Algebraic vs topological classes

National Mathematicians Meeting
in Bergen
September 13, 2018

Gereon Quick
NTNU

Algebraic topology in a nutshell

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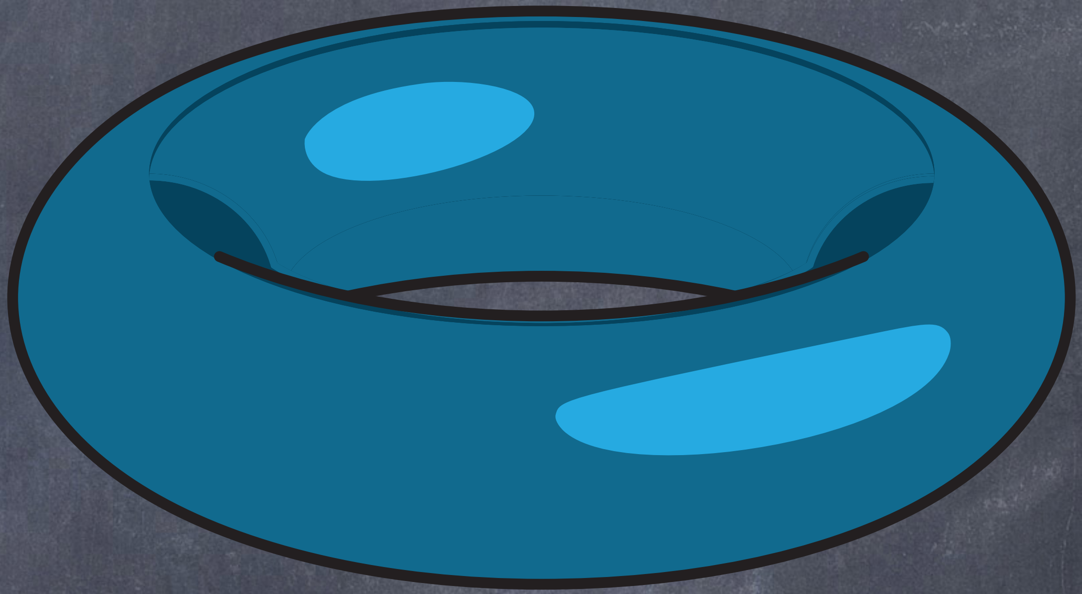
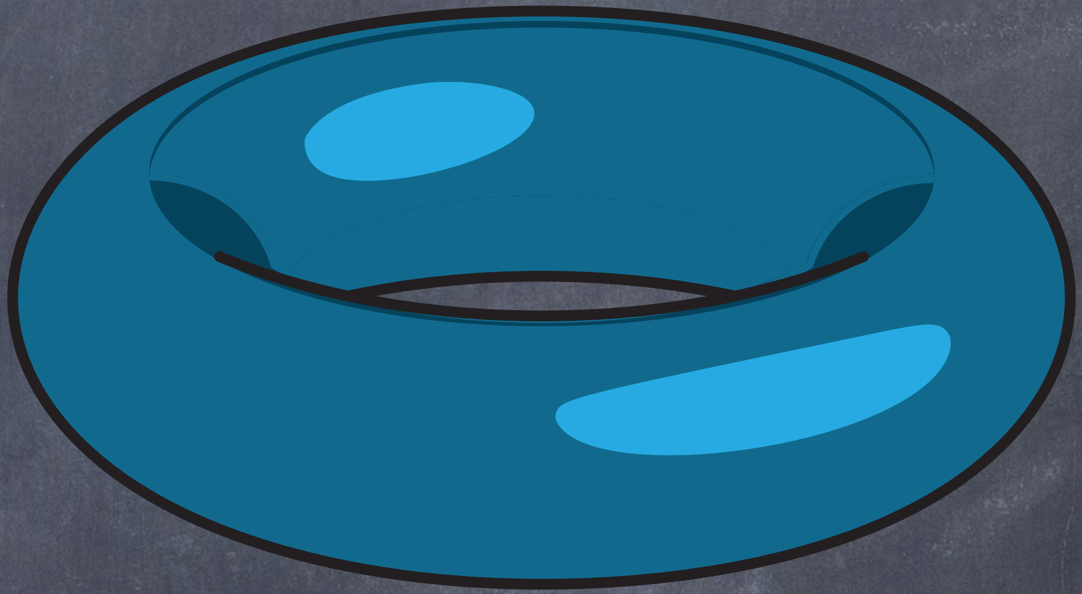
spaces



Algebraic topology in a nutshell

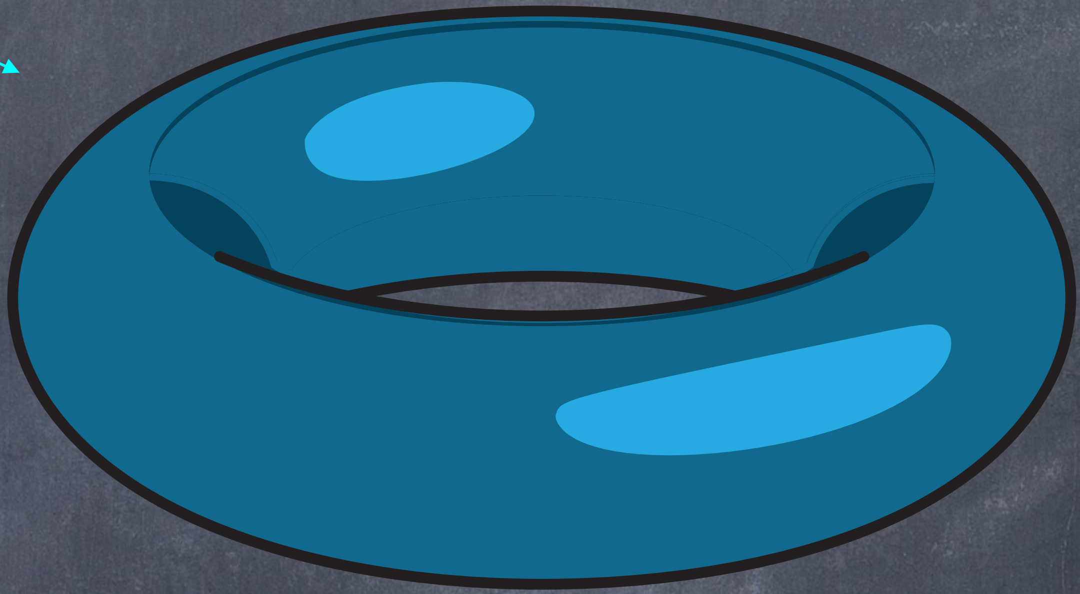
test spaces — continuous maps ⟶ spaces



Algebraic topology in a nutshell

test spaces ⟶ continuous maps ⟶ spaces

dim 0 •

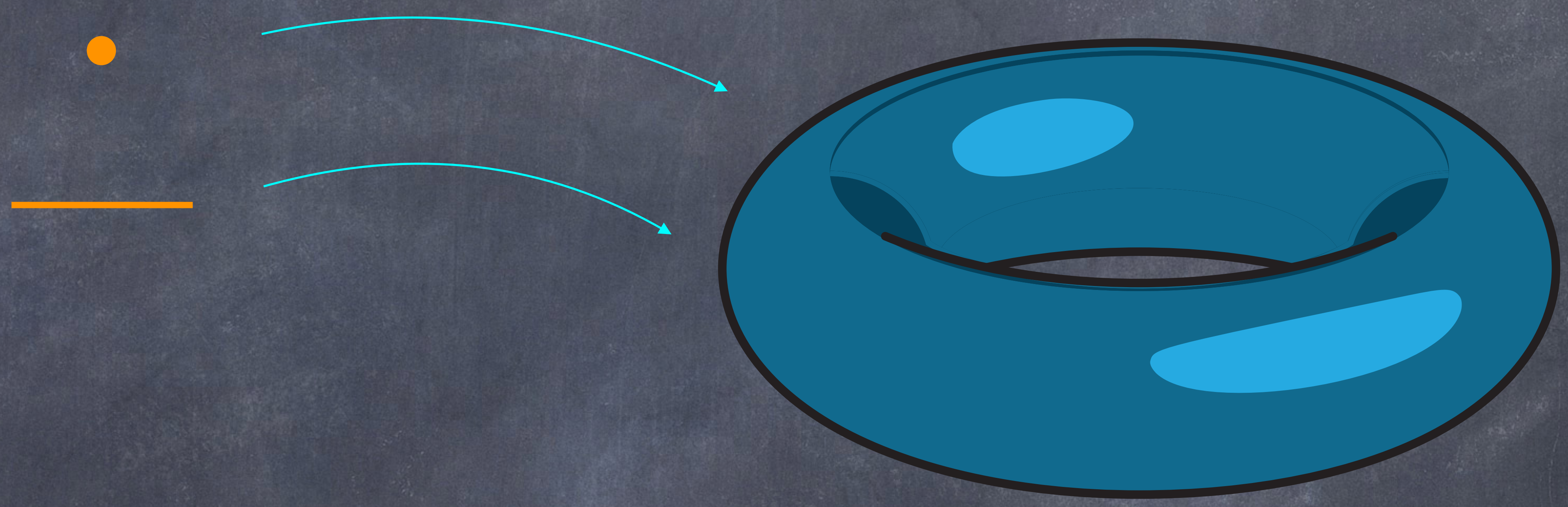


Algebraic topology in a nutshell

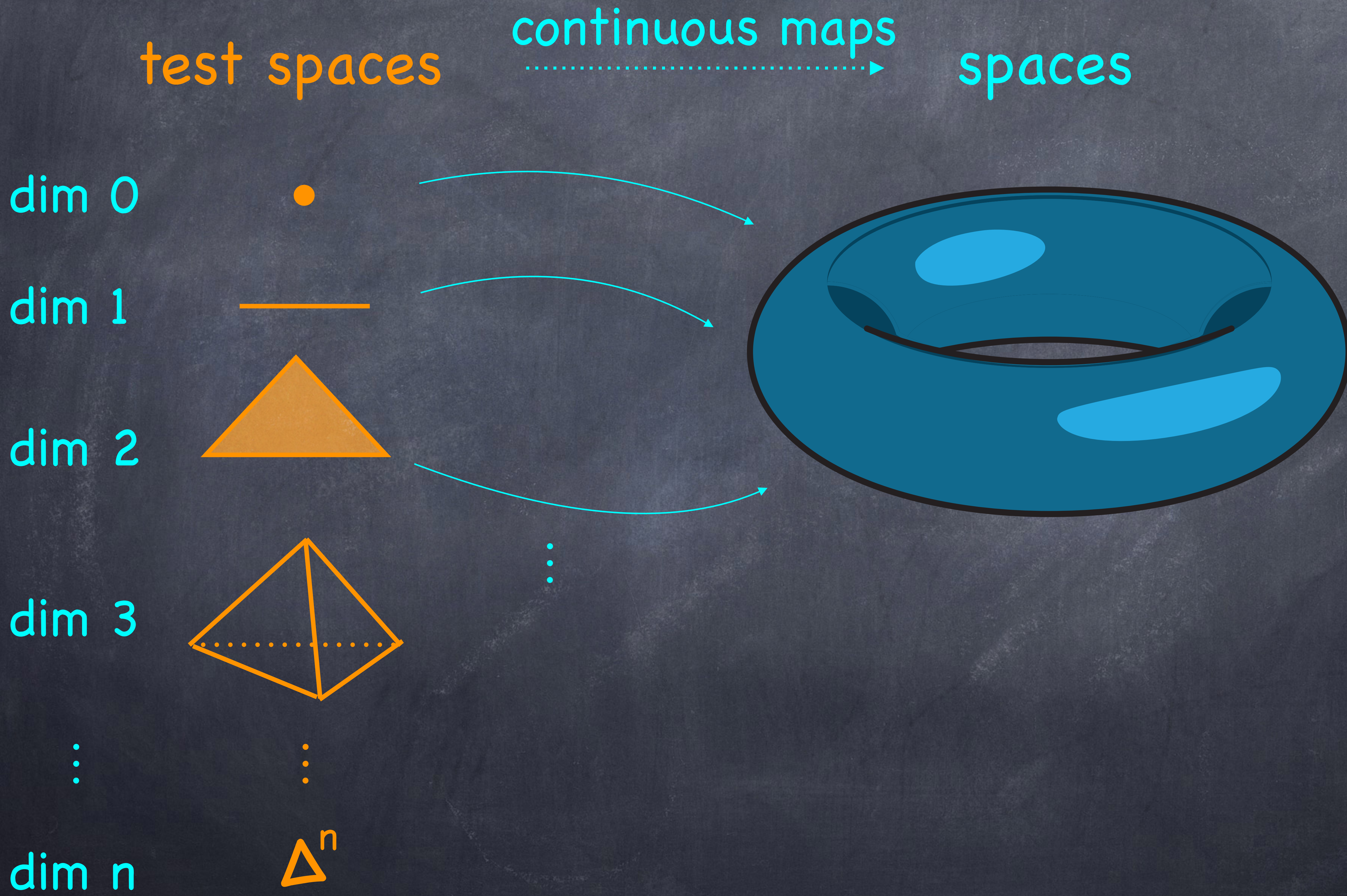
test spaces — continuous maps → spaces

dim 0

dim 1



Algebraic topology in a nutshell



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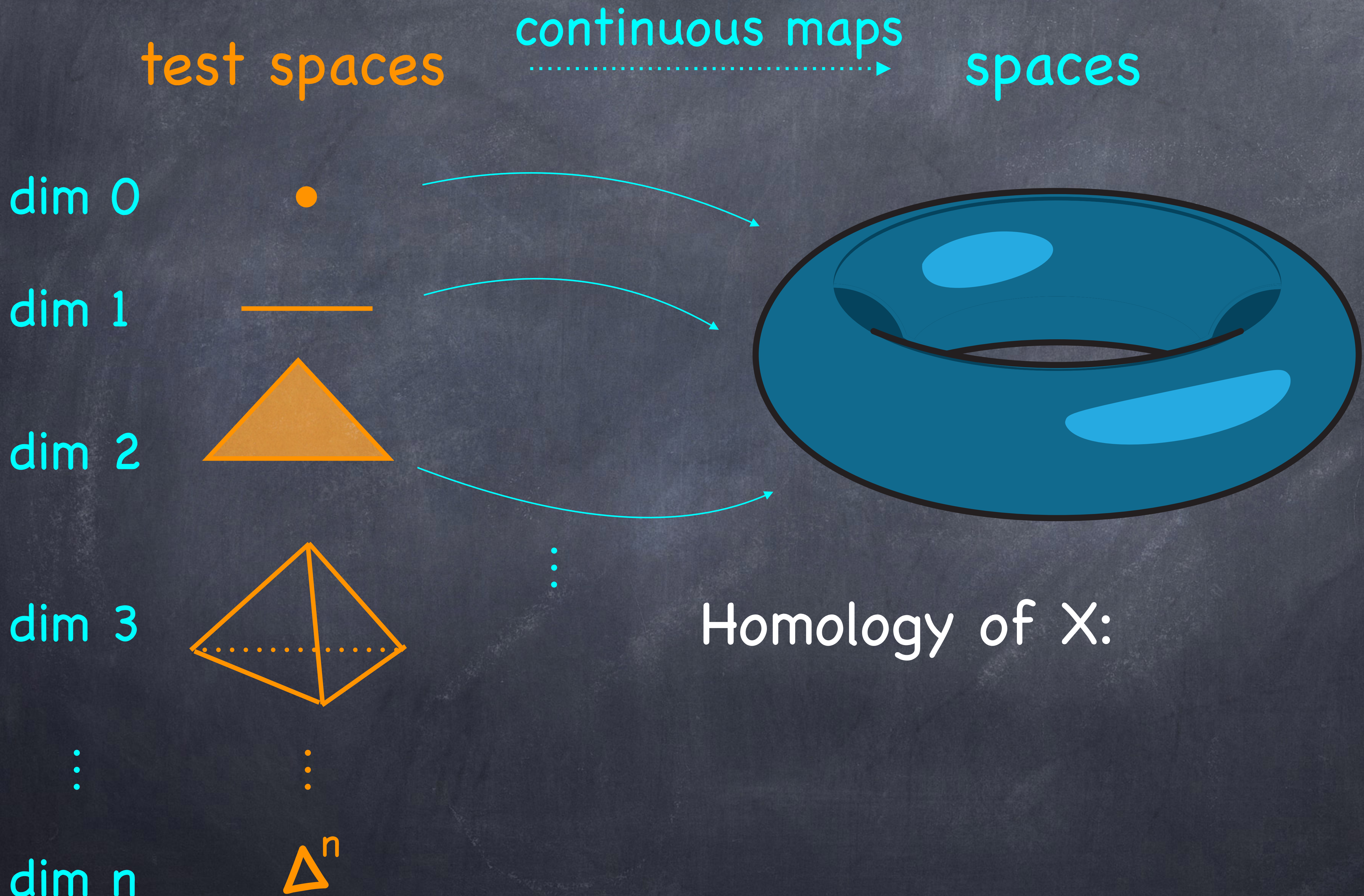
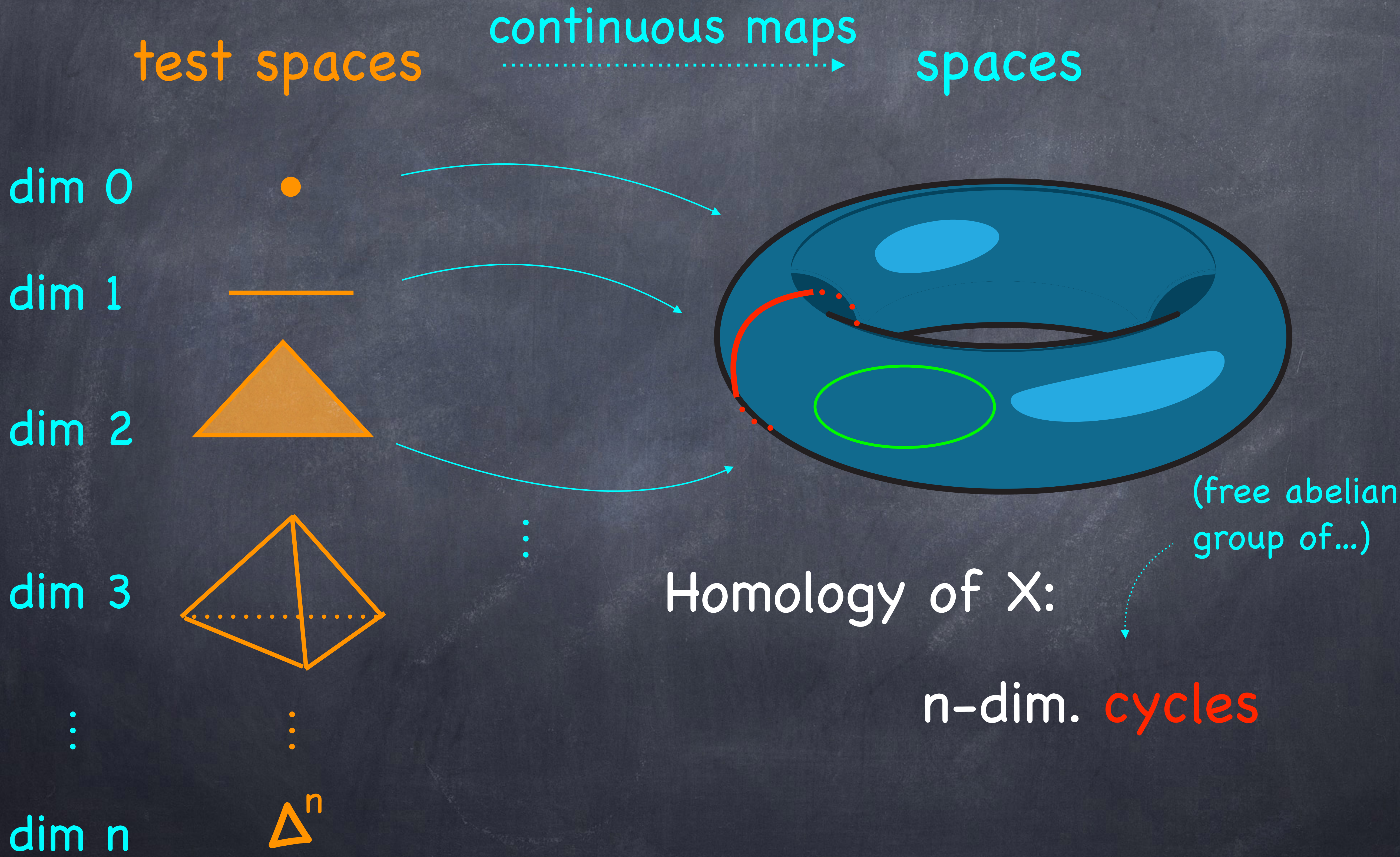


Homology of X:

Algebraic topology in a nutshell



Algebraic topology in a nutshell

test spaces ⋯ continuous maps ⋯▸ spaces

dim 0

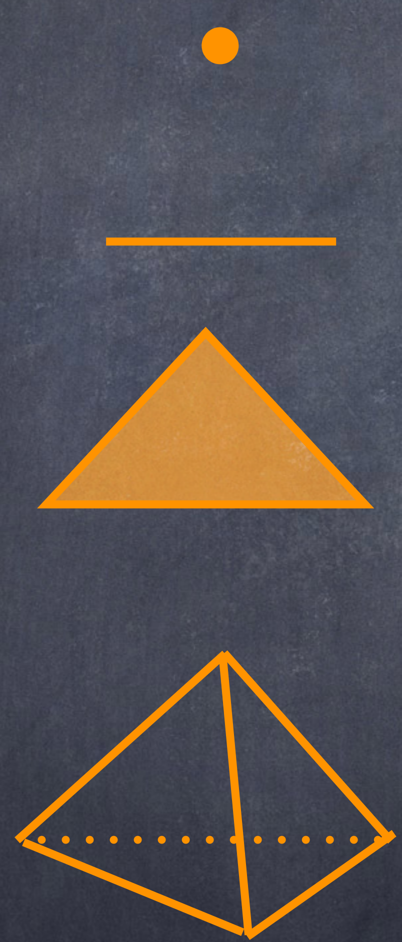
dim 1

dim 2

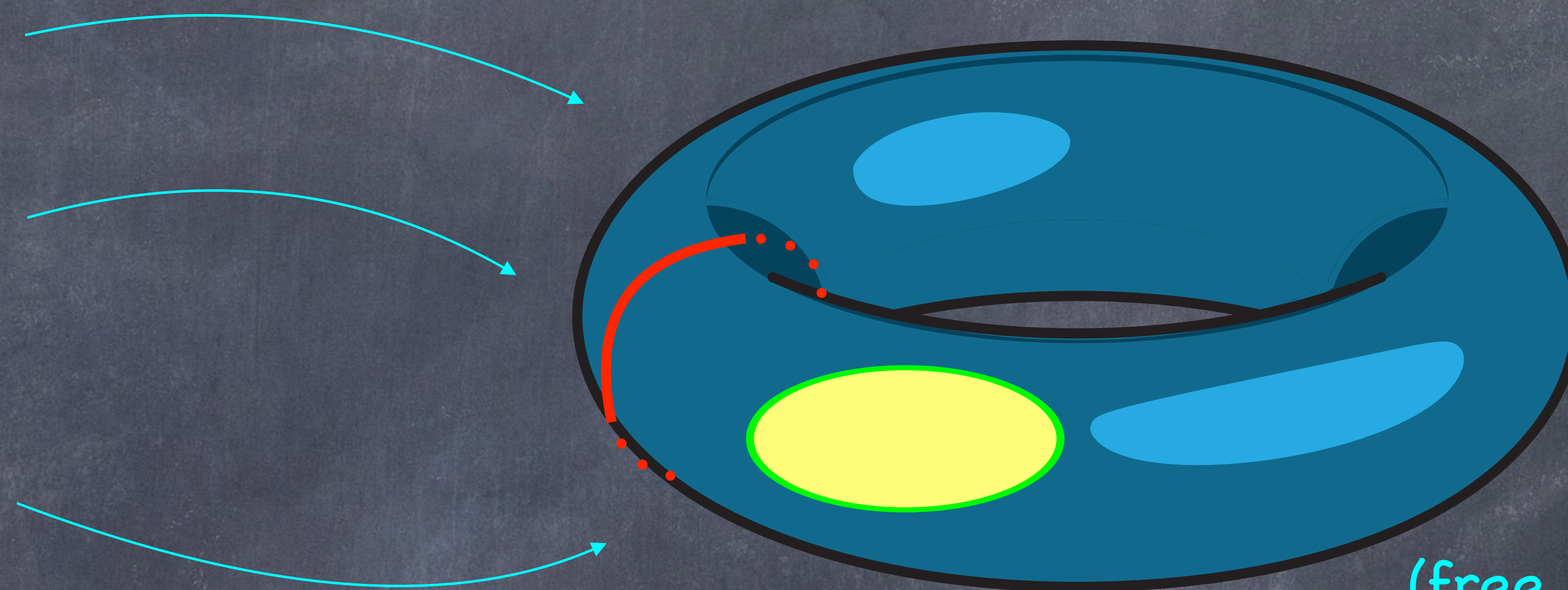
dim 3

⋮

dim n



Δ^n



⋮

(free abelian group of...)

Homology of X:

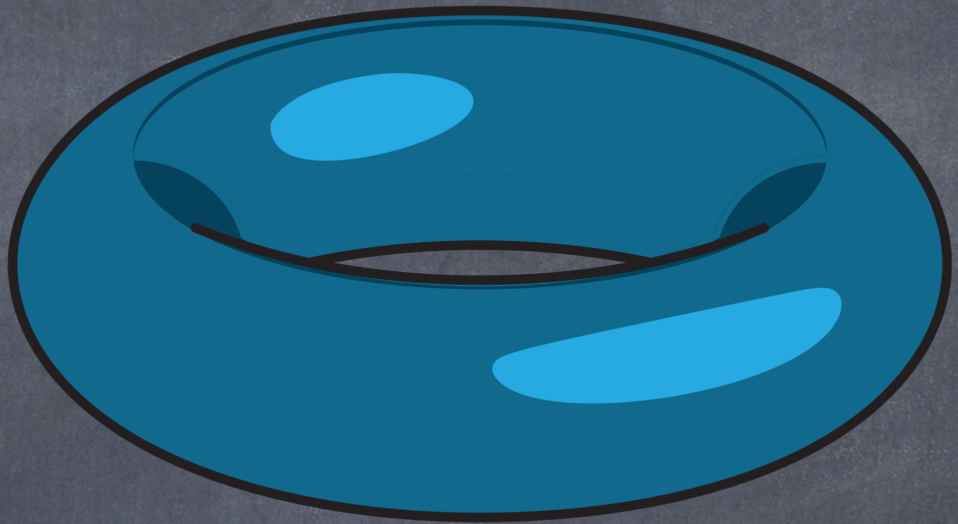
$H_n(X;Z)$ = n-dim. cycles

modulo boundaries

Fundamental classes

continuous map of spaces

$$Y \xrightarrow{f} X$$



Fundamental classes

Δ^n

$\downarrow$

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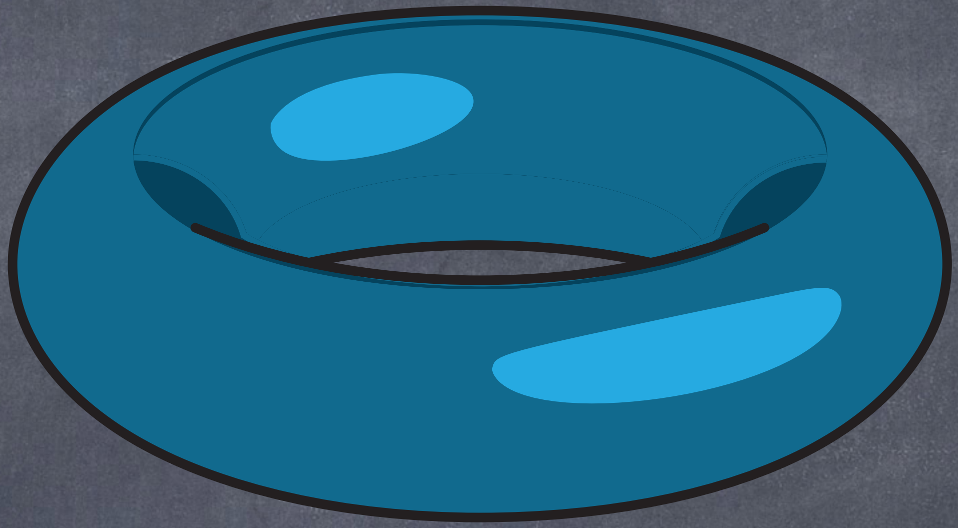


$H_n(Y;Z)$

Fundamental classes

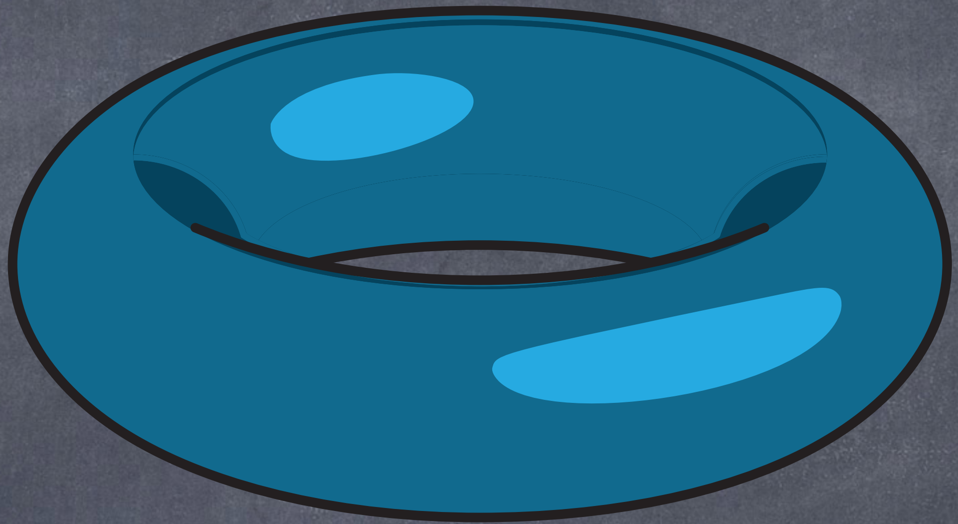
$$\Delta^n \downarrow$$

$$Y \xrightarrow{f} X$$

continuous map of spaces

induces

$$H_n(Y;Z) \xrightarrow{f_n} H_n(X;Z)$$



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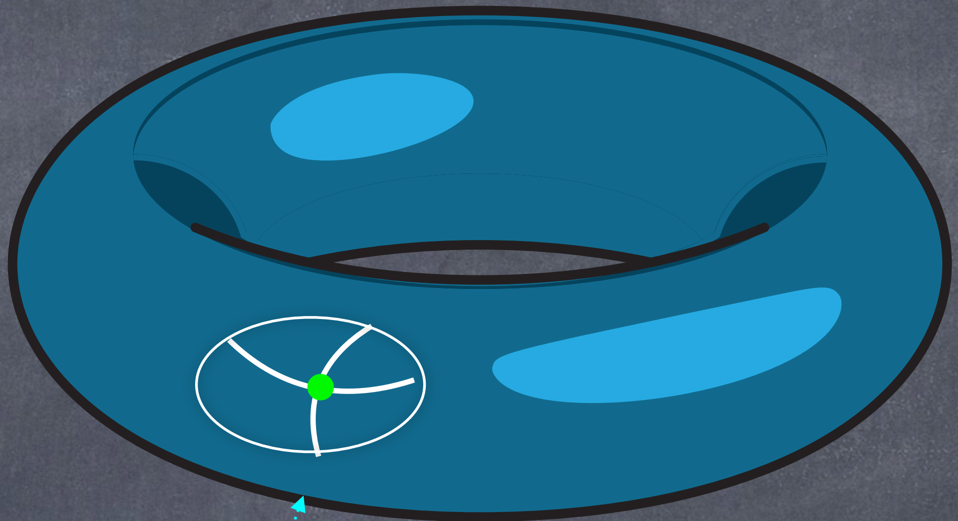

every point has a neighborhood $\approx R^n$

Assume: $Y=M$ is a smooth n-manifold.

Fundamental classes

Δ^n

↓

continuous map of spaces

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induces

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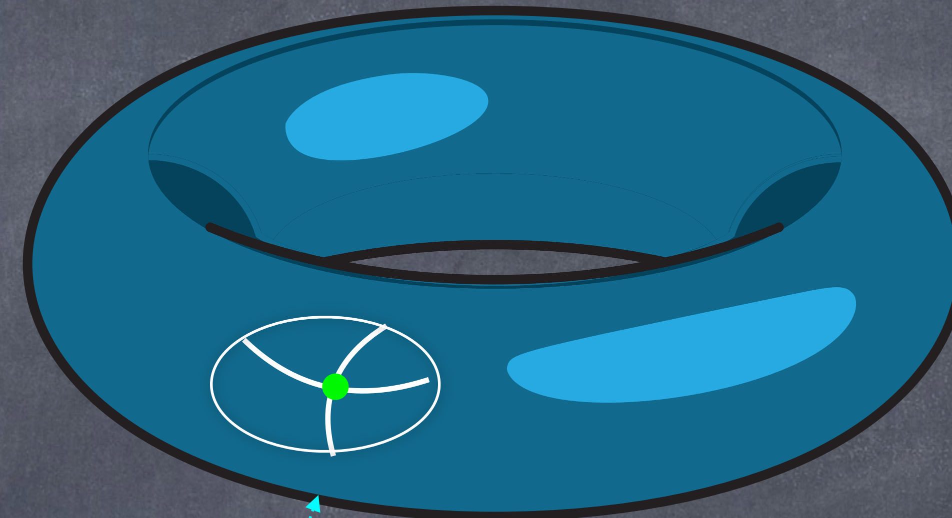

& compact, oriented

every point has a neighborhood ≈ R^n

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Fundamental classes

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induces

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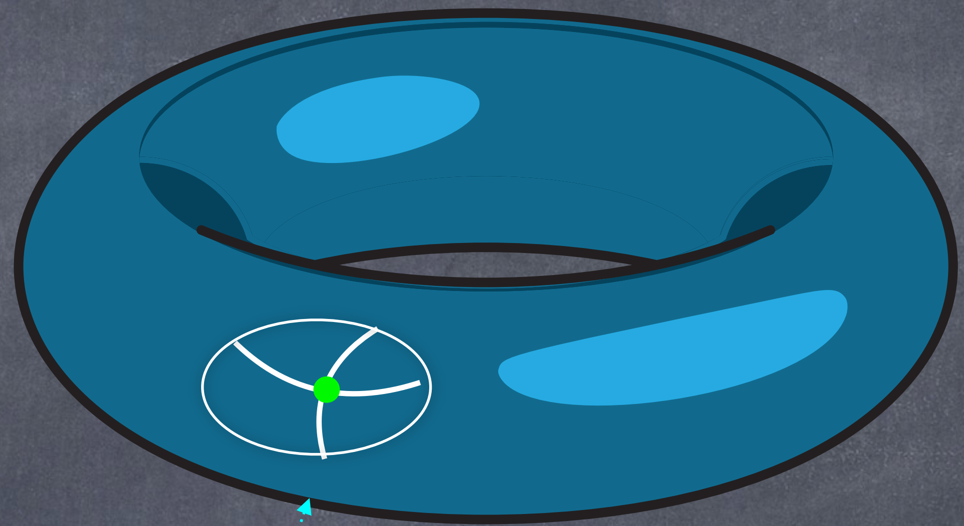

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"fundamental class of M"

$[M] \in H_n(M;Z)$

$(M,M-p)\approx(R^n,R^n-0)$

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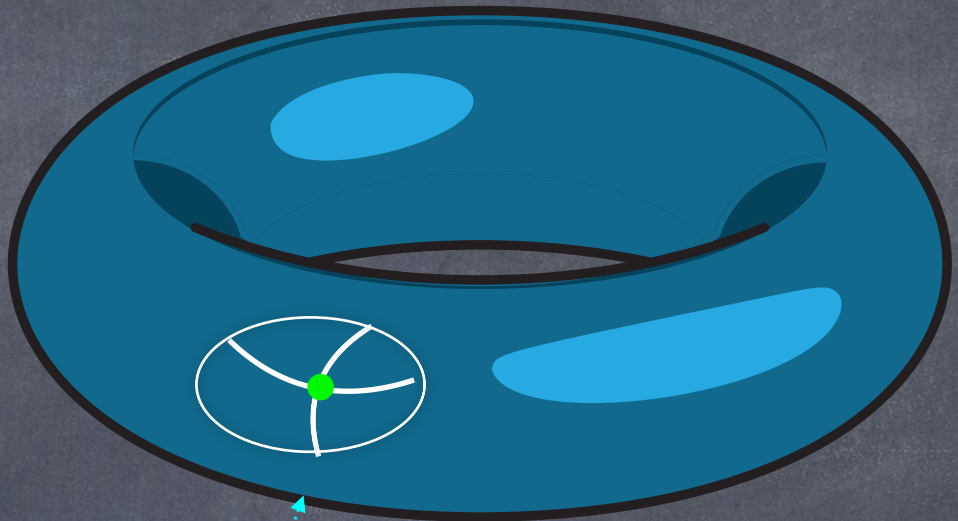

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Assume: $Y=M$ is a smooth n-manifold.

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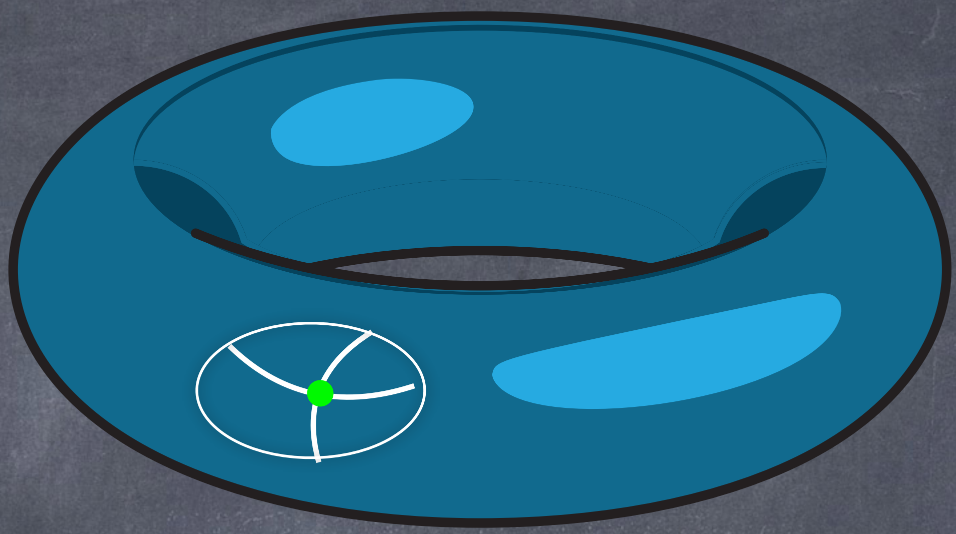
$[M] \in H_n(M;Z)$ $\mapsto$ $f_n[M] \in H_n(X;Z)$

$(M,M-p)\approx(R^n,R^n-0)$

Steenrod's question:

Can every class in $H_n(X;\mathbb{Z})$ be realized as the fundamental class of a smooth n-manifold $M \to X$?

& compact, oriented



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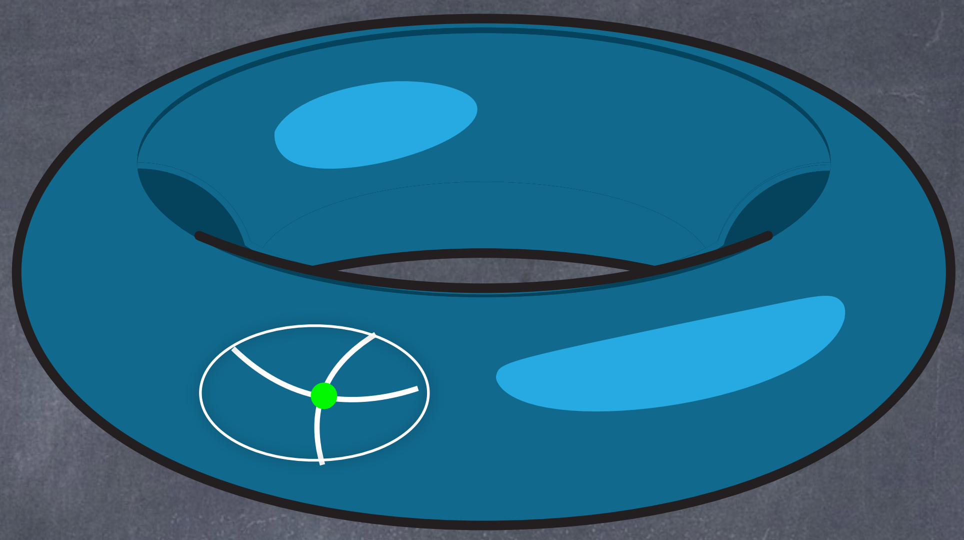

$\alpha \in H_n(X;Z)$

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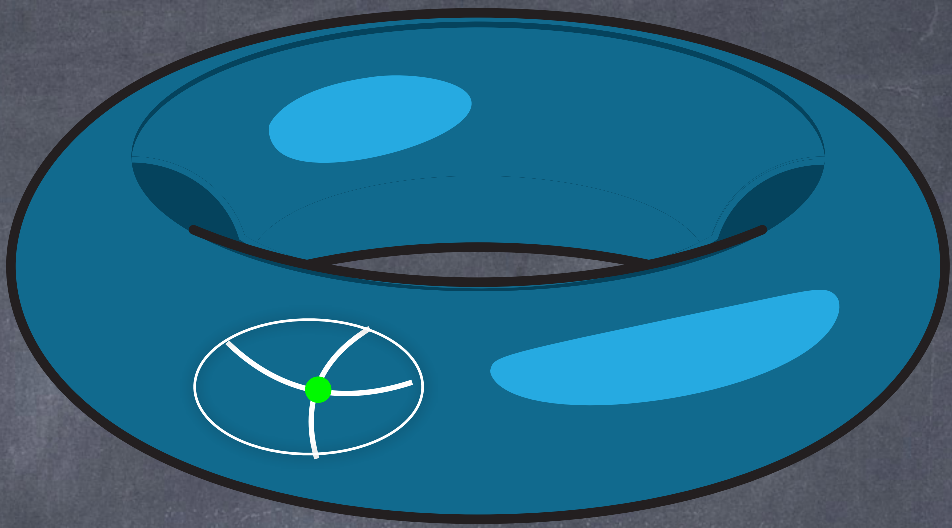

exist manifold ? M

$f_n[M] \overset{?}{=} \alpha \in H_n(X;Z)$ $\quad M \xrightarrow{f} X$

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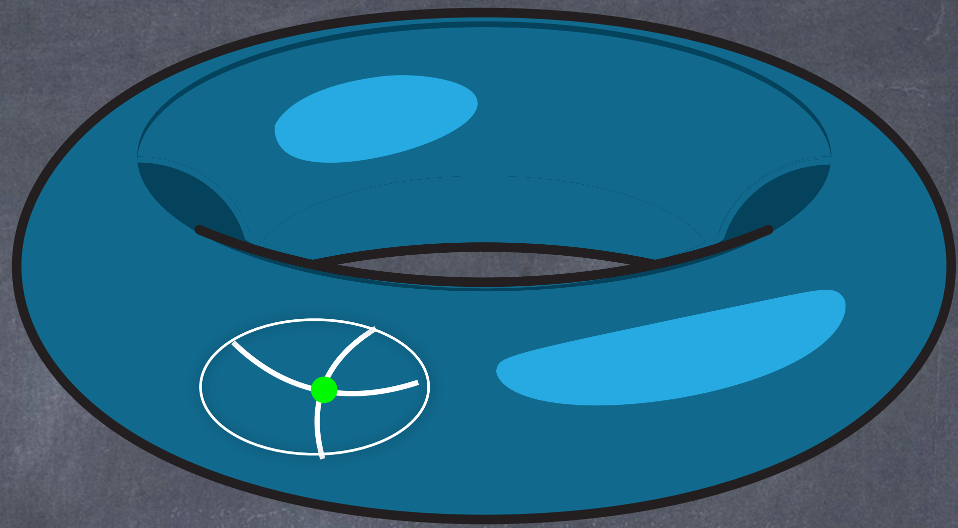

exist manifold ? $M \downarrow f X$

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Example: Every elt in $H_1(X;Z)$ is realized as $S^1 \to X$.

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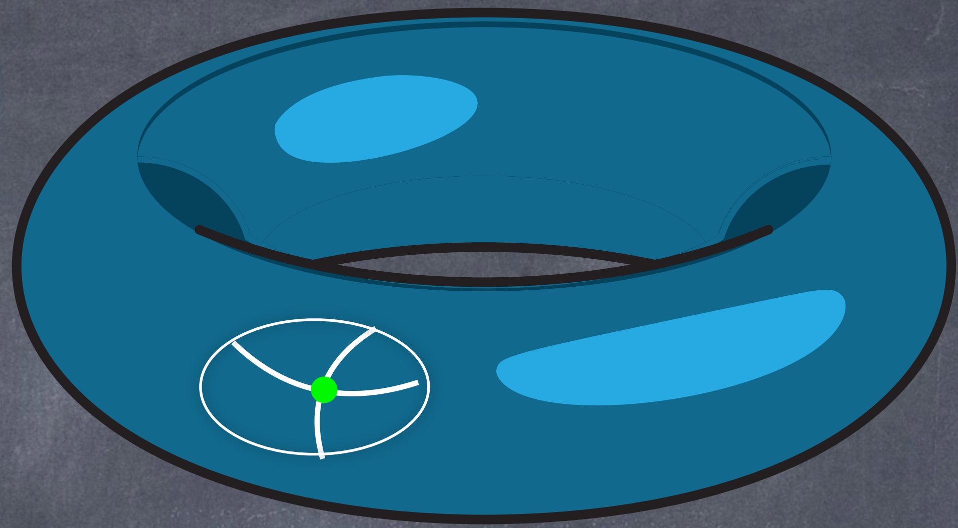

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X

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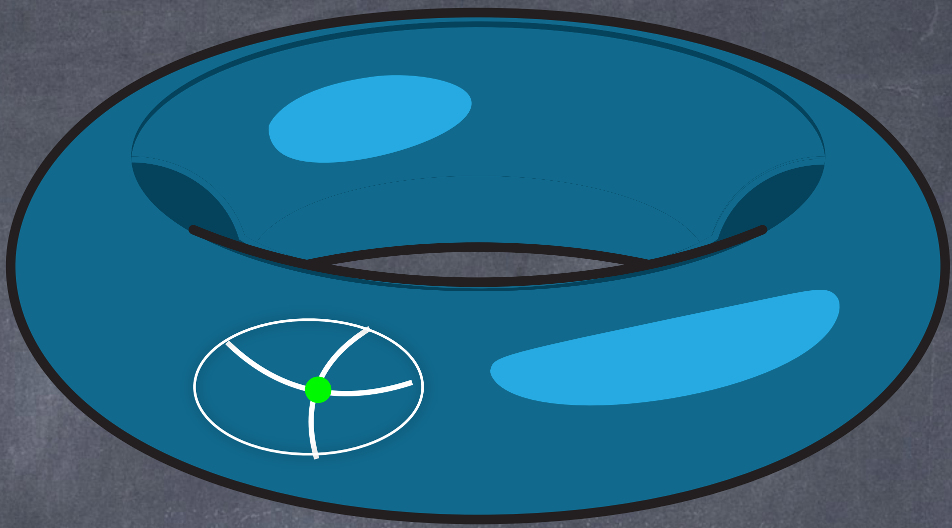
Thom's answer: In general, no!

Steenrod's question:



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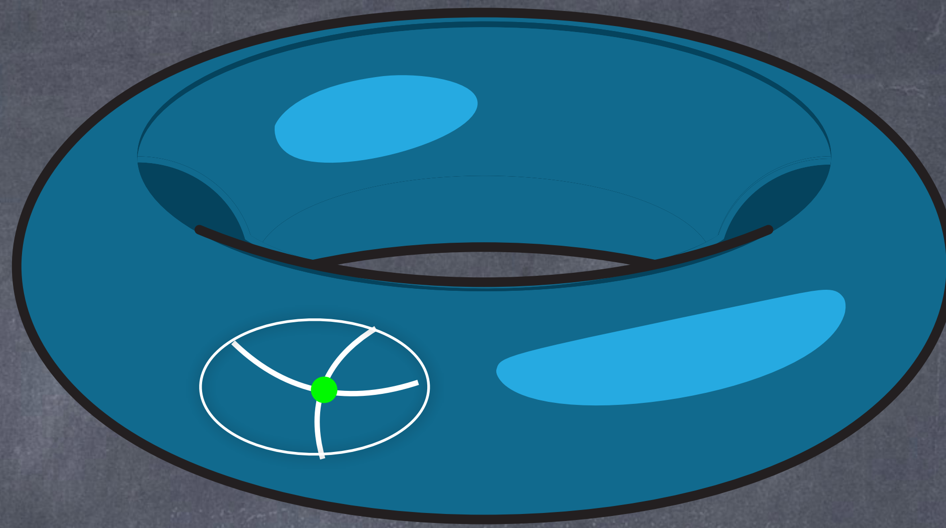

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With rational coefficients: Yes!

Modify the problem: Step 1

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- choose a nice X

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complex smooth algebraic variety X

set of solutions in some C^N of polynomial equations

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set of solutions in some $\mathbb{C}^N$ of polynomial equations

Examples:

Modify the problem: Step 1

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complex smooth algebraic variety X

set of solutions in some C^N of polynomial equations

Examples:

- $p(x,y,z) = 0$ in C^3 for $p(x,y,z) = x^n+y^n+z^n-1$.

Modify the problem: Step 1

- choose a nice X

complex smooth algebraic variety X

set of solutions in some C^N of polynomial equations

Examples:

- $p(x,y,z) = 0$ in C^3 for $p(x,y,z) = x^n+y^n+z^n-1$.
- $y^n-q(x) = 0$ in C^2, q without multiple roots, e.g. $q(x)=x^3+x+1$.

Abelian integrals $\int \frac{p(x)}{\sqrt[n]{q(x)}}dx$

Modify the problem: Step 2

- choose a nice X

complex smooth algebraic variety X

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projective

complex projective space

complex smooth algebraic variety $X \subset \mathbb{C}P^N$

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if V smooth fundamental class of V → $[V] \in H_{2n}(V;Z)$

Modify the problem: Step 2

- choose a nice X

projective

complex projective space

complex smooth algebraic variety $X \subset CP^N$

- consider an algebraic subset $V \overset{j}{\subset} X$

satisfy additional polynomial equations

if V smooth fundamental class of V

$[V] \in H_{2n}(V;Z) \mapsto j_{2n}[V] \in H_{2n}(X;Z)$

A new question:

$X \subset \mathbb{C}P^N$ smooth proj. algebraic

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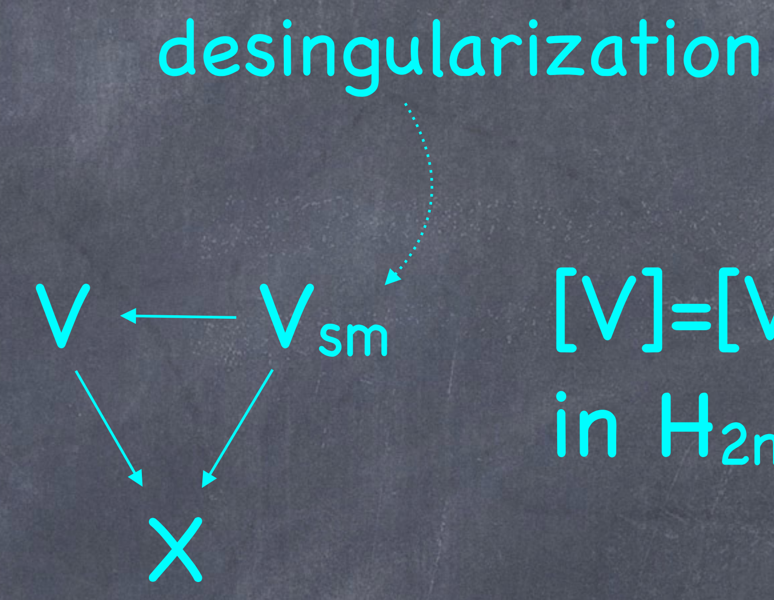

$[V]=[V_{sm}]$ in $H_{2n}(X;Z)$

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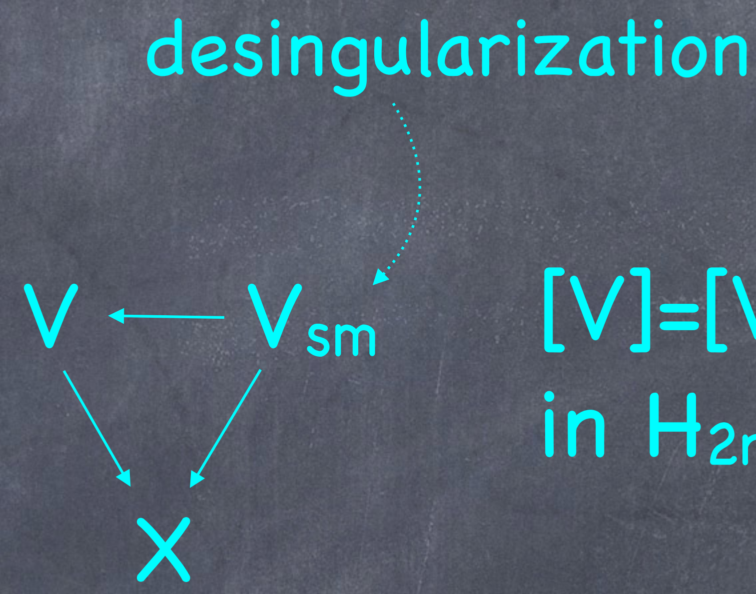

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A new question:

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Can every class in $H_{2n}(X;Z)$ be realized as the fundamental class of an algebraic subset $V \subset X$?

desingularization

$$\begin{array}{ccc} V & \leftarrow & V_{sm} \\ & \searrow \; \swarrow & \\ & X & \end{array}$$

$[V]=[V_{sm}]$ in $H_{2n}(X;Z)$

- Example:

curve $\qquad$ surface

$$V \overset{j}{\subset} X$$

A new question:

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z_1, z_2 local coordinates on X

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$V = \{z_2=0\}$

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Can every class in $H_{2n}(X;Z)$ be realized as the fundamental class of an algebraic subset $V \subset X$?

desingularization

$V \leftarrow V_{sm}$, $V \rightarrow X$, $V_{sm} \rightarrow X$

$[V]=[V_{sm}]$ in $H_{2n}(X;Z)$

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$$\int_V j^*\alpha$$

form on X

A new question: $X \subset CP^N$ smooth proj. algebraic

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desingularization

$$V \leftarrow V_{sm}, \quad V \to X, \quad V_{sm} \to X$$

$[V]=[V_{sm}]$ in $H_{2n}(X;Z)$

- Example:

curve $\to V \overset{j}{\subset} X \leftarrow$ surface

z_1, z_2 local coordinates on X

$V = \{z_2=0\}$

$\alpha = \Sigma\, g\, dz_1 \wedge dz_2$

$\alpha = \Sigma\, f\, dz_1 \wedge d\bar{z}_1$

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$$\int_V j^*\alpha$$

form on X (α)

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- Example:

curve → $V \overset{j}{\subset} X$ ← surface

z_1, z_2 local coordinates on X

$V = \{z_2 = 0\}$

$\alpha = \Sigma\, g\, dz_1 \wedge dz_2$

$\int_V j^*\alpha = 0$ ← unless → $\alpha = \Sigma\, f\, dz_1 \wedge d\bar{z}_1$

(α: form on X)

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$[V]=[V_{sm}]$ in $H_{2n}(X;Z)$

- Example:

curve

surface

z_1, z_2 local coordinates on X

$V \overset{j}{\subset} X$

$V = \{z_2=0\}$

~~$\alpha = \Sigma\, g\, dz_1 \wedge dz_2$~~

(Hodge) $\int_V j^*\alpha = 0$ unless $\alpha = \Sigma\, f\, dz_1 \wedge d\bar{z}_1$

form on X

~~$\alpha = \Sigma\, h\, d\bar{z}_1 \wedge d\bar{z}_2$~~

Hodge's question: $X \subset \mathbb{CP}^N$ smooth proj. algebraic

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With rational coefficients this is still an open question!

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Poincaré dual

α_{PD} in $H^2(X;Z)$

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$X \xrightarrow{\alpha_{PD}} CP^\infty$

$K(2;Z)$

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Poincaré dual

α_{PD} in $H^2(X;Z)$

$$X \xrightarrow{\alpha_{PD}} CP^k \subset CP^\infty$$

$K(2;Z)$

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Poincaré dual

α_{PD} in $H^2(X;Z)$

$$CP^{k-1} \downarrow \quad X \xrightarrow{\alpha_{PD}} CP^k \subset CP^\infty$$

$K(2;Z)$ (= CP^∞)

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given α in $H_{2d-2}(X;\mathbb{Z})$

Poincaré dual

α_{PD} in $H^2(X;\mathbb{Z})$

$$\begin{array}{ccccc} V & \longrightarrow & \mathbb{C}P^{k-1} & & \\ \downarrow & \text{pullback} & \downarrow & & \\ X & \xrightarrow{\ \alpha_{PD}\ } & \mathbb{C}P^k & \subset & \mathbb{C}P^\infty \end{array}$$

$K(2;\mathbb{Z})$

Atiyah-Hirzebruch-Totaro obstruction:

given

algebraic $V \subset X$

Atiyah-Hirzebruch-Totaro obstruction:

given

algebraic $V \subset X$ $\longmapsto$ $[V]_H \in H_{2n}(X;Z)$

fundamental class of V_{sm}

Atiyah-Hirzebruch-Totaro obstruction:

K-theory fundamental class ⇢ $[V]_K \in K_{2n}(X)$

given

algebraic $V \subset X$ ⟼ $[V]_H \in H_{2n}(X;Z)$

algebraic $V \subset X$ ⟼ $[V]_K$

Atiyah-Hirzebruch: $[V]_K \mapsto [V]_H$; $K_{2n}(X) \to H_{2n}(X;Z)$

fundamental class of V_{sm} ⇢ $[V]_H$

Atiyah-Hirzebruch-Totaro obstruction:

Thom/Quillen:
universal
fundamental class

$[V]_{MU} \in MU_{2n}(X)$

K-theory
fundamental
class

$[V]_K \in K_{2n}(X)$

Atiyah-
Hirzebruch

given
algebraic $V \subset X$

$[V]_H \in H_{2n}(X;Z)$

fundamental
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Atiyah-Hirzebruch-Totaro obstruction:

generators are manifolds $M \to X$

Thom/Quillen: universal fundamental class

$[V]_{MU} \in MU_{2n}(X)$

$\downarrow \quad \vdots \quad \downarrow$

K-theory fundamental class

$[V]_K \in K_{2n}(X)$

Atiyah-Hirzebruch $\downarrow$

given

algebraic $V \subset X \longmapsto [V]_H \in H_{2n}(X;Z)$

fundamental class of V_{sm}

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operations on the image

The Brown-Peterson tower: fix a prime p

p-local universal theory

Brown-Peterson spectra BP with

$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$.

$|v_i| = 2(p^i - 1)$

evaluation on point

Brown-Peterson
Quillen
Wilson
⋮

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For every n:

$$Z_{(p)}[v_1, v_2, ...] \longrightarrow Z_{(p)}[v_1, ..., v_n]$$

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"quotient map"

For every n: $BP \longrightarrow BP/(v_{n+1}, \ldots) =: BP\langle n \rangle$

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$$BP \longrightarrow \cdots \longrightarrow BP\langle n\rangle \longrightarrow \cdots \longrightarrow BP\langle 1\rangle \longrightarrow BP\langle 0\rangle \longrightarrow BP\langle -1\rangle$$

p-local connective K-theory

$HZ_{(p)} \longrightarrow HF_p$

Milnor operations:

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For every n:

stable cofibre sequence

$$\Sigma^{|v_n|} BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \longrightarrow BP\langle n-1\rangle \longrightarrow \Sigma^{|v_n|+1} BP\langle n\rangle$$

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with an induced exact sequence (for any space X)

$$BP\langle n\rangle^{*+|v_n|}(X) \longrightarrow BP\langle n\rangle^{*}(X) \longrightarrow BP\langle n-1\rangle^{*}(X) \xrightarrow{q_n} BP\langle n\rangle^{*+|v_n|+1}(X)$$

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$BP\langle n-1\rangle \xrightarrow{\text{Thom map}} HF_p$

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$BP\langle n-1\rangle \to HF_p$: Thom map

nth Milnor operation:

Q_0=Bockstein

$Q_n = P^{p^{n-1}}Q_{n-1} - Q_{n-1}P^{p^{n-1}}$

The LMT obstruction in action:

$$
\begin{array}{ccccc}
 & & BP^{2*}(X) & & \\
 & \swarrow \; ⟲ & \downarrow & & \\
BP\langle n\rangle^{2*}(X) & \longrightarrow & BP\langle n-1\rangle^{2*}(X) & \xrightarrow{q_n} & BP\langle n\rangle^{2*+|v_n|+1}(X) \\
 & & \downarrow & ⟳ & \downarrow \\
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Question: Is α algebraic?

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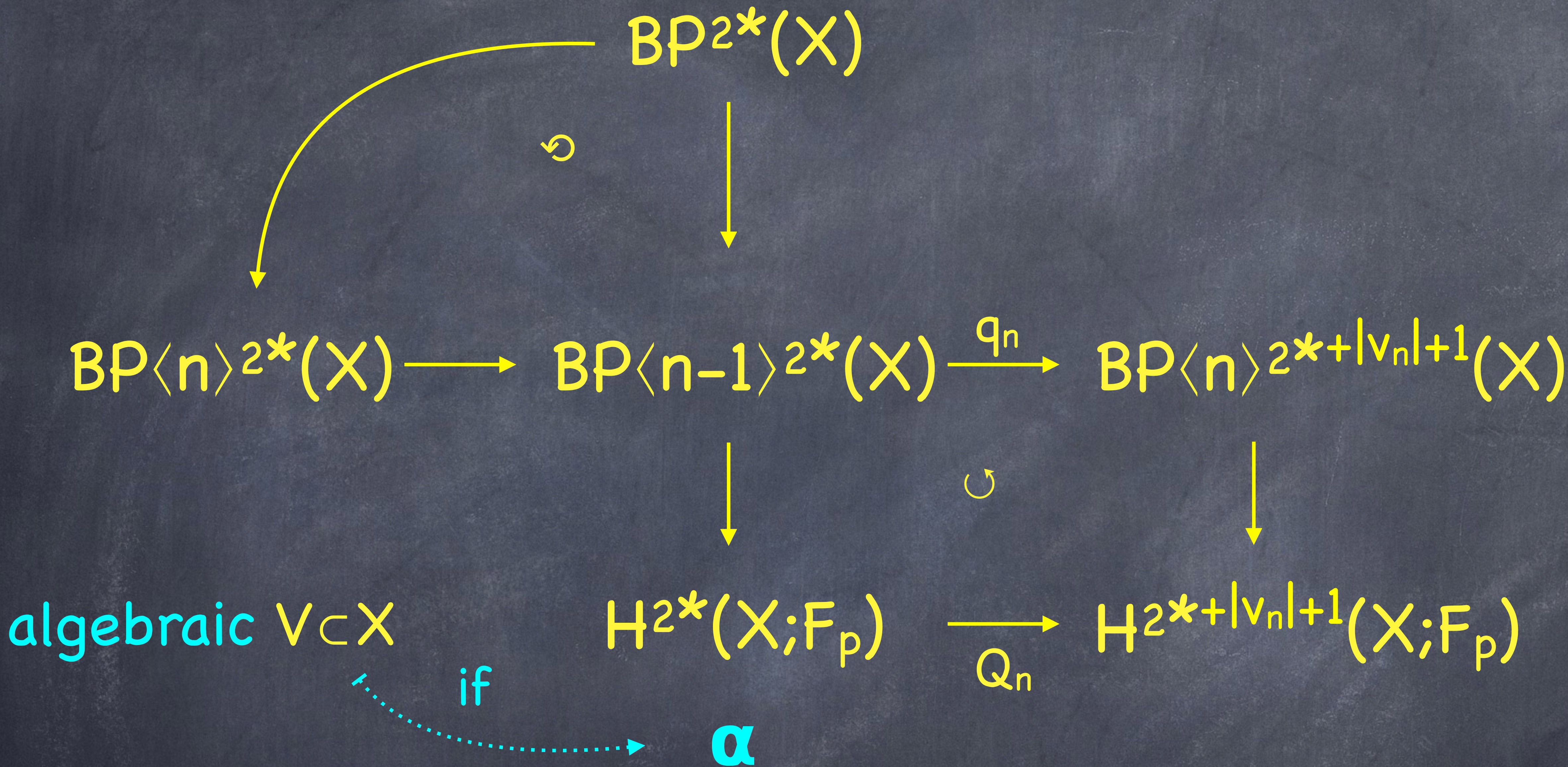

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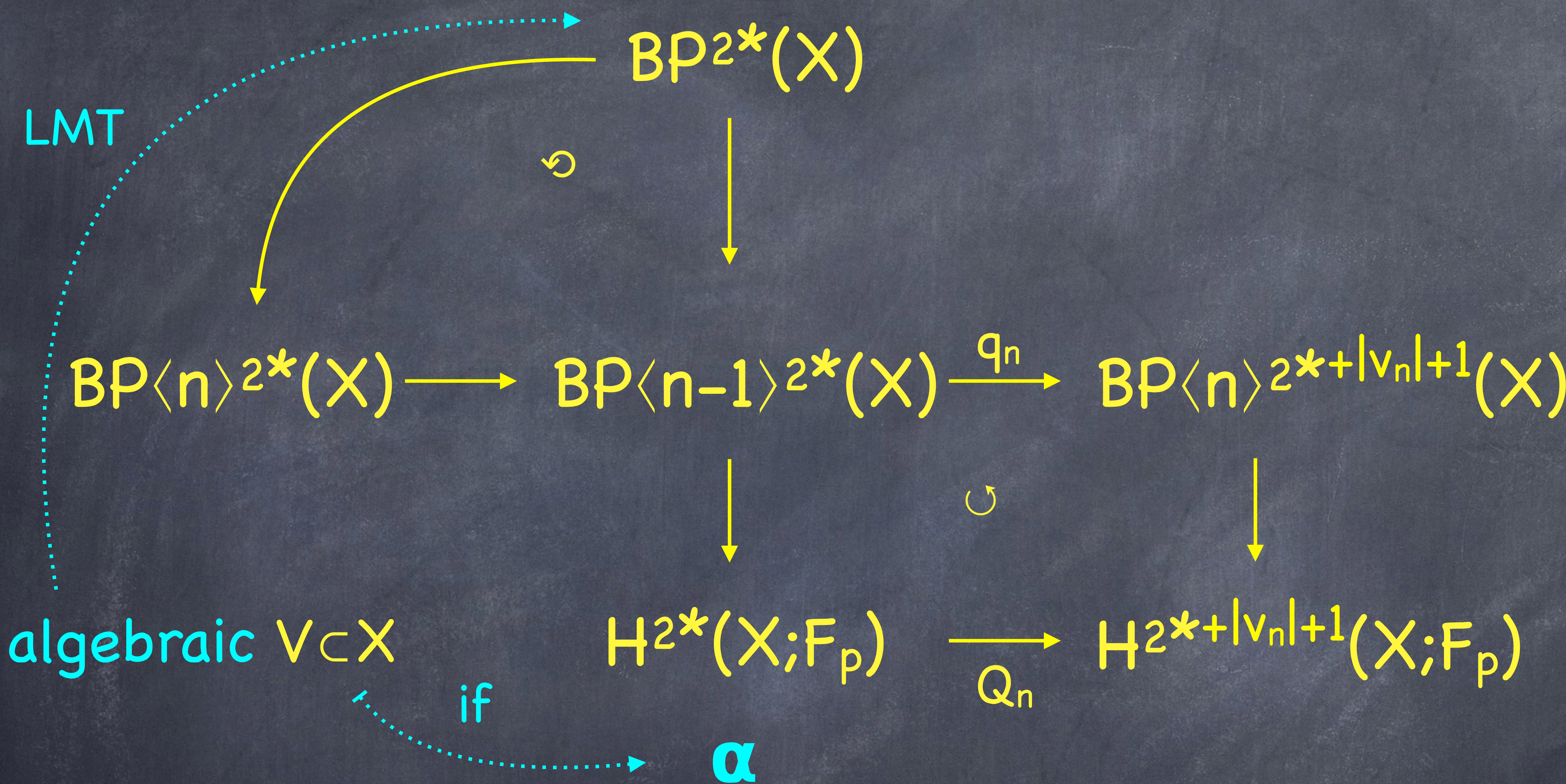

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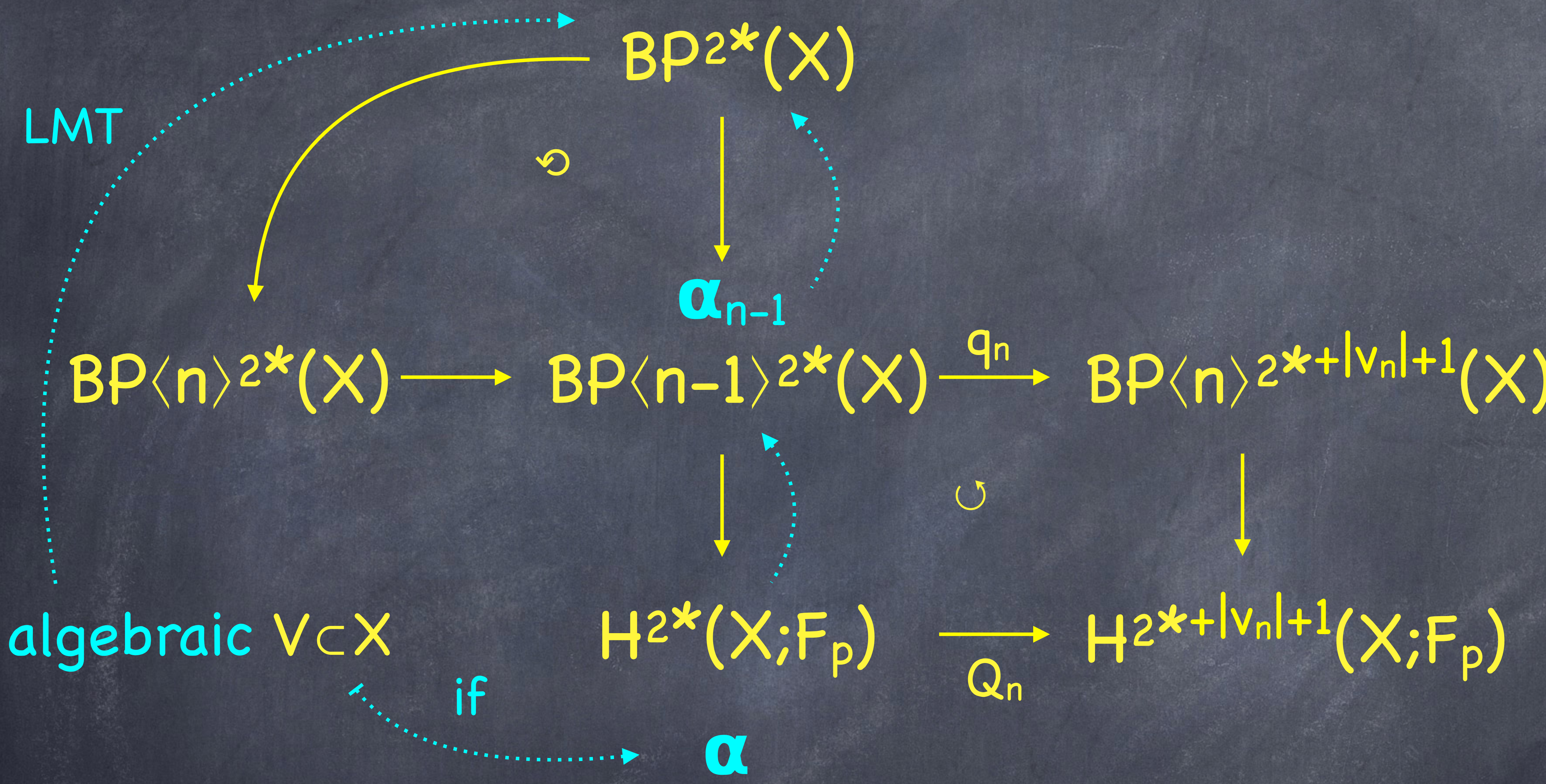

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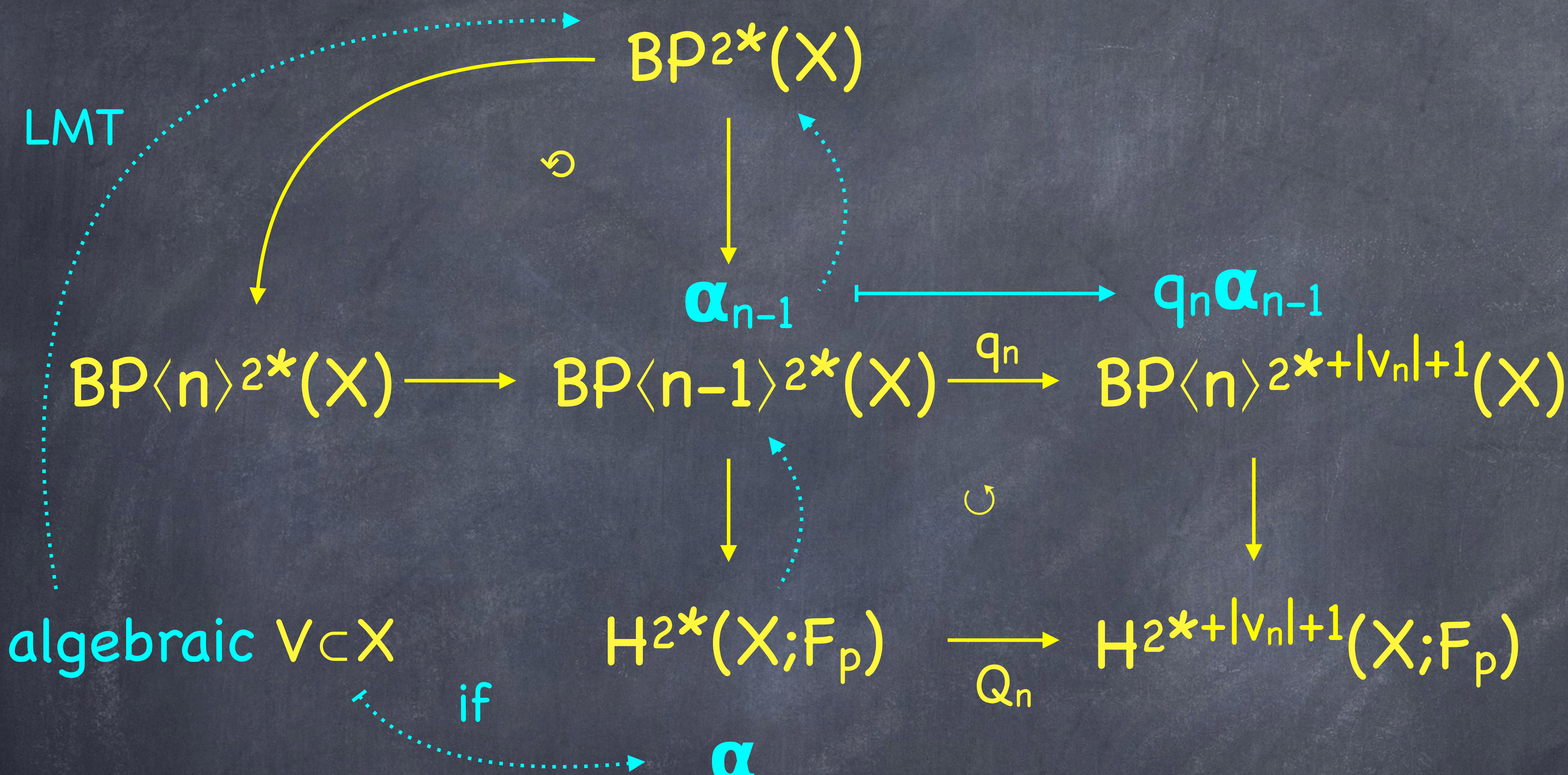

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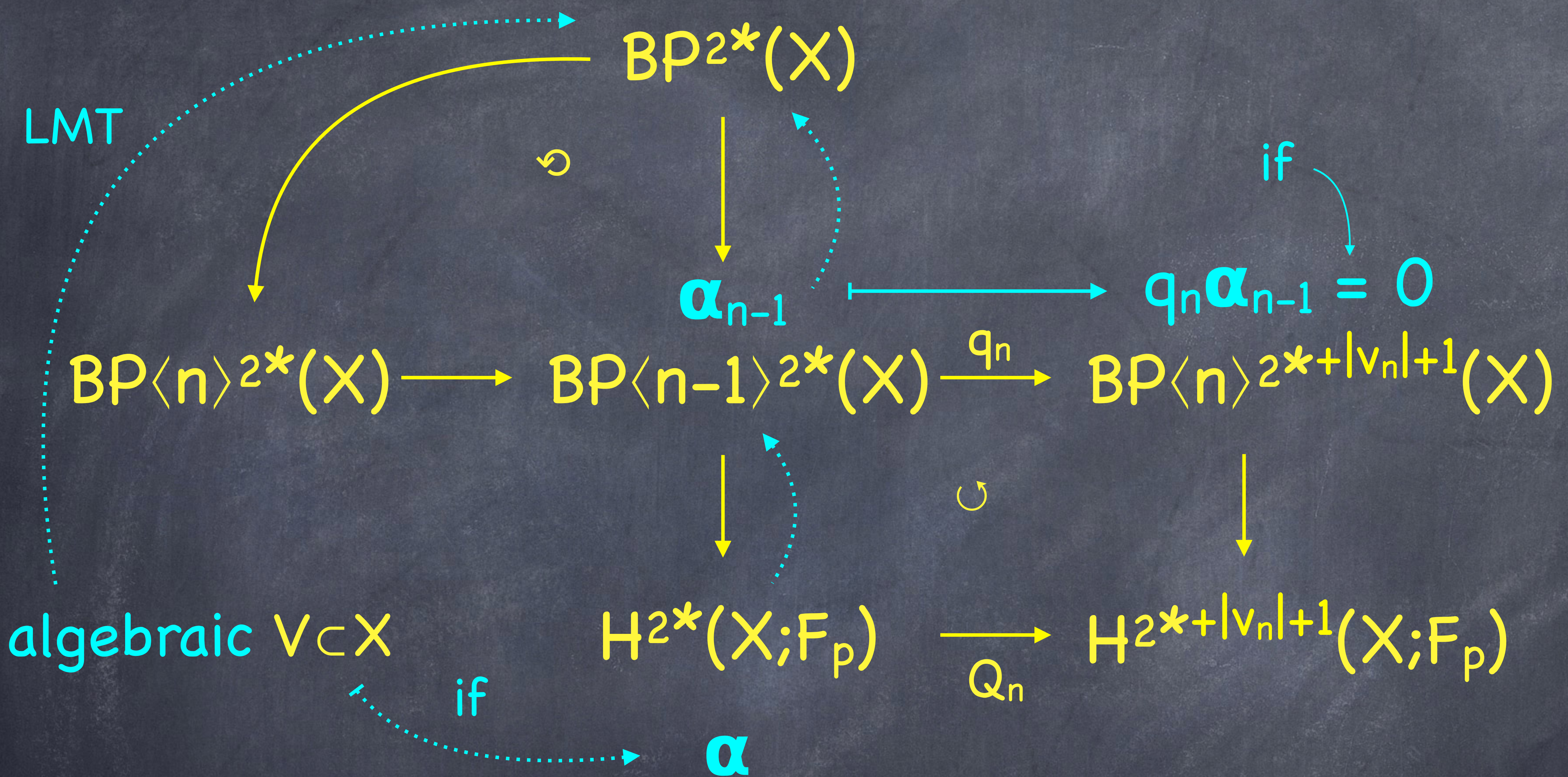

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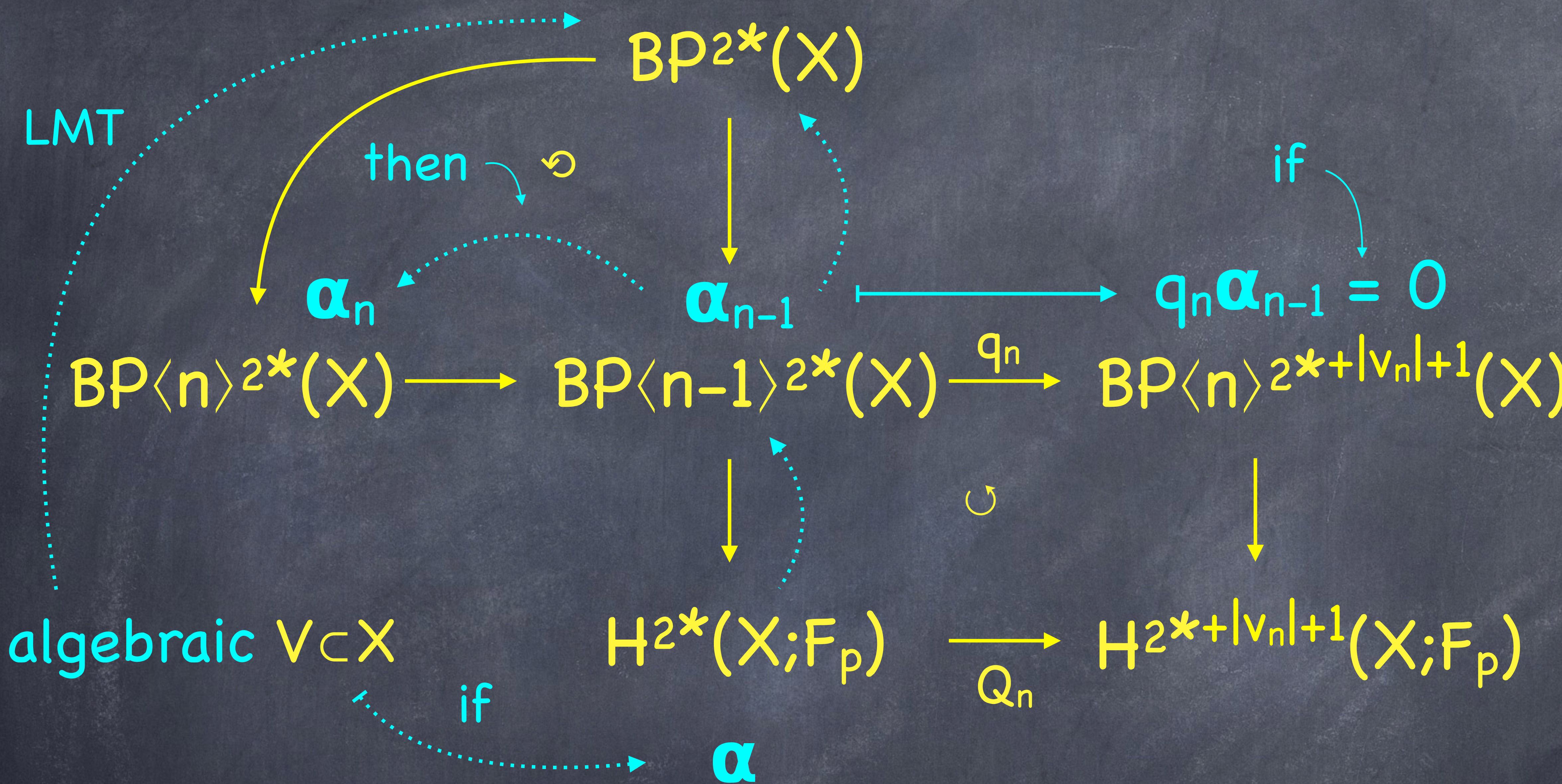

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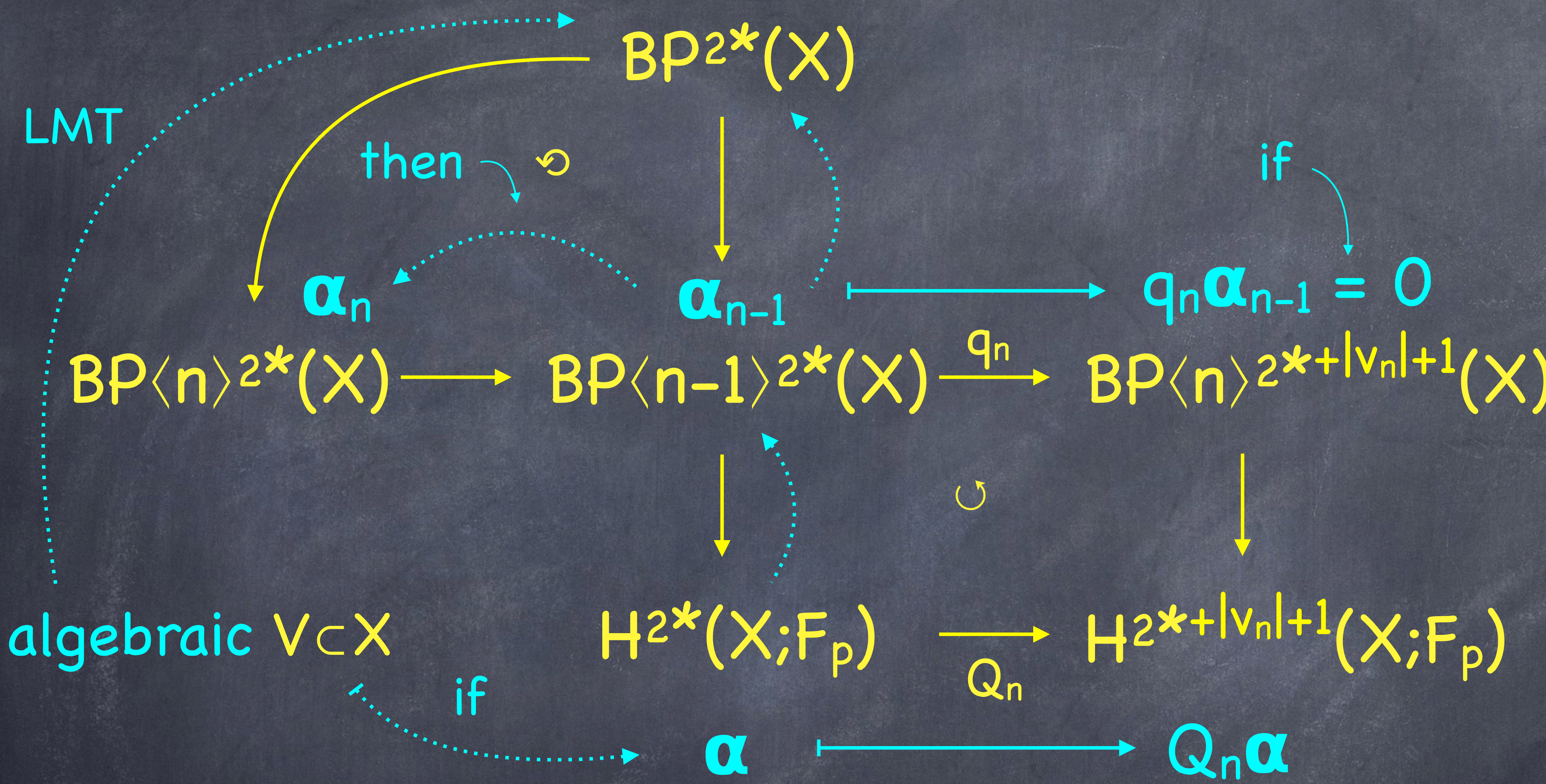

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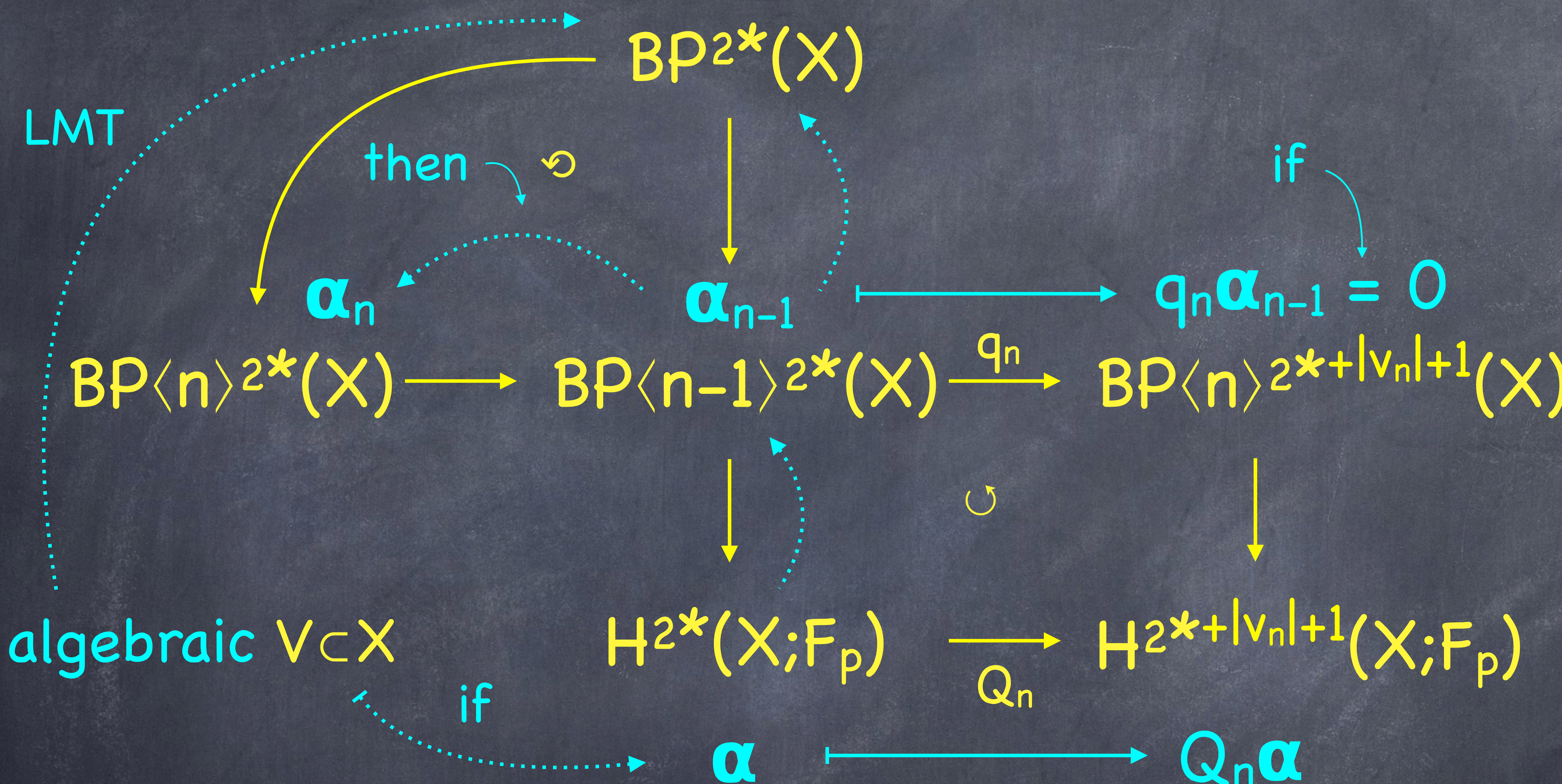

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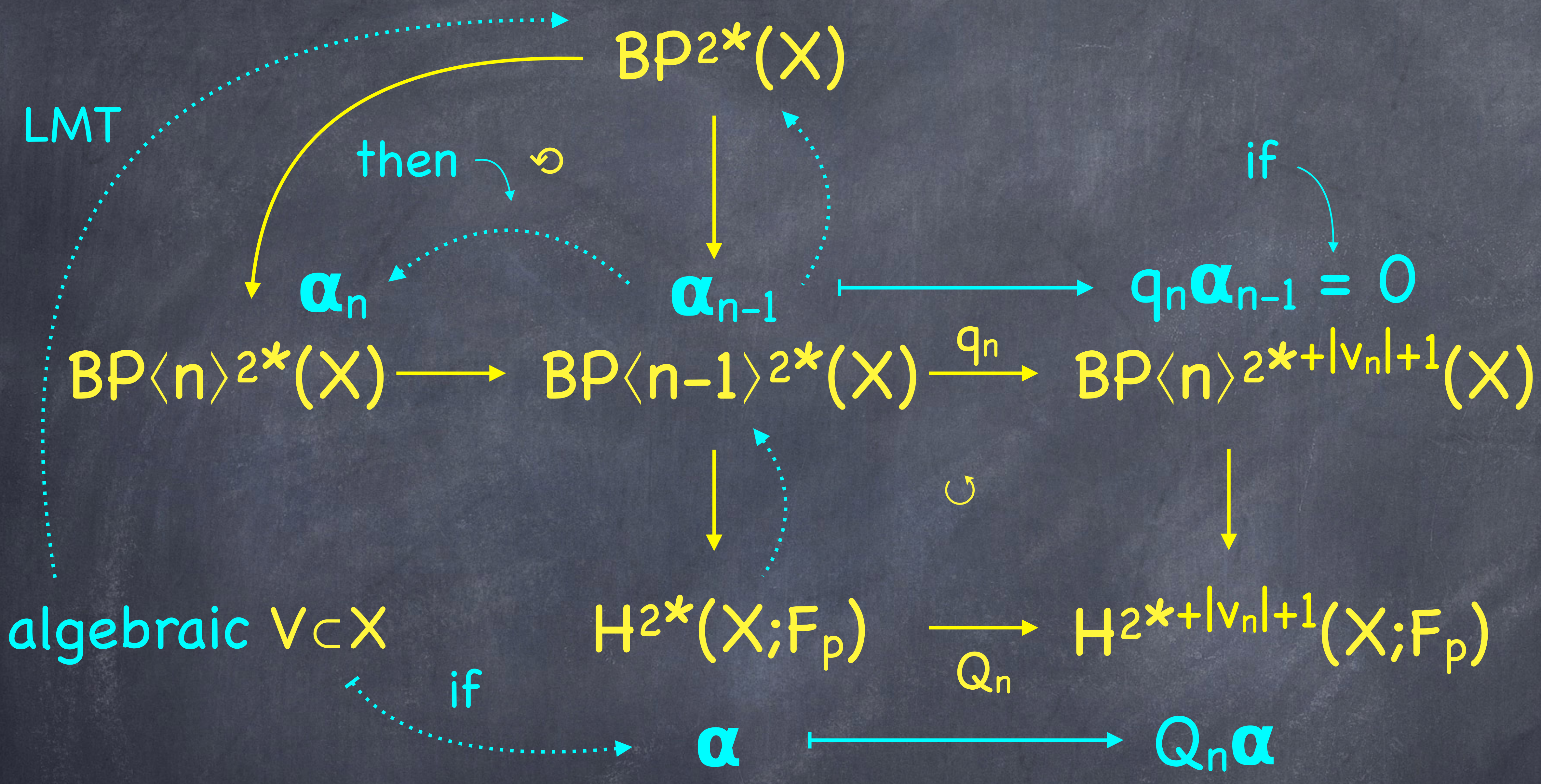

Levine-Morel-Totaro obstruction:

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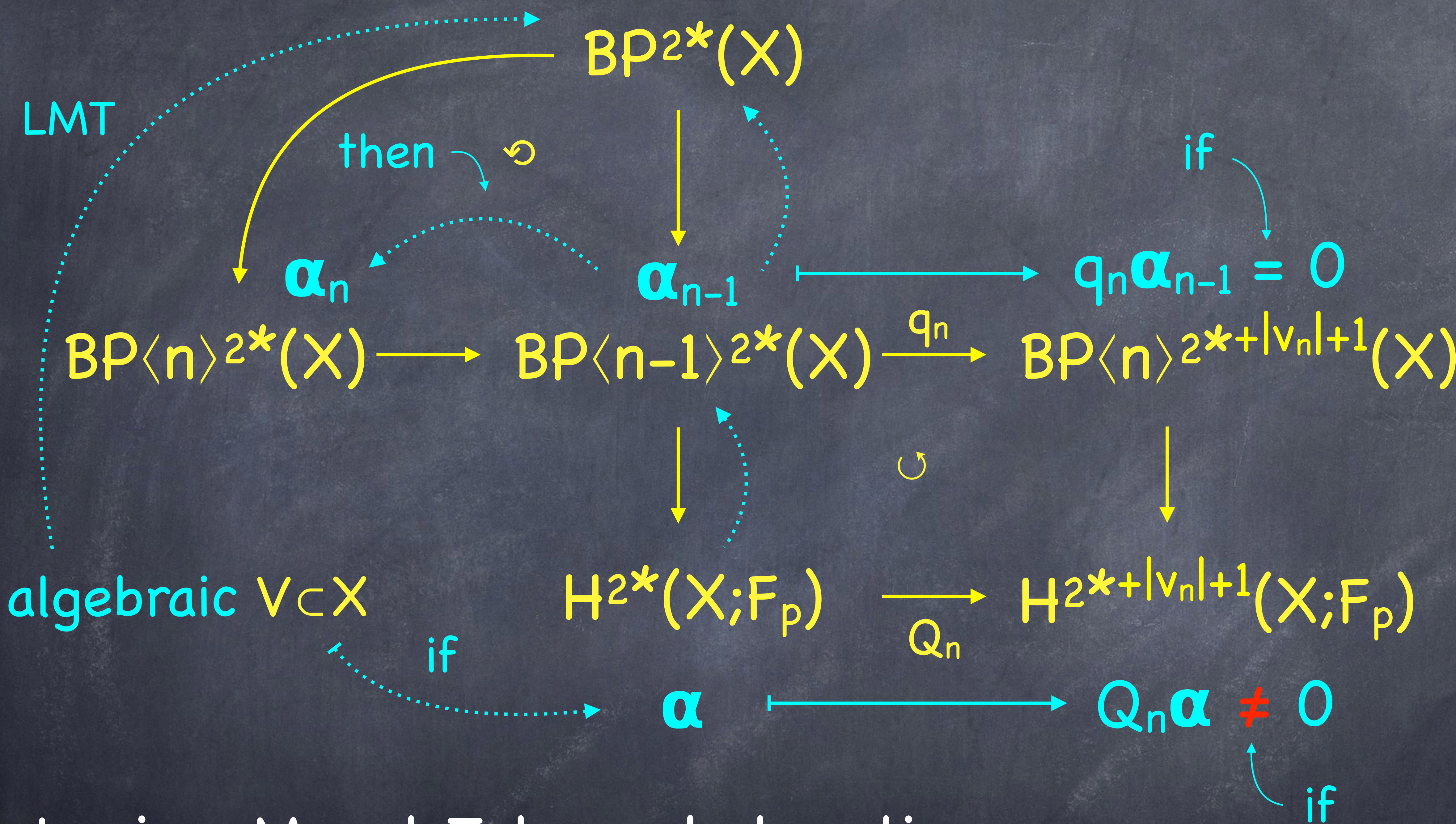

Levine-Morel-Totaro obstruction:

If $Q_n\alpha \neq 0$,

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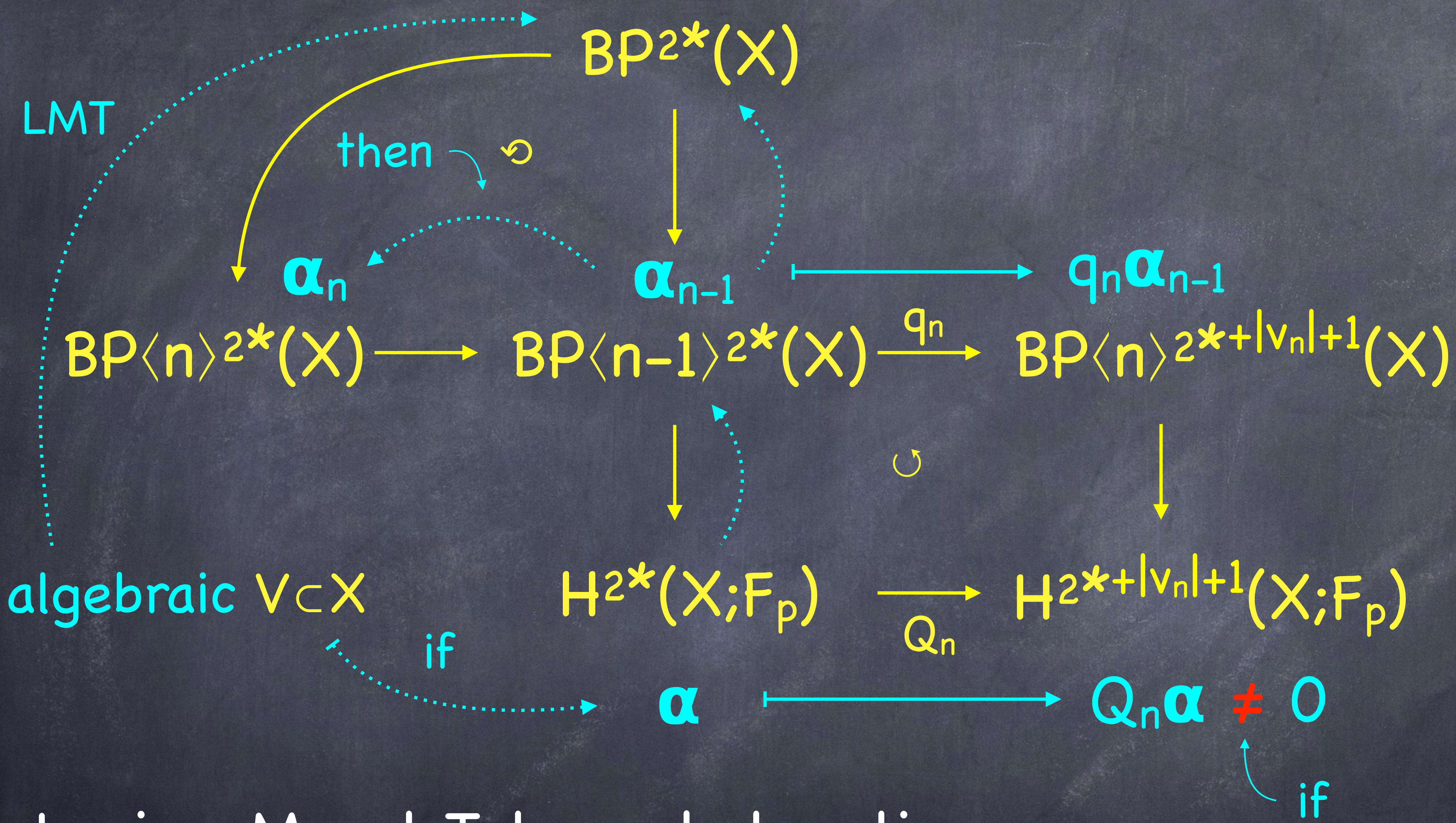

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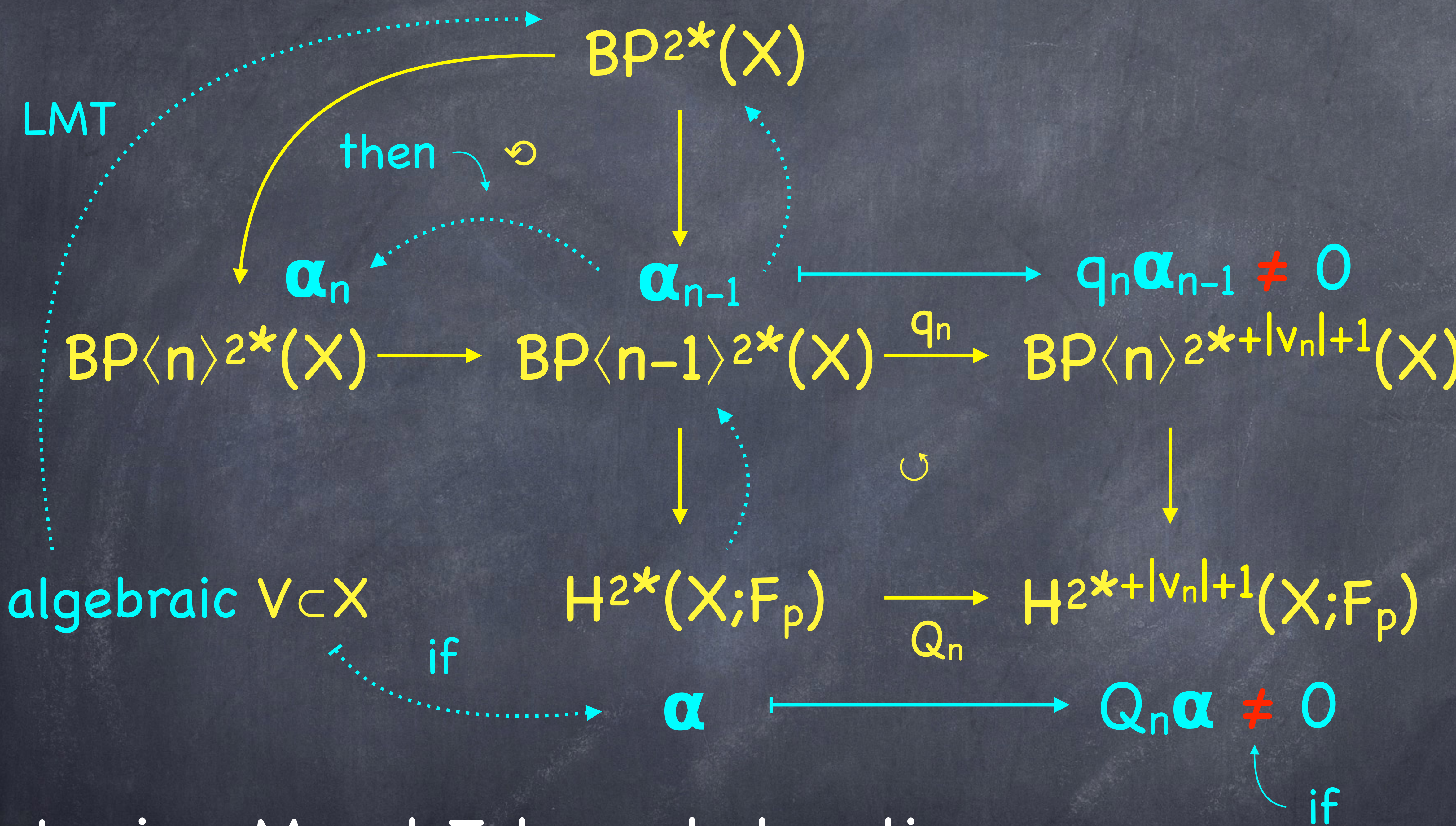

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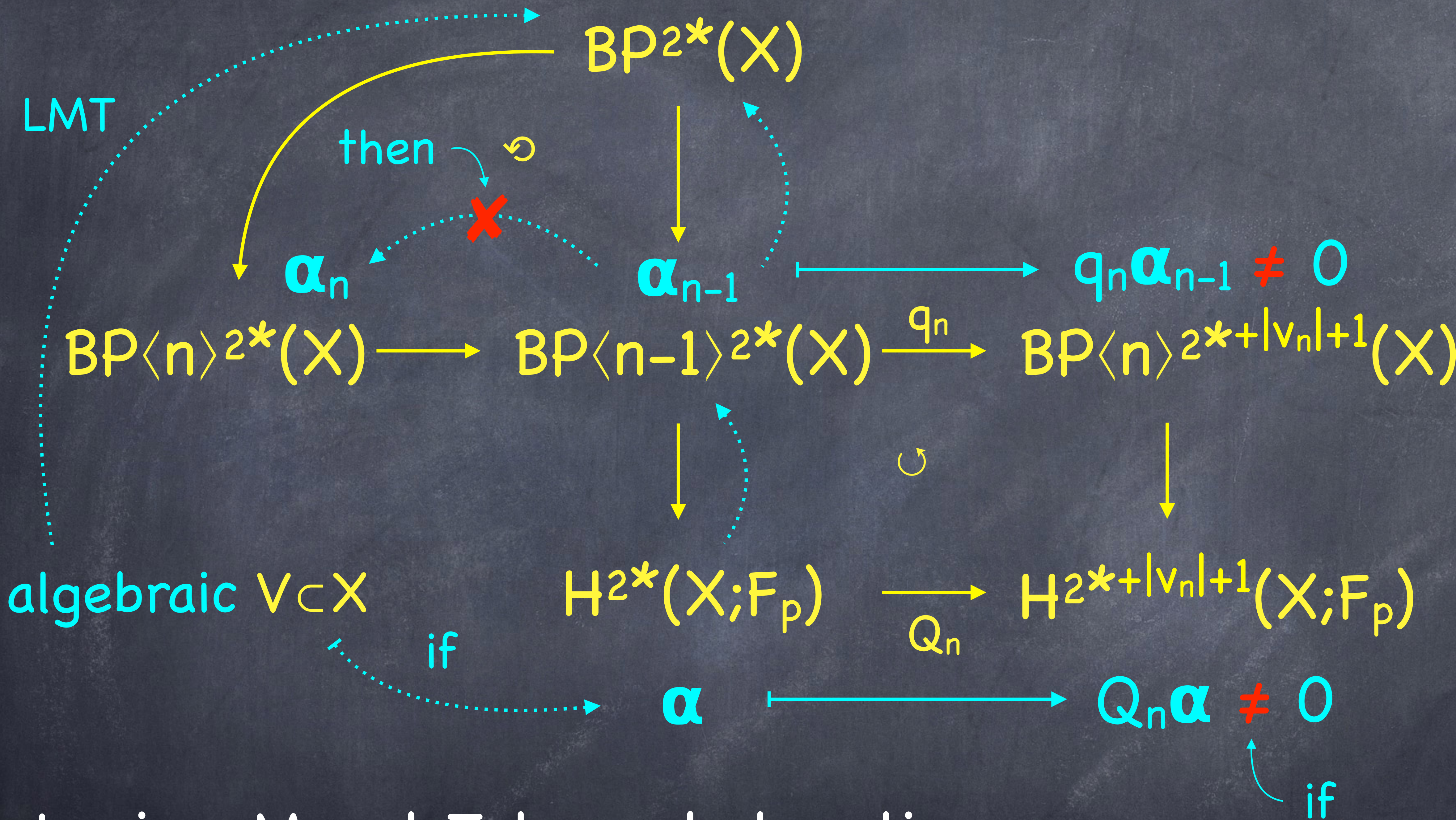

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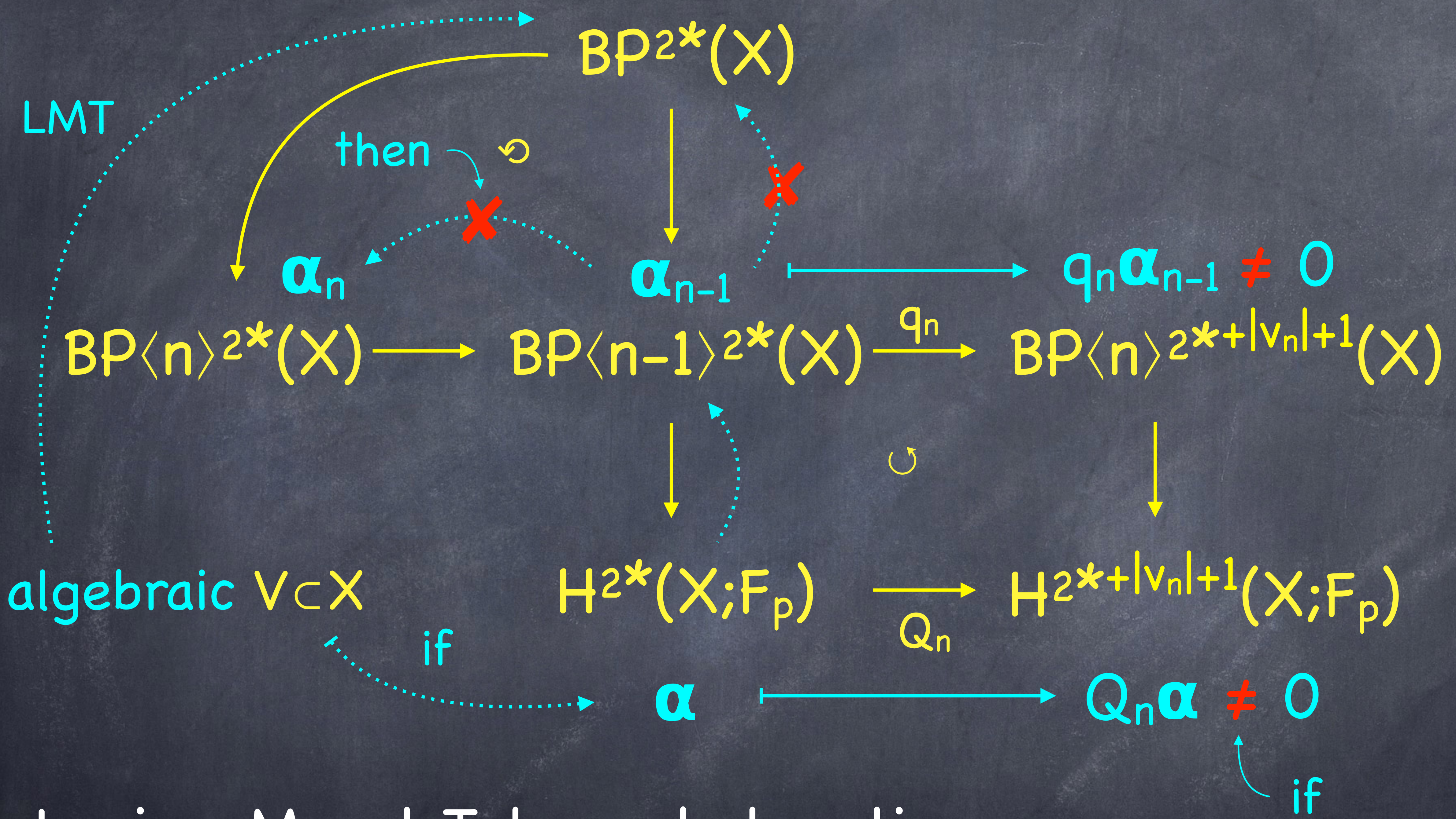

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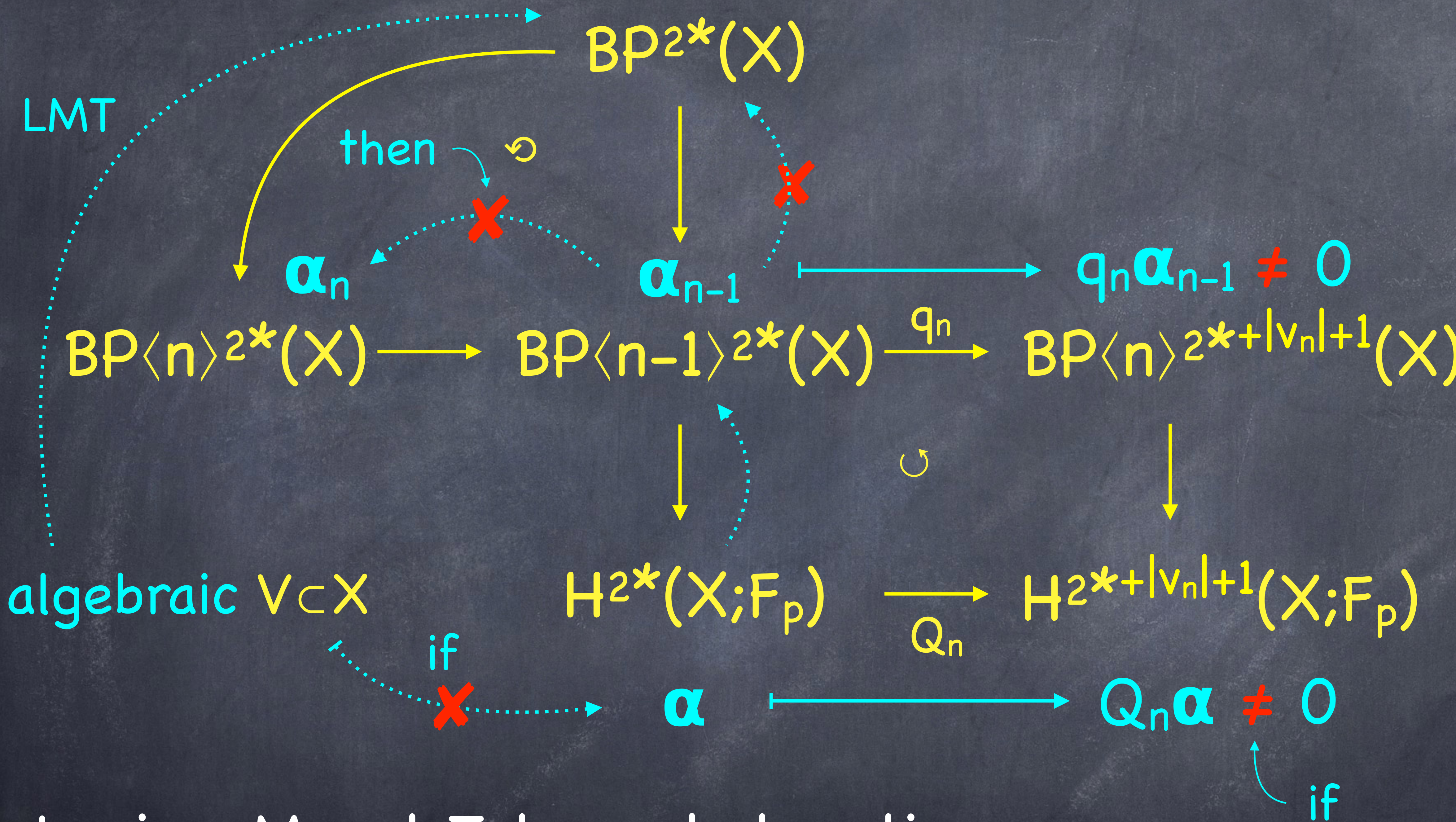

Levine-Morel-Totaro obstruction:

If $Q_n\alpha \neq 0$, then α is not algebraic.

Generalize the question:

Generalize the question:

so far

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so far

given

$\alpha \in H^{2*}(X;Z)$

Generalize the question:

so far — is there an algebraic? $V \subset X$ ↔ given $[V] = \alpha \in H^{2*}(X;\mathbb{Z})$

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so far — is there an algebraic? $V \subset X$ — $[V] = \alpha \in H^{2*}(X;\mathbb{Z})$ given

now

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now

$BP\langle n\rangle^{s}(X)$

Generalize the question:

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motivic/algebraic spectrum

$BP\langle n\rangle^{s,r}_{mot}(X)$ — $BP\langle n\rangle^{s}(X)$

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Voevodsky
Morel
Vezzosi
Hopkins
Hu-Kriz
Ormsby
Hoyois
Ormsby-Østvær
⋮

Generalize the question:

so far — is there an algebraic? $V \subset X$ — $[V] = \alpha \in H^{2*}(X;\mathbb{Z})$ given

now — topological realization

$$Sm_{\mathbb{C}} \longrightarrow Man_{\mathbb{C}}$$

induced hom.

motivic/algebraic spectrum — top. real.

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Question: How can we produce non-algebraic elements in $BP\langle n\rangle^{2*}(X)$?

We can lift...

$H^k(X;F_p)$

We can lift...

$$
\begin{array}{c}
H^{k+1}(X;\mathbb{Z}_{(p)}) \\
q_0 \uparrow \\
H^k(X;\mathbb{F}_p)
\end{array}
$$

We can lift...

$$BP\langle 1\rangle^{k+1+2p-1}(X)$$

$$\uparrow q_1$$

$$H^{k+1}(X;Z_{(p)})$$

$$\uparrow q_0$$

$$H^k(X;F_p)$$

We can lift...

$$BP\langle n\rangle^{k+|Q_0|+\ldots+|Q_n|}(X)$$

$$q_n \uparrow$$

$$\vdots$$

$$q_2 \uparrow$$

$$BP\langle 1\rangle^{k+1+2p-1}(X)$$

$$q_1 \uparrow$$

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$$\uparrow q_n$$

$$\vdots$$

$$\uparrow q_2$$

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$$H^{k+1}(X;Z_{(p)})$$

$$\uparrow q_0$$

$$H^k(X;F_p)$$

We can lift...

$$BP\langle n\rangle^{k+|Q_0|+\dots+|Q_n|}(X) \xrightarrow{q_{n+1}} BP\langle n+1\rangle^{k+|Q_0|+\dots+|Q_{n+1}|}(X)$$

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$$\vdots$$

$$\uparrow q_2$$

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$$H^k(X;F_p)$$

$$BP\langle n+1\rangle^{k+|Q_0|+\dots+|Q_{n+1}|}(X) \longrightarrow H^{k+|Q_0|+\dots+|Q_{n+1}|}(X;F_p)$$

$$BP\langle n+1\rangle \xrightarrow{\text{Thom map}} HF_p$$

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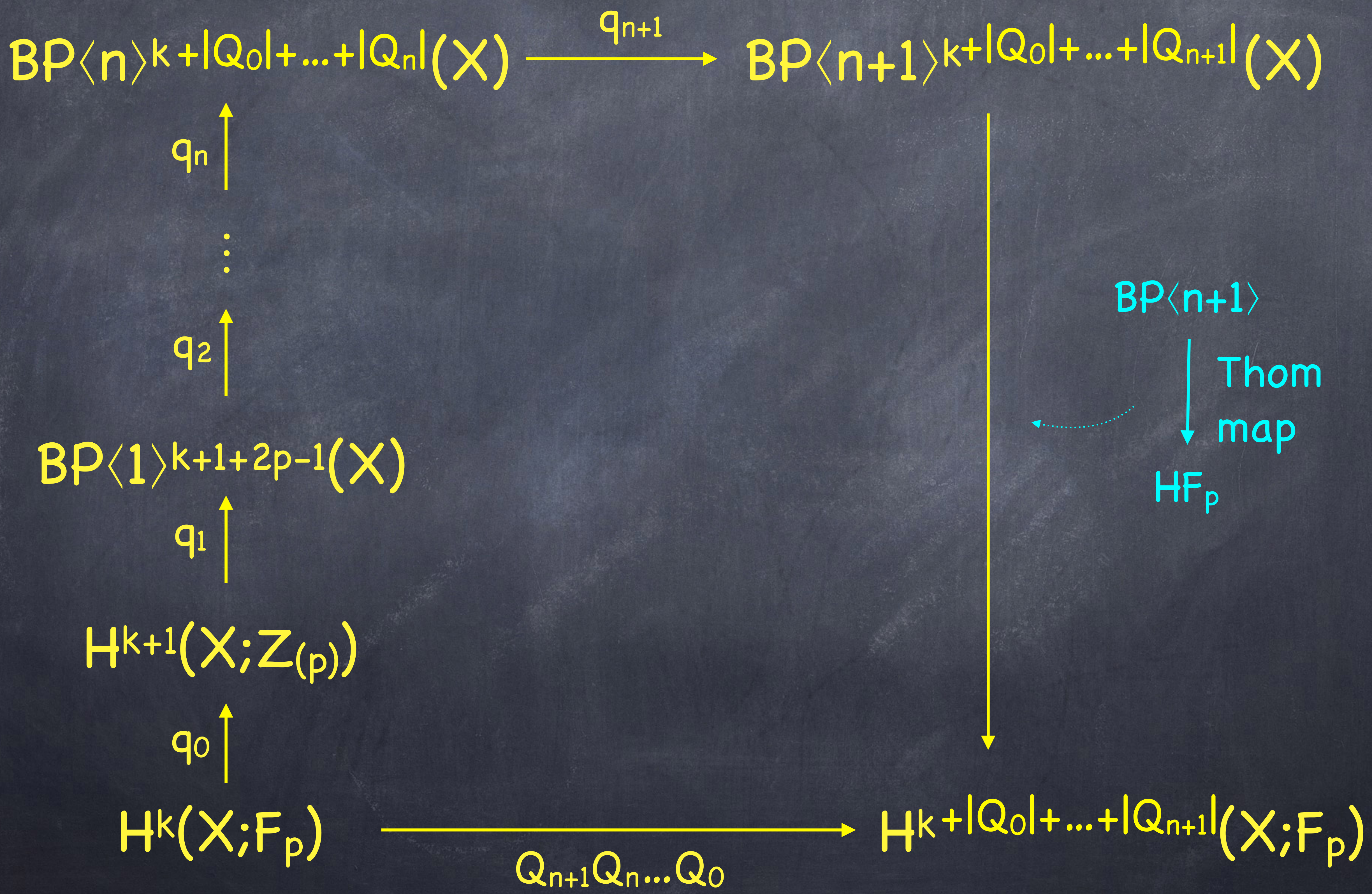

$BP\langle n\rangle^{k+|Q_0|+\dots+|Q_n|}(X)$
q_{n+1}
$BP\langle n+1\rangle^{k+|Q_0|+\dots+|Q_{n+1}|}(X)$
q_n
q_2
$BP\langle 1\rangle^{k+1+2p-1}(X)$
q_1
$H^{k+1}(X;Z_{(p)})$
q_0
$H^k(X;F_p)$
$Q_{n+1}Q_n\dots Q_0$
$H^{k+|Q_0|+\dots+|Q_{n+1}|}(X;F_p)$
$BP\langle n+1\rangle$
Thom map
HF_p

We can lift...

but we pay a price...

$$BP\langle n\rangle^{k+|Q_0|+\dots+|Q_n|}(X) \xrightarrow{q_{n+1}} BP\langle n+1\rangle^{k+|Q_0|+\dots+|Q_{n+1}|}(X)$$

$$H^k(X;F_p) \xrightarrow{q_0} H^{k+1}(X;Z_{(p)}) \xrightarrow{q_1} BP\langle 1\rangle^{k+1+2p-1}(X) \xrightarrow{q_2} \cdots \xrightarrow{q_n} BP\langle n\rangle^{k+|Q_0|+\dots+|Q_n|}(X)$$

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Examples of non-algebraic classes:

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Theorem (Q.): For every n, there is a smooth projective complex algebraic variety X and a class in

$$BP\langle n\rangle^{2(p^n+\dots+1)+2}(X)$$

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We know: • $H^*(BG_k;F_p) = F_p[y_1,\dots,y_k]\otimes\Lambda(x_1,\dots,x_k)$;

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- $H^*(BG_k;F_p) = F_p[y_1,\ldots,y_k]\otimes\Lambda(x_1,\ldots,x_k)$; with $|y_i|=2$, $|x_i|=1$
- $Q_j(x_i)=y_i^{p^j}$, $Q_j(y_i)=0$.

Proof continued: $H^*(BG_k;F_p)= F_p[y_1,\ldots,y_k]\otimes\Lambda(x_1,\ldots,x_k)$;

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Finally, set X = Godeaux-Serre variety associated to the group G_{n+3} and pullback x via

$$X \longrightarrow BG_{n+3} \times CP^\infty.$$

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Thank you!