Massey products and formality for real projective groups

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This is joint work with Ambrus Pál

Norm Residue Theorem:

for simplicity

k field with $\mathrm{char}(k) \neq p$ containing primitive pth root of unity

Voevodsky, Rost, ...

Milnor K-theory
$T(k^\times)/(u \otimes (1-u), u \neq 0,1)$

continuous cohomology of absolute Galois group

$$K^M_\bullet(k)/p \xrightarrow{\cong} H^\bullet(k, \mathbb{F}_p)$$

quadratic algebra

- generators in degree 1
- relations in degree 2

strong restriction on which $\mathbb{F}_p$-algebras can occur as the Galois cohomology of a field

Which quadratic algebras occur as $H^\bullet(k, \mathbb{F}_p)$?

Additional properties? $H^\bullet(k, \mathbb{F}_p)$ quadratic algebra

- if inclusion $K(A) \hookrightarrow B(A)$ of Koszul complex into bar complex is a quasi-isom., or

a quadratic algebra A is Koszul

- if cohomology $\mathrm{Ext}_A^\bullet(\mathbb{F}_p, \mathbb{F}_p) = A^!$ is the quadratic dual

Is $K_\bullet^M(k)/p$ Koszul?

- Conjecture of Positselski-Vishik-Voevodsky: If k contains a primitive pth root of unity, then $H^\bullet(k, \mathbb{F}_p)$ is Koszul

"The algebra $H^\bullet(k, \mathbb{F}_p)$ has a very nice and simple structure."

- Positselski: local and global fields ✓
- Mináč-Panini-Quadrelli-Tân: finite fields, pseudo algebraically closed fields, elementary type pro p-groups, Pythagorean fields if $p = 2$, ... ✓

Additional properties? $H^\bullet(k, \mathbb{F}_p)$ quadratic algebra

- Is $H^\bullet(k, \mathbb{F}_p)$ a Koszul algebra?

- Can $H^\bullet(k, \mathbb{F}_p)$ be described in "elementary terms"?

$\mathscr{C}^\bullet$ is quasi-isom as a dga to $(H^\bullet(\mathscr{C}^\bullet), \delta = 0)$

continuous Galois cochains

- Is $\mathscr{C}^\bullet(k, \mathbb{F}_p)$ a formal dg-algebra?

- Can the dga $\mathscr{C}^\bullet(k, \mathbb{F}_p)$ be described in "elementary terms" as well?

Massey products provide an obstruction to formality

Massey products: $(\mathscr{C}^\bullet, \delta)$ a differential graded algebra with coh. $H^\bullet$

$a, b, c \in H^1$ with
$ab = 0 = bc$

- $A, B, C \in \mathscr{C}^1$ represent a, b, c
- E_{ab}, E_{bc} with $\delta E_{ab} = AB, \delta E_{bc} = BC$

make a choice

triple Massey product is defined if such data exist

set of elements in H^2

$$\langle a, b, c \rangle := [AE_{bc} + E_{ab}C] \text{ in } H^2/(aH^1 + H^1c)$$

- if $\delta = 0$ then $\langle a, b, c \rangle = 0$

  we may choose $E_{ab} = 0 = E_{bc}$

- if $f: \mathscr{C}^\bullet \to \mathscr{D}^\bullet$ quasi-isom then

  $\langle a, b, c \rangle$ ⟷ (bijection) $\langle f^\bullet a, f^\bullet b, f^\bullet c \rangle$

- generalizes to n-tuple Massey product $\langle a_1, \ldots, a_n \rangle$ in all degrees

$a_1 a_2 = \cdots = a_{n-1} a_n = 0$ only a necessary condition for $\langle a_1, \ldots, a_n \rangle$ being defined

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Massey products provide an obstruction to formality

- many non-vanishing Massey products in arithm. & alg. geometry: Ekedahl, Morishita, Sharifi, Gärtner, Bleher-Chinburg-Gillibert, Deninger,...

- Hopkins and Wickelgren: $\langle a, b, c \rangle \neq \varnothing \iff 0 \in \langle a, b, c \rangle$

  k a local or global field of $\mathrm{char}(k) \neq 2$

  elements in $H^1(k, \mathbb{F}_2)$

Massey vanishing conjecture of Mináč-Tân:

for every field k, all $n \geq 3$, all primes p

Conjecture: For $a_1, \ldots, a_n \in H^1(k, \mathbb{F}_p)$:

$$\langle a_1, \ldots, a_n \rangle \neq \varnothing \iff 0 \in \langle a_1, \ldots, a_n \rangle.$$

✓

- Efrat-Matzri, Mináč-Tân: all fields, all primes, $n = 3$
- Merkurjev-Scavia: all fields, $p = 2$, $n = 4$

  - New examples of profinite groups which are not absolute Galois groups
  - Example: S = free pro-p group on generators $x_1, \ldots, x_5$

Mináč-Tân: and relation $r = [x_4, x_5][[x_2, x_3]x_1]$

Then $G = S/\langle r \rangle$ is not the maximal pro-p quotient of an absolute Galois group

Massey vanishing conjecture of Mináč-Tân:

for every field k, all $n \geq 3$, all primes p

✓ **Conjecture:** For $a_1, \ldots, a_n \in H^1(k, \mathbb{F}_p)$:

$$\langle a_1, \ldots, a_n \rangle \neq \varnothing \iff 0 \in \langle a_1, \ldots, a_n \rangle.$$

- Efrat-Matzri, Mináč-Tân: all fields, all primes, $n = 3$
- Merkurjev-Scavia: all fields, $p = 2$, $n = 4$
- Harpaz-Wittenberg: all number fields, all primes, all $n \geq 3$

and before Guillot-Mináč-Topaz-Wittenberg for $p = 2$, $n = 4$

- Pál-Szabó: fields with $\mathrm{vcd} \leq 1$
- Quadrelli: elementary type pro-p-groups

all primes, all $n \geq 3$

strong Massey vanishing conjecture

$$0 \in \langle a_1, \ldots, a_n \rangle \iff a_i \cup a_{i+1} = 0$$

for all $i = 1, \ldots, n-1$

Hopkins-Wickelgren formality:

Massey vanishing conjecture and Koszulity suggest

for every field k and all primes p?

Question: Is $\mathscr{C}^\bullet(k, \mathbb{F}_p)$ formal?

i.e., quasi-isomorphic to $(H^\bullet(k, \mathbb{F}_p), \delta = 0)$ as dgas

- Yes, for example, for pseudo-algebraically closed fields

The answer is **no** in general

- Positselski: $\mathbb{Q}_\ell$ for odd ℓ is not formal at $p = 2$
- Harpaz-Wittenberg: $\mathbb{Q}$ is not formal for $p = 2$

- formality implies all Massey products are defined and vanish when neighbouring cup products are zero

- exist $a, b, c, d \in H^1(\mathbb{Q}, \mathbb{F}_2)$ with $a \cup b = b \cup c = c \cup d = 0$ but $\langle a, b, c, d \rangle$ not defined

a $\quad E_{ab}$ $\quad E_{\langle a,b,c \rangle}$

b $\quad E_{bc}$ $\quad E_{\langle b,c,d \rangle}$

c $\quad E_{cd}$

d

cochains which witness vanishing of cup products

cochains witnessing vanishing of Massey products

- $\langle a, b, c, d \rangle$ is defined if both $\langle a, b, c \rangle$ and $\langle b, c, d \rangle$ vanish with the same choice of E_{bc}

Hopkins-Wickelgren formality:

Massey vanishing conjecture
and Koszulity suggest

for every field k and all primes p?

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- Harpaz-Wittenberg: $\mathbb{Q}$ is not formal for $p = 2$
- Merkurjev-Scavia: more examples of fields of characteristic $\neq p$ which are not formal at p

- formality implies all Massey products are defined and vanish when neighbouring cup products are zero

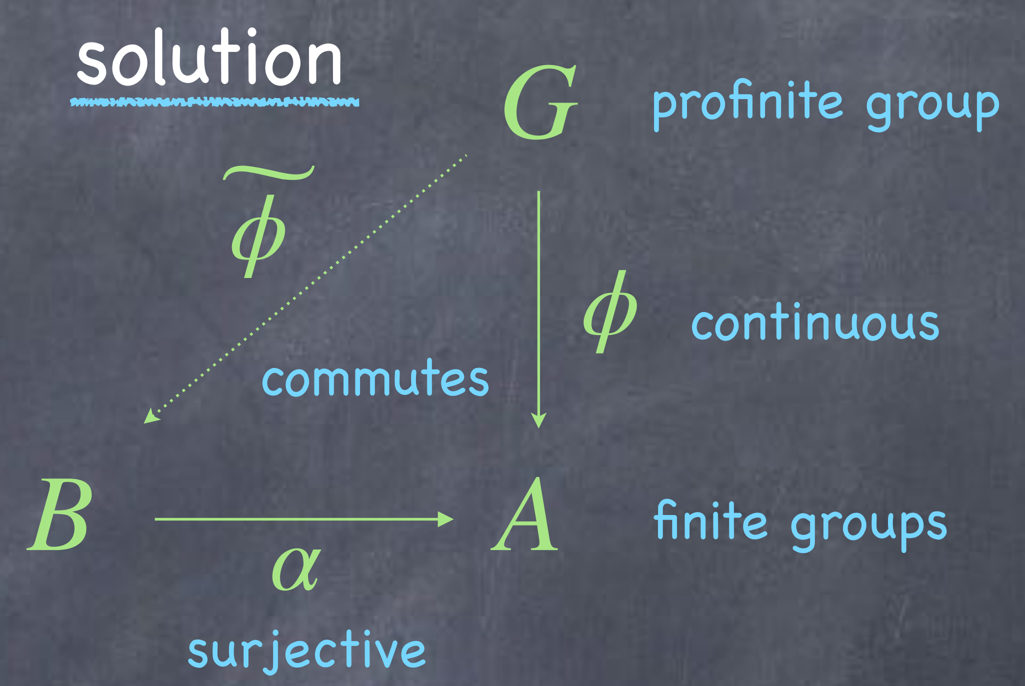
However, there are also important positive cases…

Projective groups and PAC fields:

embedding problem

G is projective if every embedding problem has a solution

→ cohomological dimension at most 1



Ax, Lubotzky-van den Dries

k a field with absolute Galois group $\Gamma(k)$

- k is called pseudo algebraically closed if every geometrically irreducible k-variety has a k-rational point

k is pseudo algebraically closed → $\Gamma(k)$ is projective

there is a PAC field k with $\Gamma(k) \cong G$ ← G is projective

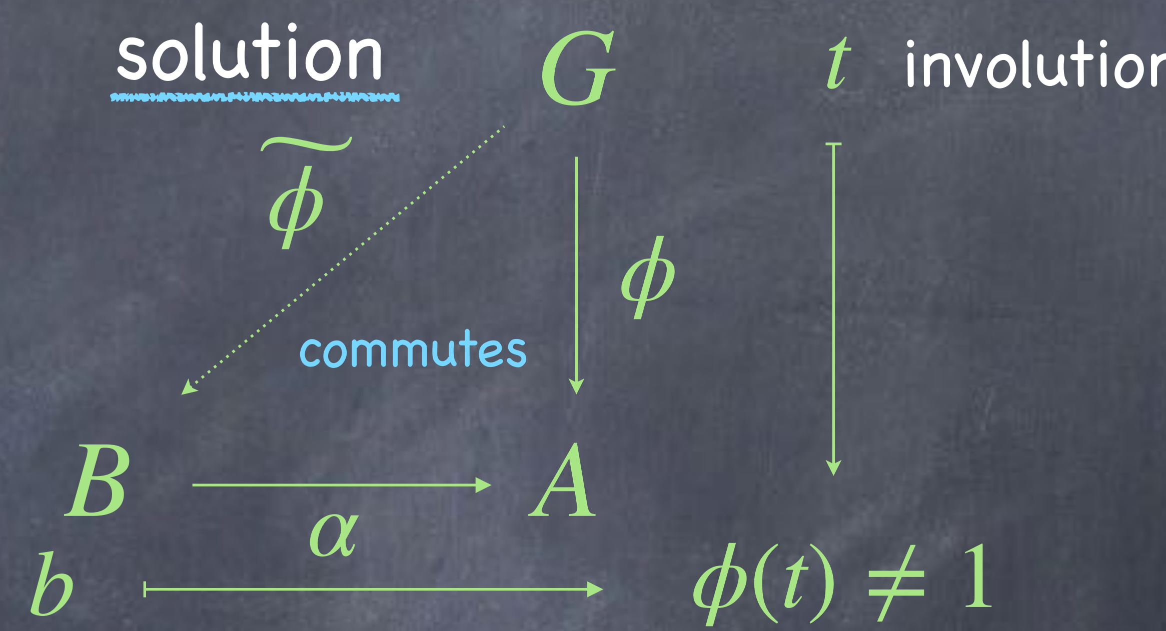
Real projective groups:

G a profinite group

real embedding problem

"involutions are no obstruction"

G is real projective if it has an open subgroup without 2-torsion and every real embedding problem has a solution



there exists an involution

Haran–Jarden

- k a field with absolute Galois group $\Gamma(k)$

$\Gamma(k)$ is real projective

k is pseudo real closed

- every geometrically irreducible k-variety, which has a $\bar{k}$-rational simple point in every real closure $\bar{k}$, has a k-rational point

k has virtual cohomological dimension at most one ≤ 1

G is real projective

there is a PRC field k with $\Gamma(k) \cong G$

every dga $\mathscr{C}^\bullet$ over $\mathbb{F}_p$ with $H^\bullet \cong H^\bullet(G, \mathbb{F}_p)$ is formal

Our main results:

- Theorem (Pál-Q.): If G is a real projective profinite group, then $H^\bullet(G, \mathbb{F}_p)$ is intrinsically formal and Koszul.

k has virtual cohomological dimension ≤ 1

Haran-Jarden

absolute Galois group $\Gamma(k)$ is real projective

- Theorem (Pál-Q.): If k has virtual cohomological dimension ≤ 1, then $H^\bullet(k, \mathbb{F}_p)$ is intrinsically formal and Koszul.

for p odd: $H^i(G, \mathbb{F}_p) = 0$ for $i \geq 2$ and intrinsic formality and Koszulity are easy

Scheiderer's theorem:

- $(B \sqcap D)^0 = \mathbb{F}_2$
- $(B \sqcap D)^i = B^i \oplus D^i$, and $B^+ \cdot D^+ = 0 = D^+ \cdot B^+$

connected sum of quadratic algebras

Scheiderer

G a real projective group $\Longrightarrow$ $H^\bullet(G, \mathbb{F}_2) = A = B^\bullet \sqcap D^\bullet$

quadratic algebra with generators in deg 1 are orthogonal

$D^i = 0$ for $i \geq 2$

graded Boolean algebra

dual algebra

- $\mathscr{X}(G)$ = set of conjugacy classes of involutions
- B = ring of continuous functions $\mathscr{X}(G) \to \mathbb{F}_2$
- $H^i(G, \mathbb{F}_2) \to B$ is surjective for $i \geq 1$ and iso for $i \geq 2$ with kernel $=: D^1$ for $i = 1$

- Theorem (Pál-Q.): The connected sum of a graded Boolean algebra and a dual algebra is Koszul.

Proof:

- $D^\bullet$ is Koszul

  Koszul complex $K(D^\bullet)$ = bar resolution

- colimits and connected sums preserve Koszulity

- $B^\bullet$ is Koszul

  if locally finite, then $B^\bullet \cong \mathbb{F}_2[x_1] \sqcap \ldots \sqcap \mathbb{F}_2[x_n]$

✓

Hochschild vanishing theorem:

Scheiderer

connected sum of quadratic algebras

G a real projective group $\Longrightarrow$ $H^\bullet(G,\mathbb{F}_2) = A = B^\bullet \sqcap D^\bullet$

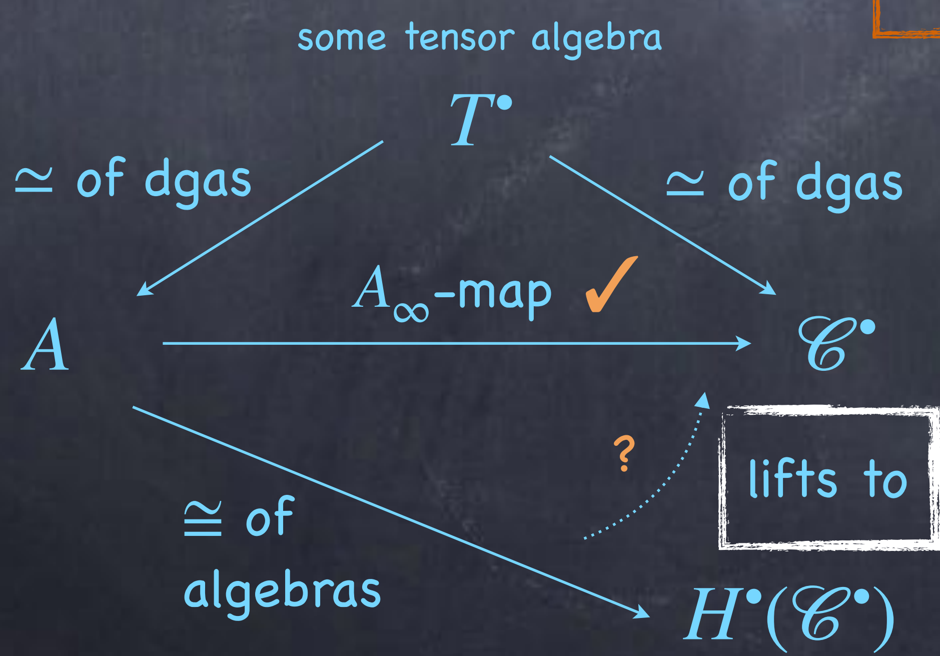
graded Hochschild cohomology

- Theorem (Pál-Q.): $\mathrm{HH}^{n,2-n}(A,A) = 0$ for all $n \geq 3$.

contain the obstructions to define an A_∞-map $A \to \mathscr{C}^\bullet$ which lifts $A \xrightarrow{\cong} H^\bullet(\mathscr{C}^\bullet)$

Kadeishvili: $\Longrightarrow$ A is intrinsically formal

i.e., every dga $\mathscr{C}^\bullet$ over $\mathbb{F}_2$ with $H^\bullet \cong H^\bullet(G,\mathbb{F}_2)$ is formal



- apply with $A = H^\bullet(G,\mathbb{F}_2)$ and get $\mathscr{C}^\bullet(G,\mathbb{F}_2)$ is formal

Proof of the Hochschild vanishing theorem:

connected sum of quadratic algebras

Scheiderer

G a real projective group $\Longrightarrow$ $H^\bullet(G, \mathbb{F}_2) = A = B^\bullet \sqcap D^\bullet$

graded Hochschild cohomology

- Theorem (Pál-Q.): $\mathrm{HH}^{n,2-n}(A, A) = 0$ for all $n \geq 3$.

Idea of proof:

finite dimensional in each degree

- Step 1: prove assertion for all $B^\bullet_{\mathrm{fin}} \subset B^\bullet$ locally finite

  explicit combinatorial computation: every cocycle is a coboundary

- Step 2: take colimit over all locally finite subalgebras

  spectral sequence and show higher lim-terms vanish

Reconstructing the group coalgebra

G a profinite group

$G(2)$:= maximal pro-2 quotient

assume

- $H^\bullet := H^\bullet(G, \mathbb{F}_2) \cong H^\bullet(G(2), \mathbb{F}_2)$ is Koszul and formal

Positselski

- can reconstruct the coalgebra $\mathbb{F}_2(G(2))$ from $H^\bullet$

if G is real projective, we can improve this ...

Quasi-Boolean groups:

- Theorem (Pál-Q.): Let G be a pro-2 group. Then

Scheiderer

  - G is real projective $\Longleftrightarrow$ $H^\bullet(G, \mathbb{F}_2) = B^\bullet \sqcap D^\bullet$
  - and we can reconstruct G from $H^\bullet$

connected sum of a graded Boolean and a dual algebra

as a pro-2 group via generators and relations

Principal construction:

- given $H^\bullet(G, \mathbb{F}_2) = B^\bullet \sqcap D^\bullet$
  - B^1 is a Boolean ring, i.e., $x^2 = x$ for all x
  - set X = spectrum of B^1
  - set Y = basis of D^1

free pro-2 group on set Y

"free pro-2 product of 2-groups over a topological space"

- then $G \cong F(Y) *_2 \mathbb{B}(X)$ is a free pro-2 product

- this uses profinite versions of Quillen's F-isom theorems on group cohomology, work of Scheiderer, Haran-Jarden on the arithmetic of field, existence of sections of principal G-bundles (Morel), ...

Thank you!