Real projective groups are formal

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This is joint work with Ambrus Pál

Milnor-Bloch-Kato conjecture:

for simplicity

k field with $\mathrm{char}(k) \neq p$ containing primitive pth root of unity

Voevodsky, Rost, Merkurjev-Suslin

Milnor K-theory

$T(k^\times)/(u \otimes (1-u), u \neq 0,1)$

continuous cohomology of absolute Galois group

$$K^M_\bullet(k)/_p \xrightarrow{\cong} H^\bullet(k, \mathbb{F}_p)$$

quadratic algebra

- generators in degree 1
- relations in degree 2

strong restriction on which $\mathbb{F}_p$-algebras can occur as the Galois cohomology of a field

Question: What other restrictions are there?

Additional properties? $H^\bullet(k, \mathbb{F}_p)$ quadratic algebra

$$A = T(V)/(R)$$

tensor algebra of vector space V

$\tau : T(V) \to A$,
$R = \ker(\tau) \cap (V \otimes V)$

- Is $H^\bullet(k, \mathbb{F}_p)$ a **Koszul algebra?**

Koszul complex: $(K(A), d)$

$K_0^0(A) = \mathbb{F}_p \quad K_1^1(A) = V \quad K_2^2(A) = R$

$$K_i^i(A) = \bigcap_{0 \le j \le i-2} V^{\otimes j} \otimes R \otimes V^{\otimes i-j-2} \subset V^{\otimes i},\ i \ge 3$$

$$K_i(A) = A \otimes K_i^i(A) \otimes A$$

- A is Koszul if multiplication $\mu : K(A) \to A$ is a quasi-isomorphism
- A is Koszul if $\mathrm{Ext}_A^{ij}(\mathbb{F}_p, \mathbb{F}_p) = 0$ for $i \neq j$

- Conjecture of Positselski-Voevodsky: If k contains a primitive pth root of unity, then $H^\bullet(k, \mathbb{F}_p)$ is Koszul

- Positselski-Vishik: Koszulity can be used for an alternative proof of the Milnor-Bloch-Kato conjecture

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- Is $H^\bullet(k, \mathbb{F}_p)$ a **Koszul algebra?**

Conjecture: If k contains a primitive pth root of unity, then $H^\bullet(k, \mathbb{F}_p)$ is Koszul.

  - Positselski: local and global fields ✓

more about PAC fields later

  - Mináč-Panini-Quadrelli-Tân: finite fields, pseudo algebraically closed fields, Pythagorean fields for $p = 2$, … ✓

"Koszul algebras are very close to their cohomology"

- Is $C^\bullet(k, \mathbb{F}_p)$ a **formal** differential graded algebra?

continuous cochains of absolute Galois group

$(C^\bullet, \delta)$ is quasi-isomorphic as a dga to its cohomology $(H^\bullet(C^\bullet), \delta = 0)$

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"Koszul algebras are very close to their cohomology"

- Is $C^\bullet(k,\mathbb{F}_p)$ a formal dg-algebra?

Non-vanishing Massey products provide an obstruction to formality

Massey products:

$(C^\bullet, \delta)$ a differential graded algebra with cohomology $H^\bullet$

a, b, c elements in H^1 with $ab = 0 = bc$

choose

- A, B, C in C^1 represent a, b, c
- E_{ab}, E_{bc} with $\delta E_{ab} = AB, \delta E_{bc} = BC$

triple Massey product is defined if such data exist

set of elements in H^2

$$\langle a, b, c \rangle := [AE_{bc} - E_{ab}C] \text{ in } H^2/(aH^1 + H^1 c)$$

- If $\delta = 0$ then $\langle a, b, c \rangle = 0$

  may choose $E_{ab} = 0 = E_{bc}$

- If $f\colon C^\bullet \to D^\bullet$ quasi-isom then

  $\langle a, b, c \rangle \longleftrightarrow \langle f^\bullet a, f^\bullet b, f^\bullet c \rangle$ (bijection)

- Generalize to n-tuple Massey product $\langle a_1, \ldots, a_n \rangle$ in all degrees

Massey product vanishing:

- $\langle a, b, c \rangle$ is defined if nonempty
- $\langle a, b, c \rangle$ vanishes if it contains 0

- Many non-vanishing Massey products in arithmetic: Morishita, Sharifi, Bleher-Chinburg-Gillibert, ...

- Hopkins and Wickelgren:

  local and global field k with $\mathrm{char}(k) \neq 2$

- Mináč and Tân:

  for every field k

triple Massey products vanish whenever they are defined

of elements in $H^1(k, \mathbb{F}_2)$

Massey vanishing conjecture of Mináč-Tân:

for every field k, all $n \geq 3$, all primes p

Conjecture: n-fold Massey products of elements in $H^1(k, \mathbb{F}_p)$ vanish whenever they are defined

- Matzri, Efrat-Matzri, Mináč-Tân: all fields, all primes, $n = 3$ ✓
- Guillot-Mináč-Topaz-Wittenberg: all number fields, $p = 2$, $n = 4$ ✓
- Harpaz-Wittenberg: all number fields, all primes, all $n \geq 3$ ✓
- Pál-Szabó: fields with $\mathrm{vcd} \leq 1$ and ppc, all primes, all $n \geq 3$ ✓
- Merkurjev-Scavia: all fields, $p = 2$, $n = 4$ ✓

more about $\mathrm{vcd} \leq 1$ later

Hopkins-Wickelgren formality:

Massey vanishing conjecture and Koszulity suggest

Question: Is $C^\bullet(k, \mathbb{F}_p)$ formal for all fields and all primes?

The answer is no in general

- Positselski: local fields of characteristic $\neq p$ which contain a primitive pth root of unity may not be formal

- Harpaz-Wittenberg: $\mathbb{Q}(\sqrt{2}, \sqrt{17})$ is not formal

- Merkurjev-Scavia: fields of characteristic $\neq 2$ have an extension which is not formal

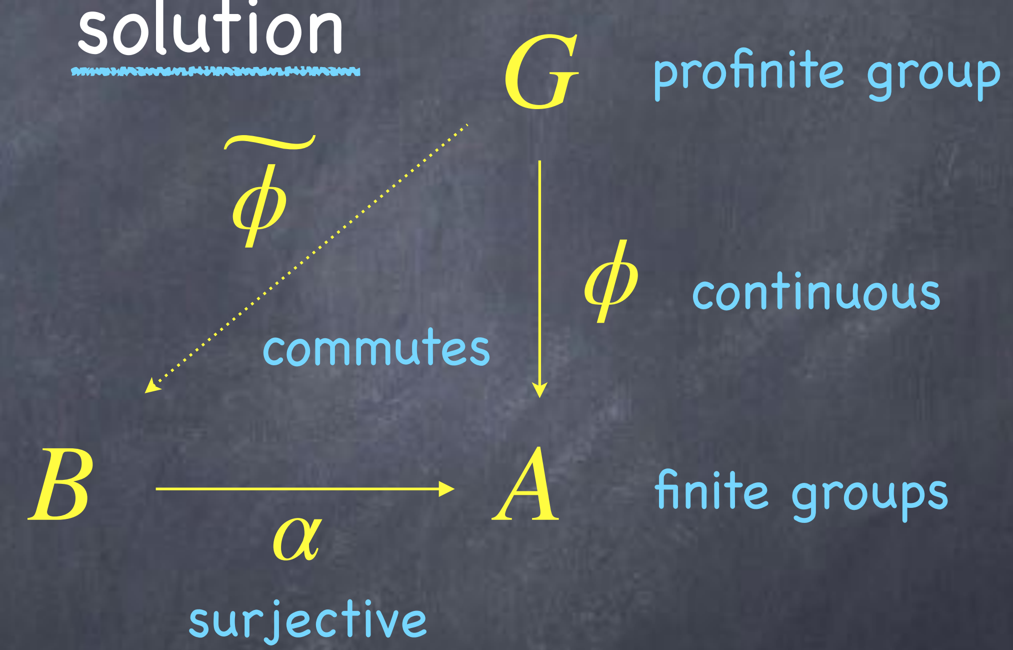
However, there are also positive cases...

PAC fields: k a field with absolute Galois group $\Gamma(k)$

- k is called **pseudo algebraically closed** if every geometrically irreducible k-variety has a k-rational point

embedding problem

solution



G is **projective** if every embedding problem has a solution

cohomological dimension at most 1

Ax, Lubotzky-van den Dries

k is pseudo algebraically closed → $\Gamma(k)$ is projective

there is a PAC field k with $\Gamma(k) \cong G$ ← G is projective

Real projective groups:

G a profinite group

real embedding problem

for every involution

solution $\tilde{\phi}$ commutes

$G \xrightarrow{\phi} A$, $B \xrightarrow{\alpha} A$

$t \mapsto \phi(t) \neq 1$

there exists an involution $b \mapsto \phi(t) \neq 1$

G is **real projective** if it has an open subgroup without 2-torsion and every real embedding problem has a solution

- k a field with absolute Galois group $\Gamma(k)$

$\Gamma(k)$ is real projective

Haran-Jarden

k has **virtual** cohomological dimension at most one ≤ 1

k is pseudo real closed $\rightarrow$ $\Gamma(k)$ is real projective

- every geometrically irreducible k-variety, which has a $\bar{k}$-rational simple point in every real closure $\bar{k}$, has a k-rational point

there is a PRC field k with $\Gamma(k) \cong G$ $\leftarrow$ G is real projective

Our main results (Pál-Q.):

Formality of $C^\bullet(G,\mathbb{F}_p)$ for p odd follows from $H^i(G,\mathbb{F}_p)=0$ for $i\geq 2$

even intrinsically formal

- If G is real projective, the dga $C^\bullet(G,\mathbb{F}_2)$ is formal

first case of fields with infinite cohomological dimension

- Fields with virtual coh. dimension ≤ 1 are formal and satisfy strong Massey vanishing for all primes

vanishing for products in all degrees

- If k has virtual coh. dimension ≤ 1, then $H^\bullet(k,\mathbb{F}_p)$ is Koszul

Scheiderer's theorem: G a real projective group

- $\mathscr{X}(G)$ = set of conjugacy classes of involutions
- B = ring of continuous functions $\mathscr{X}(G) \to \mathbb{F}_2$

connected sum of quadratic algebras

- $(B \sqcap V)^0 = \mathbb{F}_2$
- $(B \sqcap V)^i = B^i \oplus V^i$, and $B^+ \cdot V^+ = 0 = V^+ \cdot B^+$

kernel of $\pi^1 : V^1 \subset$ $H^\bullet(G, \mathbb{F}_2) \cong B^\bullet \sqcap V^\bullet$

dual algebra

- surjective in all degrees, and isom in degrees ≥ 2

$\pi^\bullet$ $\downarrow$ $B^\bullet$

- B is a Boolean ring: $x^2 = x$ for all x
- $B^\bullet = \bigoplus_{n \geq 0} B^n$: $B^0 = \mathbb{F}_2$, $B^n = B$ for $n \geq 1$

- $V^0 = \mathbb{F}_2, V^{i \geq 2} = 0$

graded Boolean algebra

- $B^\bullet$ is a Koszul algebra

if locally finite, then $B^\bullet = \mathbb{F}_2[x_1] \sqcap \ldots \sqcap \mathbb{F}_2[x_n]$

- $V^\bullet$ is a Koszul algebra

Koszul complex $K(V^\bullet)$ = bar resolution

connected sums and colimits preserve Koszulity ✓

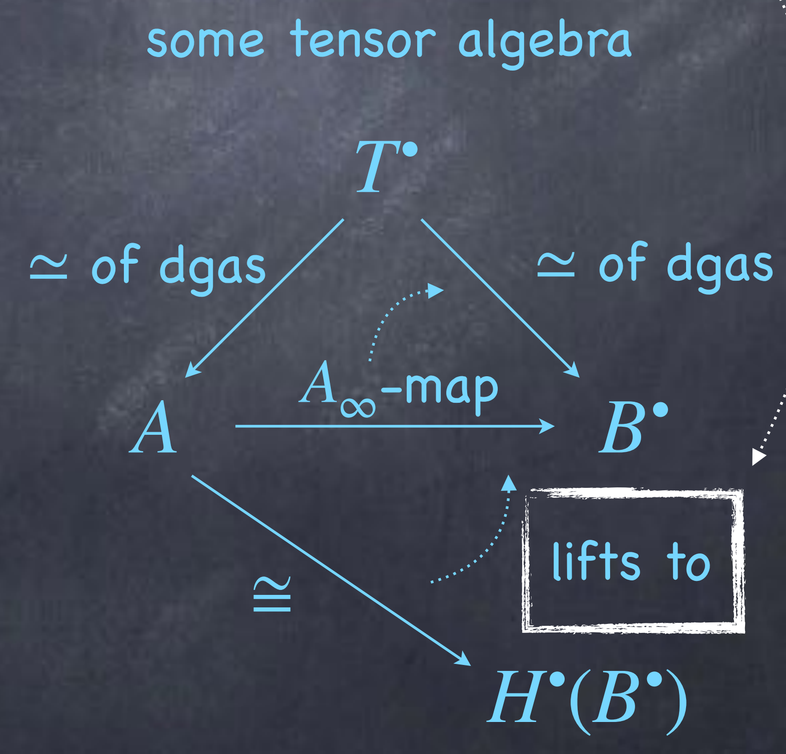
Kadeishvili's theorem: $\mathbb{F}$ field A positively graded $\mathbb{F}$-algebra with $A^0 = \mathbb{F}$

graded Hochschild cohomology

- Kadeishvili: If $\mathrm{HH}^{n,2-n}(A, A) = 0$ for all $n \geq 3$, then A is intrinsically formal

every dga with cohomology algebra equal A is formal

- apply with $A = H^\bullet(G, \mathbb{F}_2)$ and get $C^\bullet(G, \mathbb{F}_2)$ is formal



Hochschild vanishing theorem:

connected sum of graded Boolean algebra + dual algebra

$$A = B^\bullet \sqcap V^\bullet$$

graded Hochschild cohomology

Theorem (Pál-Q.): $\mathrm{HH}^{n,2-n}(A,A) = 0$ for all $n \geq 3$

Idea of proof:

- Step 1: prove assertion for all $B'^\bullet \subset B^\bullet$ locally finite

  explicit combinatorial computation: every cocycle is a coboundary

- Step 2: take colimit over all locally finite subalgebras

  spectral sequence and show higher lim-terms vanish

Thank you!