# From rational points to homotopy fixed points

HU/MPIM Online Research Seminar
Algebra, Geometry & Physics
March 23, 2021

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# Rational points

nice algebraic variety over $\mathbb{Q}$

$$X = \{x^4 - 17z^4 = 2(y^2 + 4z^2)^2\}$$

solutions over $\mathbb{Q}$ ?

solutions over $\bar{\mathbb{Q}}$

$X(\mathbb{Q})$ = rational points ?

$X(\bar{\mathbb{Q}})$ = geometric points

action by absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

fixed points

$$X(\mathbb{Q}) = X(\bar{\mathbb{Q}})^{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} \subset X(\bar{\mathbb{Q}})$$

A fundamental exact sequence: $k$ a number field, $\bar{k}$ a separable closure

$X$ smooth geometrically connected variety over $k$

$\bar{x} \in X(\bar{k})$ a geometric point

geometry arithmetic

$$1 \to \pi_1^{et}(\bar{X}, \bar{x}) \to \pi_1^{et}(X, \bar{x}) \to \mathrm{Gal}(\bar{k}, k) \to 1$$

$\bar{X} = X \times_k \bar{k}$

étale fundamental group

absolute Galois group $\cong \pi_1^{et}(\bar{k}, \bar{x})$

Sections: $y\colon \mathrm{Spec}\, k \to X$ a rational point

apply $\pi_1^{et}(\,-\,, y)$

choose étale "path"
p: y⇝x

$$\begin{tikzcd}
 & & \pi_1^{et}(X, y) \arrow[d, "p_*"'] & & \\
1 \arrow[r] & \pi_1^{et}(\bar{X}, \bar{x}) \arrow[r] & \pi_1^{et}(X, \bar{x}) \arrow[r] & \mathrm{Gal}(\bar{k}, k) \arrow[r] \arrow[ul, bend right, "s_y"'] & 1
\end{tikzcd}$$

yields a section after composition with p

unique up to conjugation by $\pi_1^{et}(\bar{X}, \bar{x})$

The Section Conjecture:

$$\begin{array}{ccc} X(k) & \longrightarrow & S(X/k) \\ y & \longmapsto & [s_y] \end{array}$$

= set of $\pi_1^{et}(\bar{X}, \bar{x})$-conjugacy classes of sections

$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{et}(\bar{X}, \bar{x}) \arrow[r] & \pi_1^{et}(X, \bar{x}) \arrow[r] & \mathrm{Gal}(\bar{k}, k) \arrow[r] \arrow[l, bend right, "{[s_y]}"'] & 1 \end{tikzcd}$$

Grothendieck: This map is a bijection if $X$ is a geom. connected projective smooth curve of genus $\geq 2$.

e.g. $X$ above by Wittenberg

Injectivity ✔ Surjectivity ?

Anabelian geometry

see work of Stix

# A motivating reformulation:

$$X(k) \longrightarrow S(X/k)$$

outer homs compatible with projection to $\mathrm{Gal}(\bar{k}/k)$

number theory

$$\cong \mathrm{Hom}_{\mathrm{out}}(\mathrm{Gal}(\bar{k}/k), \pi_1^{et}(X, \bar{x}))$$

morphisms in the homotopy category over $B\mathrm{Gal}(\bar{k}/k)$

group theory

$$\cong [B\mathrm{Gal}(\bar{k}/k), B\pi_1^{et}(X, \bar{x})]_{B\mathrm{Gal}(\bar{k}/k)}$$

$X$ is a $K(\pi,1)$-space

homotopy theory

$$\cong \pi_0((X_{\hat{et}})^{h\mathrm{Gal}(\bar{k}/k)})$$

homotopy fixed point space of the étale topological type under the action of $\mathrm{Gal}(\bar{k}/k)$

# Rational points and homotopy fixed points:

Goal: construct this map!

$$\boxed{X(k) \longrightarrow \pi_0((X_{\hat{et}})^{h\mathrm{Gal}(\bar{k}/k)})}$$

$\mathrm{Gal}(\bar{k}/k)$-homotopy fixed points of the étale topological type

This map exists in general!

"homotopical approximation" to $X(k)$

# Rational points and homotopy fixed points:

Goal:
construct
this map!

$$\boxed{X(k) \longrightarrow \pi_0((X_{\hat{et}})^{h\mathrm{Gal}(\bar{k}/k)})}$$

$\mathrm{Gal}(\bar{k}/k)$-homotopy
fixed points of
étale topological type

"homotopical approximation" to $X(k)$

might lead to

- obstructions ✔
- existence ?
- ...?

A brief detour:

with an action of a group $G$

topological space $Y$

fixed points

$$Y^G \subset Y$$

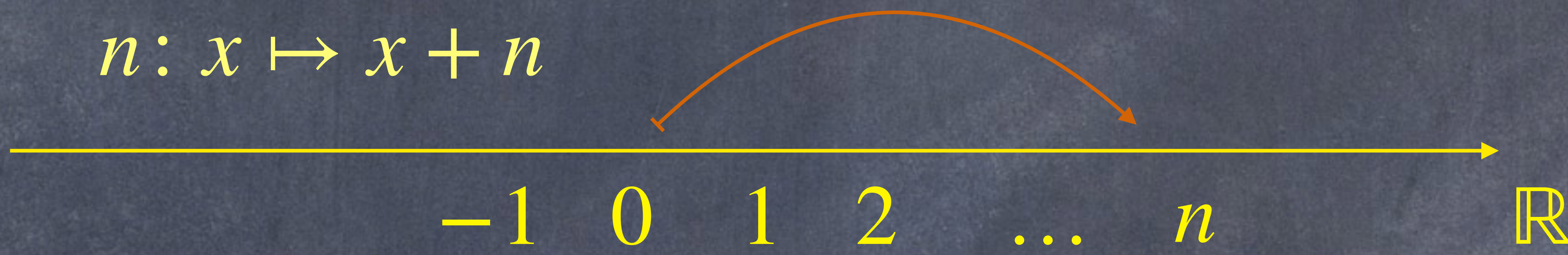
Problem: not homotopy invariant

$$X \simeq Y \text{ does not imply } X^G \simeq Y^G$$

A brief detour: $X \simeq Y$ does not imply $X^G \simeq Y^G$

Example: $G = \mathbb{Z}$ the integers, $Y = \mathbb{R}$ the real line.

$n \colon x \mapsto x + n$




- $Y \simeq \{\mathrm{pt}\}$ contractible

- $Y^G = \varnothing$

- $\{\mathrm{pt}\}^G \neq \varnothing$

action is free

Homotopy fixed points:

space of $G$-equiv. maps

$$Y = \mathrm{Map}(\mathrm{pt}, Y) \quad \text{and} \quad Y^G = \mathrm{Map}_G(\mathrm{pt}, Y)$$

Problem: point not a "well-behaved" $G$-space

Solution: replace $\mathrm{pt}$ by a cofibrant resolution $EG$

canonical $G$-bundle

a contractible space with a free $G$-action

$$\begin{array}{c} EG \\ \downarrow \\ BG = EG/G \end{array}$$

classifying space

Homotopy fixed points:

space of $G$-equiv. maps

$$Y = \mathrm{Map}(\mathrm{pt}, Y) \quad \text{and} \quad Y^G = \mathrm{Map}_G(\mathrm{pt}, Y)$$

$EG$ = contractible space with a free $G$-action

Define homotopy fixed points as

$$Y^{hG} = \mathrm{Map}_G(EG, Y)$$

Example above: $Y^{hG} \simeq \{\mathrm{pt}\}$ ✔

Fixed and homotopy fixed points:

Homotopy fixed points are homotopy invariant:

$$X \simeq Y \text{ implies } X^{hG} \simeq Y^{hG} \ ✔$$

canonical map

$EG \to \mathrm{pt}$

$$Y^G = \mathrm{Map}_G(\mathrm{pt}, Y) \to \mathrm{Map}_G(EG, Y) = Y^{hG}$$

In general not a homotopy equivalence!

Sullivan's question: $X$ an alg. variety over $\mathbb{R}$

how to recover the homotopy type ?

of $X(\mathbb{R})$ from $X(\mathbb{C})$

$C_2$ acts by complex conjugation

$$X(\mathbb{R}) = X(\mathbb{C})^{C_2}$$

But: The homotopy type of $X(\mathbb{R})$ may not be determined by the one of $X(\mathbb{C})$ by taking fixed points.

# Sullivan's question:

Example: $X = \mathbb{P}^1$ the projective line $\quad p$ odd prime

- $X(\mathbb{C}) \cong S^2$
- $\pi_n(X(\mathbb{C})_p^{hC_2}) = \pi_n(X(\mathbb{C})_p)^{C_2}$

$p$-completion of $X(\mathbb{C})$

- however $X(\mathbb{R}) \simeq S^1$ and

- $\pi_1(X(\mathbb{C})_p^{hC_2}) = \{1\} \neq \pi_1(X(\mathbb{R})_p)$

Sullivan conjecture:

$p$ a prime, $G$ a finite $p$-group, nice space $Y$ $G$-action

e.g. finite complex or $B\pi$, $\pi$ finite group

**Theorem** (Miller, Lannes, Carlsson):

$$Y_p^G \to Y_p^{hG}$$ is an equivalence

$p$-completion of $Y$

Example: $X$ a variety over $\mathbb{R}$

equivalence

$$X(\mathbb{R})_2 \simeq X(\mathbb{C})_2^{C_2} \to X(\mathbb{C})_2^{hC_2}$$

Etale homotopy in a nutshell (after Artin-Mazur, Friedlander, Sullivan,...):

if $\mathrm{char}\, k = 0$:

why not take

$k \hookrightarrow \mathbb{C}$ and $X_{\mathbb{C}}$?

Serre:
not intrinsic!

$X$ an algebraic variety over a field $k$

local diffeomorphism

$U \to X$ be an etale cover

think of as an "open" cover

Cech nerve $N_X(U) = U_\bullet$

this is the simplicial scheme

$$U \;\substack{\longleftarrow\\ \longleftarrow\\ \longrightarrow}\; U \times_X U \;\substack{\longrightarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longrightarrow}\; U \times_X U \times_X U \quad \ldots$$

$U_n$ is the $n+1$-fold fiber product of $U$ over $X$

The idea:

connected components

simplicial set $\pi_0(U_\bullet)$

Observation: variety $X$ over a field

$$\varinjlim_{U} H^s_{\text{sing}}(\pi_0(U_\bullet), F) \cong H^s_{et}(X, F)$$

singular cohomology

etale cohomology

A candidate for an etale homotopy type:

"system of all spaces $\pi_0(U_\bullet)$'s"

## Friedlander's rigidification:

A rigid cover is a disjoint union of pointed, étale, separated morphisms

$$\alpha_x : (U_x, u_x) \to (X, x)$$ one for each $x$ in $X(\bar{k})$

geometrically connected

geometric point over $x$

- Rigid covers form a filtered category $\mathrm{RC}(X)$.

crucial technical feature

Friedlander's rigidification:

The "rigid etale type of $X$" is the pro-simplicial set

functor from
a filtered
category to...

simplicial sets

$$X_{et} \colon \mathrm{RC}(X) \to \mathrm{sS}, \qquad U \mapsto \pi_0(U_\bullet).$$

Cech nerve

set of connected
components

Intrinsic topological invariant of $X$

Friedlander's rigidification:

Example: $X = \text{Spec } k$

- rigid étale covers of $k$ = finite Galois ext. $k \leq L$ in $\bar{k}$

- $N_X(L/k)_n \cong \prod_{n+1 \text{ copies}} \text{Gal}(L/k)$

finite Galois

$$k_{et} = (\text{Spec } k)_{et} \cong \{B\text{Gal}(L/k)\}_{k \leq L < \bar{k}}$$

Friedlander's rigidification:

$$X_{et}\colon \mathrm{RC}(X) \to \mathrm{sS} \qquad U \mapsto \pi_0(U_\bullet)$$

- $X$ defined over $\mathbb{C}$: $(X_{et})\hat{} \simeq X(\mathbb{C})\hat{}$

Friedlander and Cox

weak equivalence of pro-spaces after profinite completion

- $X$ defined over $\mathbb{R}$: the story is more involved...

  - $(\bar{X}_{et})\hat{} \simeq X(\mathbb{C})\hat{}$
  - $(X_{et})\hat{} \simeq X(\mathbb{C})\hat{}_{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$

Borel construction

# Profinite spaces of Lannes and Morel:

$\mathrm{s}\hat{\mathrm{S}}$ = category of simplicial profinite sets

"profinite spaces"

cofiltered limit of finite sets

profinite group

Example: $B\pi$ = classifying space of $\pi$

$B\pi_n = \pi \times \cdots \times \pi$ is a profinite set

Profinite spaces with a continuous action:

$G$ a profinite group

$s\hat{S}_G$ = category of simplicial profinite sets with a levelwise continuous $G$-action

"profinite $G$-spaces"

profinite group with a continuous $G$-action

Example: $B\pi$ = classifying space of $\pi$

$B\pi_n = \pi \times \cdots \times \pi$ is a profinite set with a continuous $G$-action

Quillen-adjunction: $G$ a profinite group

Borel construction

$$Y \times_G EG \longleftarrow\!\shortmid \; Y$$

profinite spaces with $G$-action

$$\begin{tikzcd} \mathrm{s}\hat{\mathrm{S}}_{BG} \arrow[rr, bend right, "\text{left}"'] & & \mathrm{s}\hat{\mathrm{S}}_{G} \arrow[ll, bend right, "\text{right}"'] \end{tikzcd}$$

with $G$-action induced by the one on $EG$

$$X \to BG \longmapsto X \times_{BG} EG$$

profinite spaces over $BG$

$$\begin{tikzcd} & EG \arrow[d] \\ X \arrow[r] & BG \end{tikzcd}$$

Theorem (Q.):

Both categories have fibrantly generated model structures.

# Continuous homotopy fixed points:

$G$ a profinite group $\qquad$ $Y$ in $s\hat{S}_G$

continuous homotopy fixed points

$$Y^{hG} := \mathrm{Map}_{s\hat{S}_G}(EG, R_G Y)$$

mapping space

fibrant replacement in $s\hat{S}_G$

Continuous descent spectral sequence:

$G$ profinite group

$Y$ in $s\hat{S}_G$ $\qquad Y^{hG} := \mathrm{Map}_{s\hat{S}_G}(EG, R_G Y)$

We have a descent spectral sequence:

$$E_2^{s,t} = H^s_{\mathrm{cont}}(G, \pi_t(Y)) \Rightarrow \pi_{t-s}(Y^{hG}).$$

continuous group cohomology

applications in

chromatic homotopy theory,
Lubin-Tate spectra...

# Profinite completion:

There are profinite completion functors

$$sS \longrightarrow s\hat{S}$$

levelwise profinite completion of sets

$$\text{pro}-sS \longrightarrow \text{pro}-s\hat{S} \longrightarrow s\hat{S}$$

complete objectwise

take fibrant replacements objectwise and then (homotopy) limits

# Profinite etale type:

profinite completion functor

$$\mathrm{pro-sS} \xrightarrow{\text{complete}} \mathrm{pro-s\hat{S}} \xrightarrow{\text{holim}} \mathrm{s\hat{S}}$$

For $X/k$ and $G := \mathrm{Gal}(\bar{k}/k)$:

apply etale type and complete+holim

$X_{\hat{et}}$ is an object in $\mathrm{s\hat{S}}_{BG}$,

and $\bar{X}_{\hat{et}}$ is an object in $\mathrm{s\hat{S}}_{G}$.

# Continuous Galois homotopy fixed points:

$X/k$ and $G = \mathrm{Gal}(\bar{k}/k)$ as before

We get continuous Galois homotopy fixed points:

$$(\bar{X}_{\hat{et}})^{hG} = \mathrm{Map}_{s\hat{S}_{BG}}(BG, R_G \bar{X}_{\hat{et}} \times_G EG).$$

fibrant replacement in $s\hat{S}_G$

Theorem (Q., D. A. Cox for $k = \mathbb{R}$, 1979):

The canonical map $\bar{X}_{\hat{et}} \times_G EG \to X_{\hat{et}}$ is a weak equivalence in $s\hat{S}_{BG}$.

# Back to rational points:

$X$ = geom. connected smooth projective variety over a number field $k$

$X(k) = \mathrm{Hom}_k(k, X)$ = set of rational points.

Functoriality of all constructions

homotopy category of $s\hat{\mathrm{S}}_{k_{\hat{e}t}}$

$$X(k) \to \mathrm{Hom}_{\hat{\mathcal{H}}_{k_{\hat{e}t}}}(k_{\hat{e}t}, X_{\hat{e}t})$$

$$(\mathrm{Spec}\ k \to X) \mapsto (k_{\hat{e}t} \to X_{\hat{e}t})$$

From rational to homotopy fixed points:

$$X(k) \longrightarrow \mathrm{Hom}_{\hat{\mathscr{H}}_{k_{\hat{e}t}}}(k_{\hat{e}t}, X_{\hat{e}t})$$

$G = \mathrm{Gal}(\bar{k}/k)$

$$\cong \mathrm{Hom}_{\hat{\mathscr{H}}_{BG}}(BG, X_{\hat{e}t})$$

fibrant replacement of

$$\cong \pi_0 \mathrm{Map}_{\hat{\mathrm{S}}_{BG}}(BG, X_{\hat{e}t})$$

$$\cong \pi_0(X_{\hat{e}t}^{hG})$$

# Obstructions to rational points:

$$X(k) \to \pi_0(X_{\hat{et}}^{hG})$$

different construction of $\pi_0(X_{\hat{et}}^{hG})$

- Obstruction for existence of rational points (e.g. Pal, Harpaz-Schlank, Corwin-Schlank,...)

- For $X$ and $k$ as in the Section Conjecture, we know

$$\bar{X}_{\hat{et}} \simeq B\pi_1^{et}(\bar{X}, \bar{x})$$

"$K(\pi, 1)$-space"

anabelian geometry

in this case

$$X_{\hat{et}}^{hG} \simeq (B\pi_1^{et}(\bar{X}, \bar{x}))^{hG}$$

Thank you!