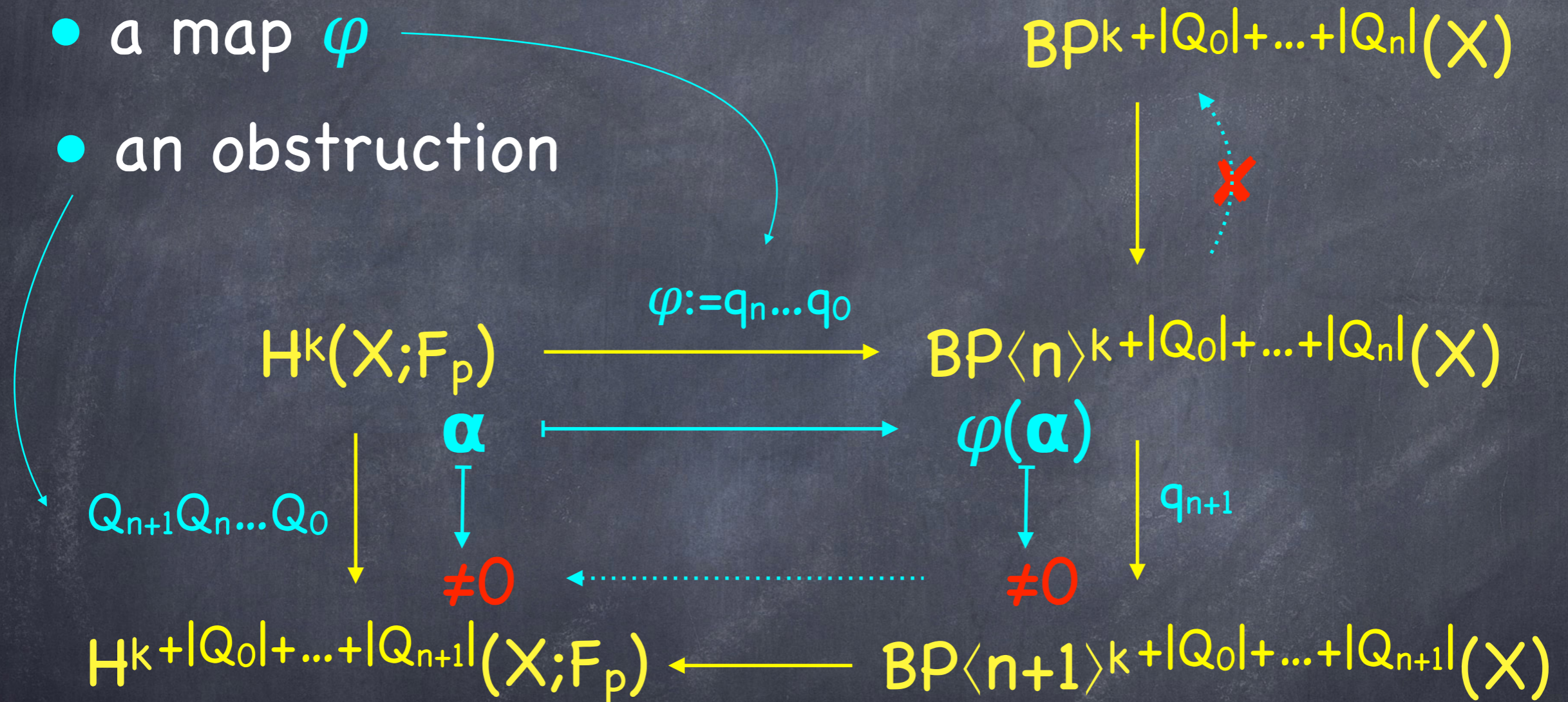
- a map φ
- an obstruction



Lifting classes:

We produced

- a map φ
- an obstruction

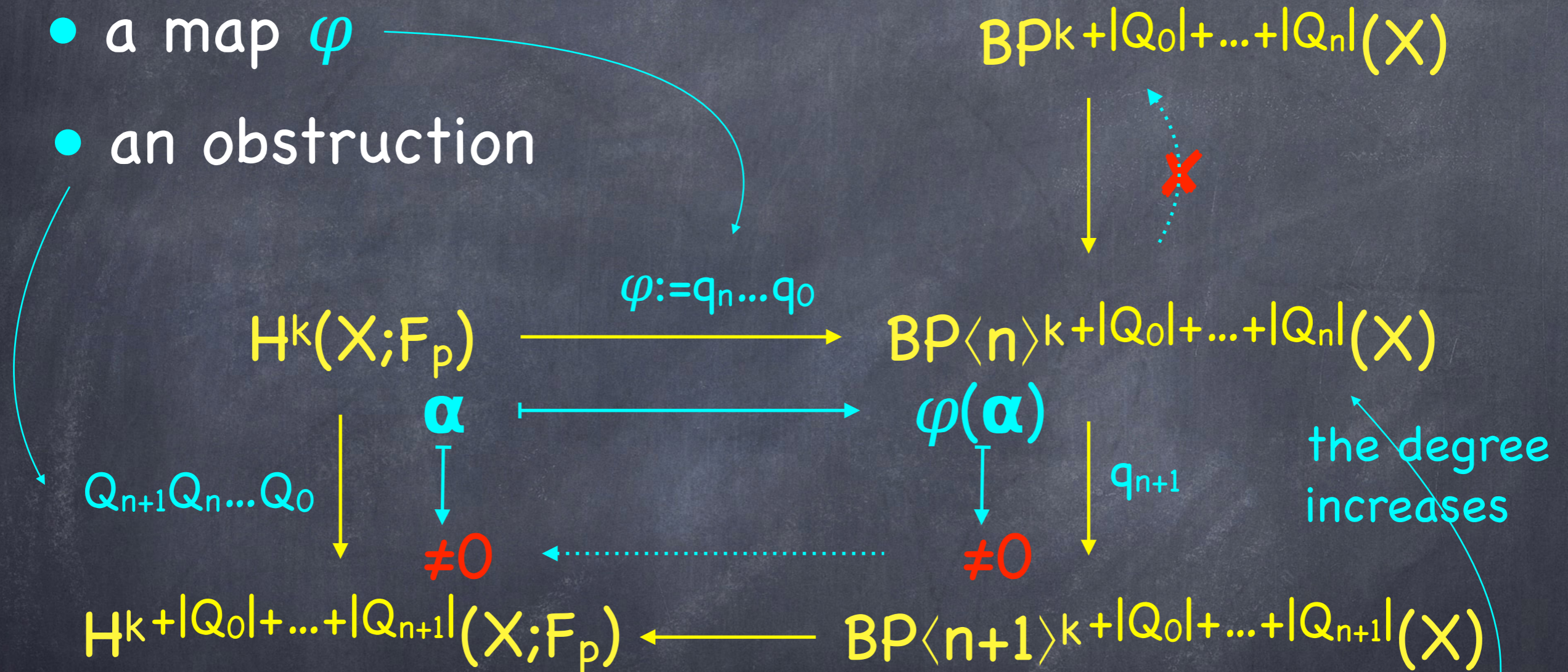


- If $Q_{n+1} \dots Q_0(\alpha) \neq 0$, then $\varphi(\alpha)$ is **not** algebraic.

Lifting classes:

We produced

- a map φ
- an obstruction



- If $Q_{n+1} \dots Q_0(\alpha) \neq 0$, then $\varphi(\alpha)$ is **not** algebraic.

But we also pay a price...

Wilson's unstable splitting:

The price is as little as possible.

Wilson's unstable splitting:

The price is as little as possible.

Theorem (Wilson): For any finite complex Y , the map

$$BP^i(Y) \longrightarrow BP\langle n \rangle^i(Y)$$

is surjective if $i \leq 2(p^n + \dots + p + 1)$.

Wilson's unstable splitting:

The price is as little as possible.

Theorem (Wilson): For any finite complex Y , the map

$$BP^i(Y) \longrightarrow BP\langle n \rangle^i(Y)$$

is **surjective** if $i \leq 2(p^n + \dots + p + 1)$.

Recall $|Q_n| = 2p^n - 1$, hence $|Q_0| + \dots + |Q_n| = 2(p^n + \dots + 1) - n - 1$.

Wilson's unstable splitting:

The price is as little as possible.

Theorem (Wilson): For any finite complex Y , the map

$$BP^i(Y) \longrightarrow BP\langle n \rangle^i(Y)$$

is **surjective** if $i \leq 2(p^n + \dots + p + 1)$.

Recall $|Q_n| = 2p^n - 1$, hence $|Q_0| + \dots + |Q_n| = 2(p^n + \dots + 1) - n - 1$.

$$\begin{array}{ccc} & & BP^{k + |Q_0| + \dots + |Q_n|}(X) \\ & & \downarrow \\ H^k(X; \mathbb{F}_p) & \xrightarrow{\varphi = q_n \dots q_0} & BP\langle n \rangle^{k + |Q_0| + \dots + |Q_n|}(X) \end{array}$$

Wilson's unstable splitting:

The price is as little as possible.

Theorem (Wilson): For any finite complex Y , the map

$$BP^i(Y) \longrightarrow BP\langle n \rangle^i(Y)$$

is **surjective** if $i \leq 2(p^n + \dots + p + 1)$.

Recall $|Q_n| = 2p^n - 1$, hence $|Q_0| + \dots + |Q_n| = 2(p^n + \dots + 1) - n - 1$.

need to pick $k \geq n + 3$

$$\begin{array}{ccc} & & BP^{k + |Q_0| + \dots + |Q_n|}(X) \\ & & \downarrow \\ H^k(X; \mathbb{F}_p) & \xrightarrow{\varphi = q_n \dots q_0} & BP\langle n \rangle^{k + |Q_0| + \dots + |Q_n|}(X) \end{array}$$

(Note: A red 'X' is drawn over the top-right map, and a dotted arrow points from the text 'need to pick k ≥ n + 3' to the top-right map.)

Examples of non-algebraic classes:

Examples of non-algebraic classes:

Theorem (Q.): For every n , there is a smooth projective complex algebraic variety X and a class in

$$BP\langle n \rangle^{2(p^n + \dots + 1) + 2}(X)$$

which is **not** in the image of the map

$$BP\langle n \rangle_{\text{mot}}^{2*,*}(X) \xrightarrow{\text{top. realization}} BP\langle n \rangle^{2*}(X).$$

Examples of non-algebraic classes:

Theorem (Q.): For every n , there is a smooth projective complex algebraic variety X and a class in

$$BP\langle n \rangle^{2(p^n + \dots + 1) + 2}(X)$$

which is **not** in the image of the map

$$BP\langle n \rangle_{\text{mot}}^{2*,*}(X) \xrightarrow{\text{top. realization}} BP\langle n \rangle^{2*}(X).$$

Proof:

Examples of non-algebraic classes:

Theorem (Q.): For every n , there is a smooth projective complex algebraic variety X and a class in

$$BP\langle n \rangle^{2(p^n + \dots + 1) + 2}(X)$$

which is **not** in the image of the map

$$BP\langle n \rangle_{\text{mot}}^{2*,*}(X) \xrightarrow{\text{top. realization}} BP\langle n \rangle^{2*}(X).$$

Proof: Let $G_k := \Pi_k(\mathbb{Z}/p)$ (following Atiyah-Hirzebruch).

Examples of non-algebraic classes:

Theorem (Q.): For every n , there is a smooth projective complex algebraic variety X and a class in

$$BP\langle n \rangle^{2(p^n + \dots + 1) + 2}(X)$$

which is **not** in the image of the map

$$BP\langle n \rangle_{\text{mot}}^{2*,*}(X) \xrightarrow{\text{top. realization}} BP\langle n \rangle^{2*}(X).$$

Proof: Let $G_k := \Pi_k(\mathbb{Z}/p)$ (following Atiyah-Hirzebruch).

We know:

Examples of non-algebraic classes:

Theorem (Q.): For every n , there is a smooth projective complex algebraic variety X and a class in

$$BP\langle n \rangle^{2(p^n + \dots + 1) + 2}(X)$$

which is **not** in the image of the map

$$BP\langle n \rangle_{\text{mot}}^{2*,*}(X) \xrightarrow{\text{top. realization}} BP\langle n \rangle^{2*}(X).$$

Proof: Let $G_k := \Pi_k(\mathbb{Z}/p)$ (following Atiyah-Hirzebruch).

We know: $\bullet H^*(BG_k; \mathbb{F}_p) = \mathbb{F}_p[y_1, \dots, y_k] \otimes \Lambda(x_1, \dots, x_k);$

$$|y_i|=2$$

$$|x_i|=1$$

Examples of non-algebraic classes:

Theorem (Q.): For every n , there is a smooth projective complex algebraic variety X and a class in

$$BP\langle n \rangle^{2(p^n + \dots + 1) + 2}(X)$$

which is **not** in the image of the map

$$BP\langle n \rangle_{\text{mot}}^{2*,*}(X) \xrightarrow{\text{top. realization}} BP\langle n \rangle^{2*}(X).$$

Proof: Let $G_k := \Pi_k(\mathbb{Z}/p)$ (following Atiyah-Hirzebruch).

We know: $\bullet H^*(BG_k; \mathbb{F}_p) = \mathbb{F}_p[y_1, \dots, y_k] \otimes \Lambda(x_1, \dots, x_k);$

$\bullet Q_j(x_i) = y_i^{p^j}, Q_j(y_i) = 0.$

Proof continued: $H^*(BG_k; F_p) = F_p[y_1, \dots, y_k] \otimes \Lambda(x_1, \dots, x_k);$
 $Q_j(x_i) = y_i^{p^j}, Q_j(y_i) = 0.$

Proof continued: $H^*(BG_k; F_p) = F_p[y_1, \dots, y_k] \otimes \Lambda(x_1, \dots, x_k);$
 $Q_j(x_i) = y_i^{p^j}, Q_j(y_i) = 0.$

Choose: $k = n+3$ and $\alpha := x_1 \dots x_{n+3}$ in $H^{n+3}(BG_k; F_p).$

Proof continued: $H^*(BG_k; F_p) = F_p[y_1, \dots, y_k] \otimes \Lambda(x_1, \dots, x_k);$
 $Q_j(x_i) = y_i^{p^j}, Q_j(y_i) = 0.$

Choose: $k = n+3$ and $\alpha := x_1 \dots x_{n+3}$ in $H^{n+3}(BG_k; F_p).$

Check: $Q_{n+1} \dots Q_0(\alpha) \neq 0.$

Proof continued: $H^*(BG_k; \mathbb{F}_p) = \mathbb{F}_p[y_1, \dots, y_k] \otimes \Lambda(x_1, \dots, x_k);$
 $Q_j(x_i) = y_i^{p^j}, Q_j(y_i) = 0.$

Choose: $k = n+3$ and $\alpha := x_1 \dots x_{n+3}$ in $H^{n+3}(BG_k; \mathbb{F}_p).$

Check: $Q_{n+1} \dots Q_0(\alpha) \neq 0.$

Hence $x := q_n \dots q_0(\alpha)$ in $BP\langle n \rangle^{2(p^n + \dots + 1) + 2}(BG_{n+3})$

is **not** in the image of the map

$$BP^{2(p^n + \dots + 1) + 2}(BG_{n+3}) \longrightarrow BP\langle n \rangle^{2(p^n + \dots + 1) + 2}(BG_{n+3}).$$

Proof continued: $H^*(BG_k; \mathbb{F}_p) = \mathbb{F}_p[y_1, \dots, y_k] \otimes \Lambda(x_1, \dots, x_k);$
 $Q_j(x_i) = y_i^{p^j}, Q_j(y_i) = 0.$

Choose: $k = n+3$ and $\alpha := x_1 \dots x_{n+3}$ in $H^{n+3}(BG_k; \mathbb{F}_p).$

Check: $Q_{n+1} \dots Q_0(\alpha) \neq 0.$

Hence $x := q_n \dots q_0(\alpha)$ in $BP\langle n \rangle^{2(p^n + \dots + 1) + 2}(BG_{n+3})$

is **not** in the image of the map

$$BP^{2(p^n + \dots + 1) + 2}(BG_{n+3}) \longrightarrow BP\langle n \rangle^{2(p^n + \dots + 1) + 2}(BG_{n+3}).$$

Finally, set $X =$ **Godeaux-Serre** variety associated to the group G_{n+3} and pullback x via

$$X \longrightarrow BG_{n+3} \times CP^\infty.$$

a $2(p^{n+1} + \dots + 1) + 1$ -connected map

□

Let's check the numbers:

Let's check the numbers:

The minimal complex dimension of X is $2^{n+3}-1$ for $p=2$.

Let's check the numbers:

The minimal complex dimension of X is $2^{n+3}-1$ for $p=2$.

For $n=1$ and $p=2$:

$BP\langle 1 \rangle$ is 2-local connective complex K-theory ku .

Let's check the numbers:

The minimal complex dimension of X is $2^{n+3}-1$ for $p=2$.

For $n=1$ and $p=2$:

$BP\langle 1 \rangle$ is 2-local connective complex K-theory ku .

There is a smooth proj. variety X of dimension 15 over \mathbb{C} with a non-algebraic class in $BP\langle 1 \rangle^8(X)$.

Some final remarks:

Some final remarks:

- For $n=0$: get example of Atiyah and Hirzebruch.

Some final remarks:

- For $n=0$: get example of Atiyah and Hirzebruch.
- Other types of non-*alg.* classes in $H^*(X;Z)$:

Some final remarks:

- For $n=0$: get example of Atiyah and Hirzebruch.
- Other types of non-*alg.* classes in $H^*(X;Z)$: "not topological"
 - Kollar: non-torsion classes on hypersurface in P^4 .
 - Voisin: torsion classes based on Kollar's example.

Some final remarks:

- For $n=0$: get example of Atiyah and Hirzebruch.
- Other types of non-*alg.* classes in $H^*(X;Z)$: *not* topological"
 - Kollar: non-torsion classes on hypersurface in \mathbb{P}^4 .
 - Voisin: torsion classes based on Kollar's example.
 - Benoist-Ottem: torsion classes of degree 4 on 3-folds.

Some final remarks:

- For $n=0$: get example of Atiyah and Hirzebruch.
- Other types of non-*alg.* classes in $H^*(X;Z)$: ^{“not} *topological*”
 - Kollar: non-torsion classes on hypersurface in \mathbb{P}^4 .
 - Voisin: torsion classes based on Kollar's example.
 - Benoist-Ottem: torsion classes of degree 4 on 3-folds.
 - Yagita, Pirutka-Yagita, Kameko, ... :
 - non-torsion class in $H^*(BG;Z)$ for alg. group G with $(Z/p)^3 \subset G$

Some final remarks:

- For $n=0$: get example of Atiyah and Hirzebruch.
- Other types of non-*alg.* classes in $H^*(X;Z)$: *not* topological"
 - Kollar: non-torsion classes on hypersurface in P^4 .
 - Voisin: torsion classes based on Kollar's example.
 - Benoist-Ottem: torsion classes of degree 4 on 3-folds.
- Yagita, Pirutka-Yagita, Kameko, ... :
 - non-torsion class in $H^*(BG;Z)$ for alg. group G with $(Z/p)^3 \subset G$
 - Antieau: class in $H^*(BG;Z)$ for alg. group G , represent. theory on which the Q_i 's vanish, but a higher differential in the AH-spectral sequence is nontrivial.

Some final remarks:

- For $n=0$: get example of Atiyah and Hirzebruch.
- Other types of non-*alg.* classes in $H^*(X;Z)$: *not topological*
 - Kollar: non-torsion classes on hypersurface in P^4 .
 - Voisin: torsion classes based on Kollar's example.
 - Benoist-Ottem: torsion classes of degree 4 on 3-folds.
 - Yagita, Pirutka-Yagita, Kameko, ... :
 - non-torsion classes in $BP\langle n \rangle^*(BG)$
 - non-torsion class in $H^*(BG;Z)$ for alg. group G with $(Z/p)^3 \subset G$
 - Antieau: class in $H^*(BG;Z)$ for alg. group G , represent. theory on which the Q_i 's vanish, but a higher differential in the AH-spectral sequence is nontrivial.

Thank you!