Examples of non-algebraic classes in the Brown-Peterson tower

Derived algebraic geometry and chromatic homotopy theory
Isaac Newton Institute
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Steenrod's question:

Can every class in $H_n(X;Z)$ be realized as the fundamental class of a smooth n-manifold $M \to X$?

& compact, oriented

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Thom's answer: In general, no!

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desingularization

$[V]=[V_{sm}]$ in $H_{2n}(X;\mathbb{Z})$

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$$X \xrightarrow{\alpha_{PD}} CP^k \subset CP^\infty$$

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$$\begin{tikzcd} & CP^{k-1} \arrow[d] & \\ X \arrow[r, "\alpha_{PD}"'] & CP^k \subset & CP^{\infty} \end{tikzcd}$$

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$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & CP^{k-1} \arrow[d] & \\ X \arrow[r, "\alpha_{PD}"'] & CP^k \subset CP^\infty & \end{tikzcd}$$

$K(2;Z)$

How to do homotopy on $\mathrm{Man}_{\mathbb{C}}$?

category of complex manifolds

How to do homotopy on Man_C?

category of complex manifolds

$$\begin{array}{ccc} Man_C & \longrightarrow & Pre_\Delta \\ M & \longmapsto & F_M \end{array}$$

presheaves of simplicial sets

F_M: $X \longmapsto$ discrete simplicial set $Hom_{Man}(X,M)$

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- Given $n \geq 0$, the n-dimensional stalk of $F_\bullet$

$$F_\bullet^{(n)} = \operatorname{colim}_{r \to 0} F_\bullet(B^n(r)) \quad \text{in } \mathrm{Set}_{\Delta}$$

ball of radius r in n-dim. complex affine space

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- A map $F_\bullet \to G_\bullet$ is a weak equivalence in Pre_{Δ} if $F_\bullet^{(n)} \to G_\bullet^{(n)}$ is a weak equivalence in Set_{Δ} for all $n \geq 0$.

Homotopy category of Man_C:

- $$\mathrm{Man} \longrightarrow \mathrm{Pre}_\Delta$$

Homotopy category of Man_C: homotopy category of simplicial presheaves on Man_C

- $$\mathrm{Man} \longrightarrow \mathrm{hoPre}_\Delta = \mathrm{Pre}_\Delta[\mathrm{w.e.}^{-1}]$$

Homotopy category of Man_C: homotopy category of simplicial presheaves on Man_C

- $$Man \longrightarrow hoPre_\Delta = Pre_\Delta[w.e.^{-1}]$$

- Given M with an open cover $\{U_\alpha\}$:

  $F_{U_\bullet} \to F_M$ is a weak equivalence.

  $\coprod U_\alpha \rightrightarrows \coprod U_\alpha \times_X U_\beta \; \substack{\to\\\to\\\to} \; \ldots$

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- Can replace Set_Δ with Spectra and get a stable homotopy category $\mathrm{hoPre}_{\mathrm{Spectra}}$ of Man_C.

  - $S^1 \wedge -$ with S^1 viewed as a simplicial (constant) presheaf is made invertible.

Homotopy category of $\mathrm{Sm}_{\mathbb{C}}$:

smooth complex varieties

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Homotopy category of $Sm_{\mathbb{C}}$:

simplicial presheaves on $Sm_{\mathbb{C}}$

smooth complex varieties

$$Sm_{\mathbb{C}} \longrightarrow Pre_{\Delta}$$

Morel
Voevodsky
Jardine
Joyal
Isaksen
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...

Homotopy category of Sm_C:

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simplicial presheaves on Sm_C

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- Nisnevich covers (replacing open covers)

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- Localize with respect to

Homotopy category of $Sm_{\mathbb{C}}$:

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$F_{U_\bullet} \to F_X$ for any X and any (hyper)cover $U \to X$

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smooth complex varieties

$$Sm_C \longrightarrow hoPre_\Delta$$

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Homotopy category of Sm_C: motivic homotopy category of simplicial presheaves on Sm_C

smooth complex varieties

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$A^1 x X \to X$ for any X

affine line over C

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affine line over C

- stable motivic homotopy category of Sm_C
  - $P^1 \wedge$- the projective line
  - $S^1 \wedge$- the "simplicial circle" and $(A^1 - 0)\wedge$- the "Tate circle"

Topological realization: $Sm_{\mathbb{C}} \xrightarrow{\rho} Man_{\mathbb{C}}$

$$X \longmapsto X(\mathbb{C})$$

complex manifold of solutions in $\mathbb{C}$

e.g. $P^1 \longmapsto P^1(\mathbb{C}) = \mathbb{C}P^1 \simeq S^2$

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motivic spectrum

induced map

top. realization of E

$$E^{a,b}_{mot}(X) \xrightarrow{\rho} E^{a}_{top}(X(\mathbb{C}))$$

"algebraic" "topological"

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Example:

$$[V \subset X] \in \quad H\mathbb{Z}^{2n,n}_{mot}(X) \longrightarrow H\mathbb{Z}^{2n}_{top}(X(\mathbb{C})) = H^{2n}(X;\mathbb{Z})$$

$[V_{sm}]$

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Question: How can we produce classes in $E^{2*}_{top}(X(\mathbb{C}))$ which are not algebraic, i.e., are not in the image of ρ?

Atiyah-Hirzebruch obstruction:

given

algebraic $V \subset X$

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algebraic $V \subset X \longmapsto [V]_H \in H^{2n}(X;Z)$

fundamental class of V_{sm}

Atiyah-Hirzebruch obstruction:

ku-fundamental class

$$\begin{tikzcd}
 & {[V]_{ku} \in ku^{2n}(X)} \arrow[d] \\
\text{given algebraic } V \subset X \arrow[ur, mapsto, bend left] \arrow[r, mapsto] & {[V]_H \in H^{2n}(X;\mathbb{Z})}
\end{tikzcd}$$

Atiyah-Hirzebruch

fundamental class of V_{sm}

Atiyah-Hirzebruch obstruction:

Thom/Quillen:
universal
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ku-
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Atiyah-
Hirzebruch

given

$$
\begin{tikzcd}
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 & \vdots \arrow[d, mapsto] & & \vdots \arrow[d] \\
 & {[V]_{ku}} \arrow[d, mapsto] & \in & ku^{2n}(X) \arrow[d] \\
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\text{algebraic } V \subset X \arrow[r, mapsto] \arrow[uuur, bend left, mapsto, "\text{Totaro}"] \arrow[ur, bend left, mapsto, "\text{Atiyah-Hirzebruch}"'] & {[V]_H \in H^{2n}(X;Z)}
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$$\begin{tikzcd}
 & {[V]_{MU}} \arrow[d, mapsto] \arrow[r, phantom, "\in"] & MU^{2n}(X)/MU^{*<0}\cdot MU^*(X) \arrow[d] \\
 & \vdots \arrow[d, mapsto] & \vdots \arrow[d] \\
 & {[V]_{ku}} \arrow[d, mapsto] \arrow[r, phantom, "\in"] & ku^{2n}(X)/\beta\cdot ku^*(X) \arrow[d] \\
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alg. Thom spectrum

$$\begin{tikzcd}
\mathrm{MGL}^{2n,n}(X) \arrow[d] \\
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MGL^{2n,n}(X)/MU^{*<0}.MGL^{2*,*}(X) \arrow[d, "\approx\ \text{Levine-Morel}"] & & {[V]_{MU} \in MU^{2n}(X)/MU^{*<0}.MU^{*}(X)} \arrow[d, dotted] \\
H^{2n,n}_{mot}(X;Z) & & {[V]_{ku} \in ku^{2n}(X)/\beta.ku^{*}(X)} \arrow[d] \\
\text{algebraic } V \subset X \arrow[rr, mapsto] \arrow[urr, mapsto, bend left, "\text{Atiyah-Hirzebruch}"] \arrow[uurr, mapsto, bend left, "\text{Totaro}"] & & {[V]_H \in H^{2n}(X;Z)}
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$MGL^{2n,n}(X)/MU^{*<0}\cdot MGL^{2*,*}(X)$

$\approx$ Levine-Morel

$H^{2n,n}_{mot}(X;Z)$

$[V]_{MU} \in MU^{2n}(X)/MU^{*<0}\cdot MU^{*}(X)$

$\vdots$

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Atiyah-Hirzebruch

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algebraic $V \subset X$ $\longmapsto$ $[V]_H \in H^{2n}(X;Z)$

$\not\approx$ in general

fundamental class of V_{sm}

The Brown-Peterson tower: fix a prime p

p-local universal theory

Brown-Peterson spectra BP with

$BP_* = Z_{(p)}[v_1, v_2, \ldots]$.

$|v_i| = 2(p^i - 1)$

Brown-Peterson
Quillen
Wilson
⋮

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quotient map

For every n:

$$\mathbb{Z}_{(p)}[v_1, v_2, \ldots] \longrightarrow \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$$

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For every n: $BP \xrightarrow{\text{quotient map}} BP/(v_{n+1}, \ldots) =: BP\langle n \rangle$

$\mathbb{Z}_{(p)}[v_1, v_2, \ldots] \longrightarrow \mathbb{Z}_{(p)}[v_1, \ldots, v_n] = BP\langle n \rangle_*$

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The Brown-Peterson tower:

$$BP \to \cdots \to BP\langle n\rangle \to \cdots \to BP\langle 1\rangle \to BP\langle 0\rangle \to BP\langle -1\rangle$$

$$H\mathbb{Z}_{(p)} \longrightarrow H\mathbb{F}_p$$

Milnor operations:

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For every n:

stable cofibre sequence

$$\Sigma^{|v_n|}BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \longrightarrow BP\langle n-1\rangle \longrightarrow \Sigma^{|v_n|+1}BP\langle n\rangle$$

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with an induced exact sequence (for any space Y)

$$BP\langle n\rangle^{*+|v_n|}(Y) \longrightarrow BP\langle n\rangle^{*}(Y) \longrightarrow BP\langle n-1\rangle^{*}(Y) \xrightarrow{q_n} BP\langle n\rangle^{*+|v_n|+1}(Y)$$

Milnor operations:

For every n:

stable cofibre sequence

$$\Sigma^{|v_n|}BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \longrightarrow BP\langle n-1\rangle \longrightarrow \Sigma^{|v_n|+1}BP\langle n\rangle$$

with an induced exact sequence (for any space Y)

$$\begin{tikzcd}
 & BP\langle n\rangle^{*+|v_n|}(Y) \arrow[r] & BP\langle n\rangle^{*}(Y) \arrow[dl, bend left] & \\
BP\langle n-1\rangle \arrow[d, "\text{Thom map}"] & BP\langle n-1\rangle^{*}(Y) \arrow[r, "q_n"] \arrow[d] & BP\langle n\rangle^{*+|v_n|+1}(Y) \arrow[d] \\
HF_p & H^{*}(Y;F_p) \arrow[r] & H^{*+|v_n|+1}(Y;F_p)
\end{tikzcd}$$

Milnor operations:

For every n:

stable cofibre sequence

$$\Sigma^{|v_n|}BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \longrightarrow BP\langle n-1\rangle \longrightarrow \Sigma^{|v_n|+1}BP\langle n\rangle$$

with an induced exact sequence (for any space Y)

$$BP\langle n\rangle^{*+|v_n|}(Y) \longrightarrow BP\langle n\rangle^{*}(Y) \longrightarrow$$

$$\begin{tikzcd}
BP\langle n-1\rangle \arrow[d, "\text{Thom map}"] & BP\langle n-1\rangle^{*}(Y) \arrow[r, "q_n"] \arrow[d] & BP\langle n\rangle^{*+|v_n|+1}(Y) \arrow[d] \\
HF_p & H^{*}(Y;F_p) \arrow[r, "Q_n"'] & H^{*+|v_n|+1}(Y;F_p)
\end{tikzcd}$$

nth Milnor operation:

Q_0=Bockstein

$Q_n = P^{p^{n-1}}Q_{n-1} - Q_{n-1}P^{p^{n-1}}$

The LMT obstruction in action:

$X \subset CP^N$ smooth projective algebraic

$$
\begin{tikzcd}
 & BP^{2*}(X) \arrow[dl, bend right] \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] & \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
 & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p)
\end{tikzcd}
$$

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\end{tikzcd}
$$

α

The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth projective algebraic

$$
\begin{tikzcd}
 & BP^{2*}(X) \arrow[d] \arrow[dl, bend right] \arrow[dl, phantom, "\circlearrowleft"] & \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
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α

Question: Is α algebraic?

The LMT obstruction in action:

$X \subset \mathbb{CP}^N$ smooth projective algebraic

$$
\begin{tikzcd}
 & BP^{2*}(X) \arrow[dl, bend right] \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] & \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
\text{algebraic } V \subset X \arrow[dr, dotted, bend right] & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p) \\
 & \alpha &
\end{tikzcd}
$$

Question: Is α algebraic?

The LMT obstruction in action:

$X \subset CP^N$ smooth projective algebraic

$$
\begin{tikzcd}
 & BP^{2*}(X) \arrow[dl, bend right] \arrow[d] \arrow[dl, phantom, shift left=6, "\circlearrowleft"] & \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
\text{algebraic } V\subset X \arrow[uur, dotted, bend left, "\text{LMT}"] \arrow[dr, dotted, bend right] & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p) \\
 & \boldsymbol{\alpha} &
\end{tikzcd}
$$

Question: Is $\boldsymbol{\alpha}$ algebraic?

The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth projective algebraic

$$
\begin{tikzcd}
 & BP^{2*}(X) \arrow[dl, bend right] \arrow[d, "\alpha_{n-1}"] \arrow[dl, phantom, "\circlearrowleft"] & \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
\text{algebraic } V \subset X \arrow[uur, dotted, bend left, "\text{LMT}"] \arrow[dr, dotted, bend right] & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p) \\
 & \alpha &
\end{tikzcd}
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 & H^{2*}(X;\mathbb{F}_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;\mathbb{F}_p)
\end{tikzcd}$$

LMT

$\alpha_{n-1} \longmapsto q_n\alpha_{n-1}$

algebraic $V \subset X$

α

Question: Is α algebraic?

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 & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p)
\end{tikzcd}
$$

LMT

if

$\alpha_{n-1} \longmapsto q_n\alpha_{n-1} = 0$

algebraic $V \subset X$

α

Question: Is α algebraic?

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$$\begin{tikzcd}
 & BP^{2*}(X) \arrow[dl, bend right] \arrow[d] & \\
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 & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p)
\end{tikzcd}$$

LMT

then $\circlearrowleft$

α_n

$\alpha_{n-1} \longmapsto q_n\alpha_{n-1} = 0$

if

algebraic $V \subset X$

α

Question: Is α algebraic?

The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth projective algebraic

$$\begin{tikzcd}
 & BP^{2*}(X) \arrow[d] \arrow[ddl, bend right] & \\
\alpha_n & \alpha_{n-1} \arrow[l, dotted, bend right, "\text{then } \circlearrowleft"'] \arrow[r, mapsto] \arrow[u, dotted, bend right] & q_n\alpha_{n-1} = 0 \ \text{(if)} \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
\text{algebraic } V \subset X \arrow[uuur, dotted, bend left, "\text{LMT}"] \arrow[dr, dotted, bend right] & H^{2*}(X;\mathbb{F}_p) \arrow[r, "Q_n"'] \arrow[u, dotted, bend right] & H^{2*+|v_n|+1}(X;\mathbb{F}_p) \\
 & \alpha \arrow[r, mapsto] & Q_n\alpha
\end{tikzcd}$$

Question: Is α algebraic?

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$$
\begin{tikzcd}
 & BP^{2*}(X) \arrow[ddl, bend right] \arrow[dd] & \\
\alpha_n & \alpha_{n-1} \arrow[l, dotted, bend right, "\text{then}\ \circlearrowleft"'] \arrow[u, dotted, bend right] \arrow[r, mapsto, "\text{if}"] & q_n\alpha_{n-1} = 0 \\
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 & \alpha \arrow[r, mapsto] & Q_n\alpha
\end{tikzcd}
$$

LMT

then $\circlearrowleft$

if

Levine-Morel-Totaro obstruction:

The LMT obstruction in action:

$X \subset \mathbb{C}P^N$ smooth projective algebraic

LMT

then $\circlearrowleft$

α_n $\qquad$ $\alpha_{n-1} \longmapsto q_n\alpha_{n-1} = 0$ $\quad$ if

$$\begin{tikzcd}
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BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
 & H^{2*}(X;\mathbb{F}_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;\mathbb{F}_p)
\end{tikzcd}$$

algebraic $V \subset X$

$\alpha \longmapsto Q_n\alpha \neq 0$

if

Levine-Morel-Totaro obstruction:

If $Q_n\alpha \neq 0$,

The LMT obstruction in action:

$X \subset \mathbb{CP}^N$ smooth projective algebraic

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LMT

then ↺

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↺

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$\boldsymbol{\alpha} \longmapsto Q_n\boldsymbol{\alpha} \neq 0$

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LMT

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$\circlearrowleft$

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 & H^{2*}(X;F_p) \arrow[r, "Q_n"'] & H^{2*+|v_n|+1}(X;F_p)
\end{tikzcd}$$

LMT

then ⟲

$\boldsymbol{\alpha}_n$ ✗ $\boldsymbol{\alpha}_{n-1} \longmapsto q_n\boldsymbol{\alpha}_{n-1} \neq 0$

algebraic $V \subset X$

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If $Q_n\boldsymbol{\alpha} \neq 0$,

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 & H^{2*}(X;F_p) \arrow[r, "Q_n"'] \arrow[ur, phantom, "\circlearrowleft"] & H^{2*+|v_n|+1}(X;F_p)
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LMT

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α_n $\quad$ $\alpha_{n-1} \longmapsto q_n\alpha_{n-1} \neq 0$

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$$\begin{tikzcd}
 & BP^{2*}(X) \arrow[dl, bend right] \arrow[d] & \\
\alpha_n & \alpha_{n-1} \arrow[r, mapsto] \arrow[l, dotted, "then"'] & q_n\alpha_{n-1} \neq 0 \\
BP\langle n\rangle^{2*}(X) \arrow[r] & BP\langle n-1\rangle^{2*}(X) \arrow[r, "q_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & BP\langle n\rangle^{2*+|v_n|+1}(X) \arrow[d] \\
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 & \alpha \arrow[r, mapsto] & Q_n\alpha \neq 0
\end{tikzcd}$$

if

Levine-Morel-Totaro obstruction:

If $Q_n\alpha \neq 0$, then α is not algebraic.

Voevodsky’s motivic Milnor operations:

Voevodsky's motivic Milnor operations:

There are motivic operations

$$Q_n^{mot} \in \mathscr{A}^{2p^n-1,p^n-1}$$

mod p-motivic Steenrod algebra

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There are motivic operations

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mod p-motivic
Steenrod algebra

For a smooth complex variety X:

$$H^{i,j}_{mot}(X;F_p) \xrightarrow{Q_n^{mot}} H^{i+2p^n-1,j+p^n-1}_{mot}(X;F_p)$$

mod p-motivic
cohomology

Voevodsky's motivic Milnor operations:

There are motivic operations

$$Q_n^{mot} \in \mathscr{A}^{2p^n-1,p^n-1}$$

mod p-motivic Steenrod algebra

For a smooth complex variety X:

$$H^{i,j}_{mot}(X;F_p) \xrightarrow{Q_n^{mot}} H^{i+2p^n-1,j+p^n-1}_{mot}(X;F_p)$$

mod p-motivic cohomology

Recall: $H^{i,j}_{mot}(X;F_p) = 0$ if $i>2j$.

Obstructions revisited: X smooth complex variety

$$
\begin{tikzcd}
H^{2i,i}_{mot}(X;\mathbb{F}_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;\mathbb{F}_p) \arrow[d] \\
H^{2i}(X;\mathbb{F}_p) \arrow[r, "Q_n"'] & H^{2i+2p^n-1}(X;\mathbb{F}_p)
\end{tikzcd}
$$

topological realization

Obstructions revisited: X smooth complex variety

$$
\begin{tikzcd}
H^{2i,i}_{mot}(X;F_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;F_p) = 0 \arrow[d] \\
H^{2i}(X;F_p) \arrow[r, "Q_n"'] & H^{2i+2p^n-1}(X;F_p)
\end{tikzcd}
$$

topological realization

Obstructions revisited:

X smooth complex variety

$$
\begin{tikzcd}
H^{2i,i}_{mot}(X;\mathbb{F}_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[rr, bend left, "Q_n^{mot}"] \arrow[dr, phantom, "\circlearrowleft"] & H^{2i+2p^n-1,\,i+p^n-1}_{mot}(X;\mathbb{F}_p) \arrow[d] \arrow[r, phantom, "="] & 0 \\
H^{2i}(X;\mathbb{F}_p) \arrow[r, "Q_n"'] & H^{2i+2p^n-1}(X;\mathbb{F}_p) &
\end{tikzcd}
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topological realization

α

Obstructions revisited: X smooth complex variety

$$
\begin{tikzcd}
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\end{tikzcd}
$$

topological realization

$$\boldsymbol{\alpha} \longmapsto Q_n\boldsymbol{\alpha} \neq 0$$

if

Obstructions revisited:

X smooth complex variety

$$\begin{tikzcd}
H^{2i,i}_{mot}(X;F_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[rr, bend left, "Q_n^{mot}"] \arrow[dr, phantom, "\circlearrowleft"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;F_p) \arrow[d] & = 0 \\
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\end{tikzcd}$$

topological realization

$$\alpha \longmapsto Q_n\alpha \neq 0$$

if

Obstructions revisited: X smooth complex variety

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H^{2i,i}_{mot}(X;\mathbb{F}_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[rr, bend left, "Q_n^{mot}"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;\mathbb{F}_p) \arrow[d] & = 0 \\
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\alpha \arrow[r, mapsto] & Q_n\alpha \neq 0 &
\end{tikzcd}
$$

topological realization

if

Observation: The LMT-obstruction is particular to smooth varieties and bidegrees $(2i,i)$.

Obstructions revisited: X smooth complex variety

$$
\begin{tikzcd}
H^{2i,i}_{mot}(X;\mathbb{F}_p) \arrow[r, "Q_n^{mot}"] \arrow[d] \arrow[rr, bend left, "Q_n^{mot}"] \arrow[dr, phantom, "\circlearrowright"] & H^{2i+2p^n-1,i+p^n-1}_{mot}(X;\mathbb{F}_p) \arrow[d] & = 0 \\
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\end{tikzcd}
$$

topological realization

$$\alpha \longmapsto Q_n\alpha \neq 0$$

if

Observation: The LMT-obstruction is particular to smooth varieties and bidegrees $(2i,i)$.

Example: $Q_n\iota\neq 0$ for ι the fundamental class of a suitable Eilenberg-MacLane space, though ι is algebraic.

Back to our task:

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Find non-algebraic classes in $E^{2*}_{top}(X)$.

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For example: $E=BP\langle n\rangle$?

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Recall: BP and $BP\langle n\rangle$ exist in the motivic world (e.g. Hopkins, Vezzosi, Hu-Kriz, Ormsby, Hoyois, Ormsby-Østvær).

Back to our task:

Find non-algebraic classes in $E^{2*}_{top}(X)$.

For example: $E = BP\langle n\rangle$?

Recall: BP and $BP\langle n\rangle$ exist in the motivic world (e.g. Hopkins, Vezzosi, Hu-Kriz, Ormsby, Hoyois, Ormsby-Østvær).

Question: How can we produce non-algebraic elements in $BP\langle n\rangle^{2*}_{top}(X)$?

We can lift...

$H^k(X;F_p)$

We can lift...

$$\begin{array}{c} H^{k+1}(X;Z_{(p)}) \\ \Big\uparrow{\scriptstyle q_0} \\ H^k(X;F_p) \end{array}$$

We can lift...

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But we also pay a price...

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Finally, set X = Godeaux-Serre variety associated to the group G_{n+3} and pullback x via

$$X \longrightarrow BG_{n+3} \times CP^\infty.$$

a $2(p^{n+1}+\ldots+1)+1$-connected map

□

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There is a smooth proj. variety X of dimension 15 over C with a non-algebraic class in $BP\langle 1\rangle^8(X)$.

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Thank you!