Geometric Hodge filtered complex cobordism

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Euler, Abel and Jacobi:

We would like to evaluate the integral

$$I(t) = \int_0^t 1/\sqrt{f(x)}dx.$$

polynomial of degree 3
with simple roots

Reformulation: $E : y^2 = f(x)$

point and path in space $E(\mathbb{C})$ of complex solutions

$\omega = dx/y$

$$I(p) = \int_0^p \omega$$

depends on choice of homotopy class of paths

$$E(\mathbb{C}) \overset{?}{\dashrightarrow} \mathbb{C}$$

$$E(\mathbb{C}) \leftarrow \widetilde{E(\mathbb{C})} \rightarrow \mathbb{C}$$

universal covering

Euler: group homomorphism

The Jacobian and the Abel-Jacobi map:

$\approx \mathbb{S}^1 \times \mathbb{S}^1$

$$H_1(E(\mathbb{C}); \mathbb{Z}) \approx \mathbb{Z} \times \mathbb{Z}$$

basis: loops γ_1, γ_2

periods

$$\lambda_1 := \int_{\gamma_1} \omega$$

$$\lambda_2 := \int_{\gamma_2} \omega$$

isomorphism

$$E(\mathbb{C}) \longrightarrow \mathbb{C}/(\mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2) \approx \mathrm{Jac}(E)$$

$$p \longmapsto \int_0^p \omega$$

Abel-Jacobi map

Lefschetz's theorem on (1,1)-classes:

X compact Kähler manifold

$$\left\{\begin{array}{l}\text{isom. classes of}\\ \text{holomorphic}\\ \text{line bundles on } X\end{array}\right\} \xrightarrow{\boxed{\text{surjective}}} H^2(X;\mathbb{Z}) \cap H^{1,1}(X;\mathbb{C})$$

$$L \longmapsto c_1(L)$$

Lefschetz's proof: $X \subset \mathbb{CP}^N$ surface

$\mathscr{C}$ family of curves on X with common point p_0

$\downarrow$

$\mathbb{CP}^1$

"normal function" ν_D

$\mathrm{Jac} = \cup_t \mathrm{Jac}(C_t)$

$\downarrow \pi$

$\mathbb{CP}^1$

$H^0(C_t; \Omega^1_{\mathrm{hol}})^\vee / H_1(C_t; \mathbb{Z})$

line bundle on X $\leftrightarrow$ D curve on X

$p_1(t), \ldots, p_d(t)$ intersection points with C_t

$D \longmapsto \nu_D(t) \in \mathrm{Jac}(C_t)$

$$\omega \mapsto \sum_i \int_{p_0}^{p_i(t)} \omega$$

Abel-Jacobi map

Lefschetz's proof continued: $X \subset \mathbb{CP}^N$ surface

$\mathscr{C}$ family of curves on X with common point p_0

$\mathscr{C} \to \mathbb{CP}^1$

"normal function" ν_D

$\mathrm{Jac} = \cup_t \mathrm{Jac}(C_t)$

$\pi: \mathrm{Jac} \to \mathbb{CP}^1$ holomorphic

$H^0(C_t; \Omega^1_{\mathrm{hol}})^\vee / H_1(C_t; \mathbb{Z})$

D curve on X $\mapsto$ $\nu_D(t) \in \mathrm{Jac}(C_t)$

Abel-Jacobi map

Poincaré's Existence Theorem:

For every normal function $\nu = \nu_D$ for some curve D

$\eta(\nu) \in H^2(X; \mathbb{Z})$ of Hodge type (1,1) such that $\eta(\nu_D) = [D]$ Poincaré dual of fundamental class

Every class of Hodge type (1,1) arises as $\eta(\nu)$.

Griffiths' Jacobian: X compact Kähler manifold

complex submanifold of codim p: $Z \subset X$

$[Z]_{MU}$ in $MU^{2p}(X)$

fundamental class in complex cobordism

Thom morphism

not always surjective

if $[Z] = 0$

Abel-Jacobi invariant: $AJ(Z) \longrightarrow [Z]_{\mathscr{D}} \longrightarrow [Z]$

Poincaré dual of fundamental class

$$0 \longrightarrow \mathrm{Jac}^{2p-1}(X) \longrightarrow H^{2p}_{\mathscr{D}}(X;\mathbb{Z}(p)) \longrightarrow \mathrm{Hdg}^{2p}(X;\mathbb{Z}) \longrightarrow 0$$

Griffiths' intermediate Jacobian

Deligne cohomology

integral Hodge classes

Deligne cohomology: integer $p \geq 0$ complex manifold X

Deligne complex

locally constant sheaf

complexes of sheaves on X

$$
\begin{array}{ccc}
\mathbb{Z}_{\mathscr{D}}(p) & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \cdot (2\pi i)^p \\
F^p \mathscr{A}^\star & \longrightarrow & \mathscr{A}^\star
\end{array}
$$

homotopy cartesian

at least p many dz's in local coord.

exact sequence

sheaf of smooth forms with complex coeff.

Deligne cohomology of X

hyper-cohomology

$$H^n_{\mathscr{D}}(X; \mathbb{Z}(p)) := \mathbb{H}^n(X; \mathbb{Z}_{\mathscr{D}}(p))$$

Hodge filtered complex cobordism: jt w/ M. Hopkins

$$\begin{array}{ccccc} MU_{\mathscr{D}}(p) & \longrightarrow & MU & \longrightarrow & MU \wedge H\mathbb{C} \\ \downarrow & & \downarrow ? & & \downarrow \simeq \\ H(F^p\mathscr{A}^\star(V_\star)) & \longrightarrow & H(\mathscr{A}^\star(V_\star)) & \longleftarrow & H(V_\star) \end{array}$$

define as homotopy pullback

constant presheaf

$\cdot\,(2\pi i)^{p+n}$ on $\pi_{2n}(MU)\otimes\mathbb{C}$

"fundamental cocycle"

integer p

homotopy cartesian

forms with coeff. $V_\star = MU_\star \otimes_{\mathbb{Z}} \mathbb{C}$

$= \oplus_j F^{p+j}\mathscr{A}^\star(V_{2j})$

presheaves of spectra on $\mathbf{Man}_{\mathbb{C}}$

Eilenberg-MacLane spectrum functor

$$MU^n_{\mathscr{D}}(p)(X) := \mathrm{Hom}_{\mathrm{hosPre}}(\Sigma^\infty(X_+), \Sigma^n MU_{\mathscr{D}}(p))$$

complex manifold

stable homotopy category of simplicial presheaves

Hodge filtered complex cobordism:

$$\begin{array}{ccc} MU_{\mathscr{D}}(p) & \longrightarrow & MU \\ \downarrow & \lrcorner h & \downarrow \\ H(F^p \mathscr{A}^\star(V_\star)) & \longrightarrow & H(\mathscr{A}^\star(V_\star)) \end{array}$$

$$MU^n_{\mathscr{D}}(p)(X) := \mathrm{Hom}_{\mathrm{hosPre}}(\Sigma^\infty(X_+), \Sigma^n MU_{\mathscr{D}}(p))$$

- pullbacks along holomorphic maps ✓
- multiplicative structure ✓
- long exact sequence ✓

A short exact sequence:

$$\begin{array}{ccc} MU_{\mathscr{D}}(p) & \longrightarrow & MU \\ \downarrow & \lrcorner h & \downarrow \\ H(F^p \mathscr{A}^{\star}(V_{\star})) & \longrightarrow & H(\mathscr{A}^{\star}(V_{\star})) \end{array}$$

X compact Kähler manifold

$$MU^n_{\mathscr{D}}(p)(X) := \mathrm{Hom}_{\mathrm{hosPre}}(\Sigma^{\infty}(X_+), \Sigma^n MU_{\mathscr{D}}(p))$$

"MU-Jacobian"
$\approx MU^{2p-1}(X) \otimes \mathbb{R}/\mathbb{Z}$
as a real Lie group

"MU-Hodge classes"

Thom morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Jac}^{2p-1}_{MU}(X) & \longrightarrow & MU^{2p}_{\mathscr{D}}(p)(X) & \longrightarrow & \mathrm{Hdg}^{2p}_{MU}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Jac}^{2p-1}(X) & \longrightarrow & H^{2p}_{\mathscr{D}}(X; \mathbb{Z}(p)) & \longrightarrow & \mathrm{Hdg}^{2p}(X; \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

Hodge filtered complex cobordism:

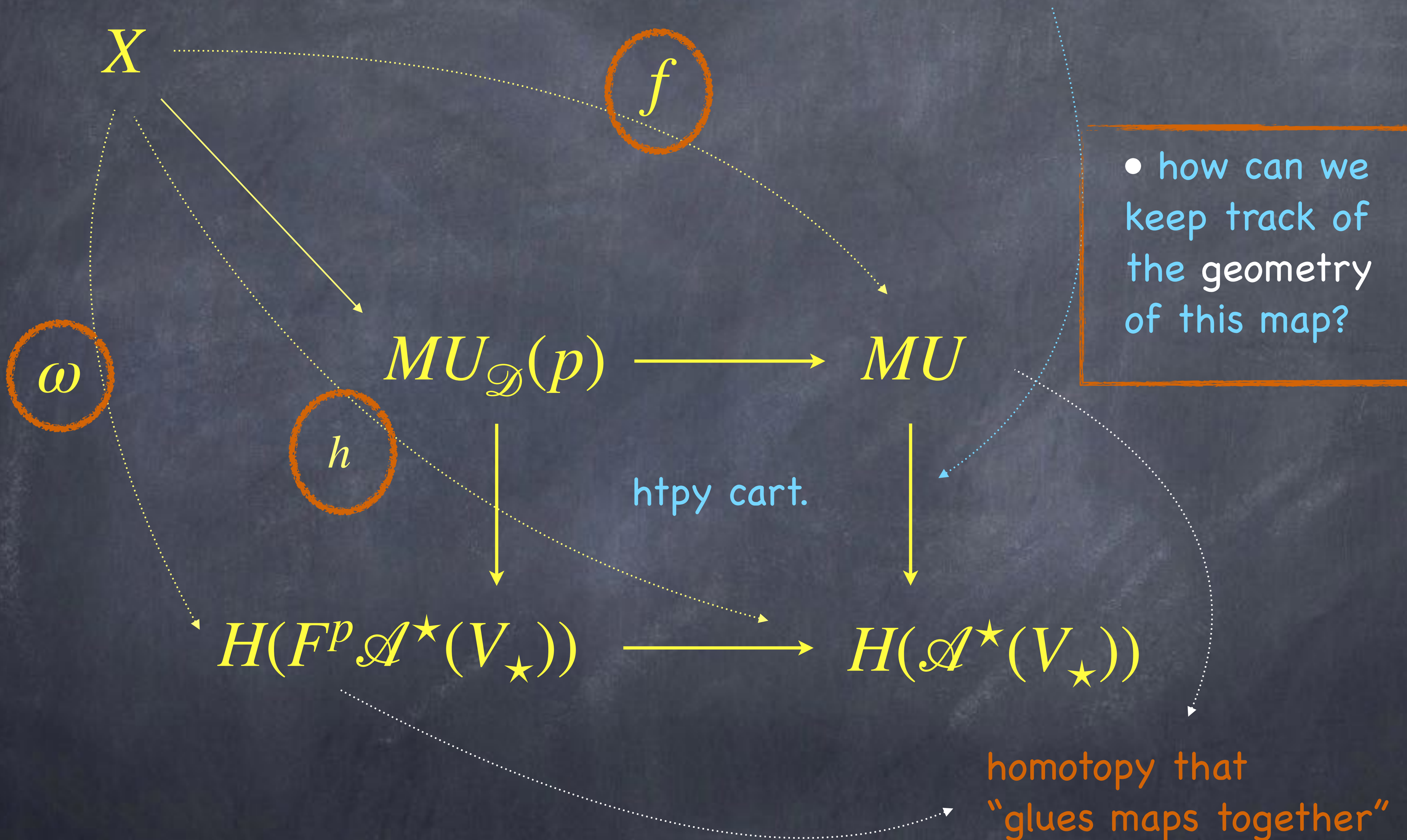
$$\begin{array}{ccc} MU_{\mathscr{D}}(p) & \longrightarrow & MU \\ \big\downarrow & \lrcorner h & \big\downarrow \\ H(F^p\mathscr{A}^\star(V_\star)) & \longrightarrow & H(\mathscr{A}^\star(V_\star)) \end{array}$$

$$MU^n_{\mathscr{D}}(p)(X) := \mathrm{Hom}_{\mathrm{hosPre}}(\Sigma^\infty(X_+), \Sigma^n MU_{\mathscr{D}}(p))$$

- pullbacks along holomorphic maps ✓
- multiplicative structure ✓
- long exact sequence ✓
- pushforward along proper oriented holomorphic maps ?
- a geometric way to describe elements ?

Geometric interpretation?

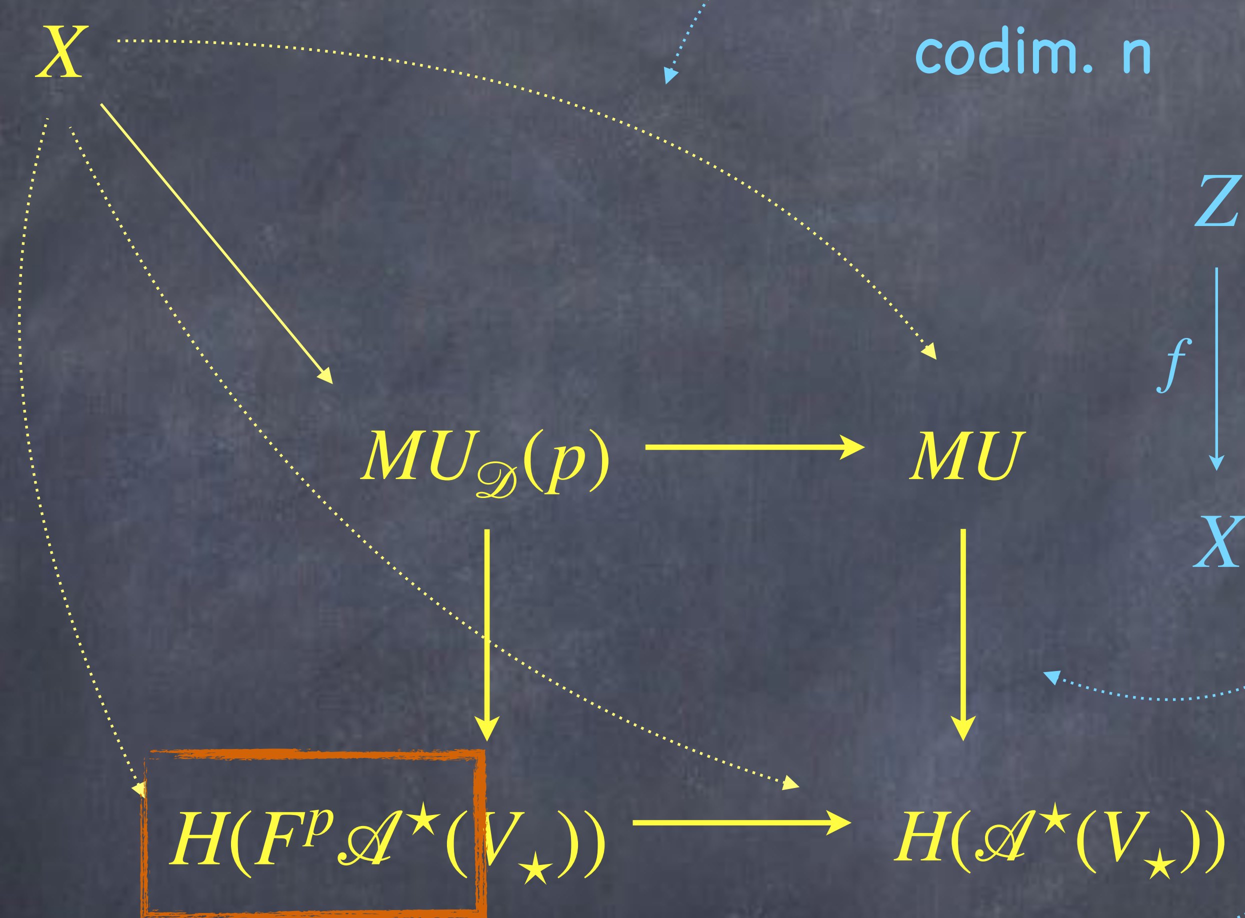
triples (f, ω, h) ?



Geometric interpretation?

jt w/ K. Haus

Quillen



- proper complex oriented map of codim. n

- (stable) normal bundle N_f on Z
- connection ∇_f on N_f

$Z \xrightarrow{f} X$

$K(\nabla_f) = K(c_1(\nabla_f), c_2(\nabla_f), \ldots)$

pushforward along f

Chern-Weil

$f_* K(\nabla_f)$

$[K(\nabla_f)] = K(N_f)$

current on X

?

Hodge filtration

- genus $K\colon MU_\star \longrightarrow MU_\star \otimes \mathbb{C}$

  $\cdot\,(2\pi i)^{p+n}$ on MU_{2n}

- choose current h such that $f_* K(\nabla_f) - dh$ is a closed form in $F^p\mathscr{A}^n(X; V_\star))$

Geometric Hodge filtered complex cobordism:

- geometric Hodge filtered cobordism cycles ← $ZMU^n(p)(X)$

complex oriented codim n

$$Z \xrightarrow{f} X \qquad N_f \qquad \nabla_f \qquad h$$

such that

$$f_*K(\nabla_f) - dh \in F^p\mathscr{A}^n(X; V_\star))_{\mathrm{cl}}$$

$MU^n(p)(X) := ZMU^n(p)(X)$ modulo a bordism relation

How can we compare these theories?

Uniqueness ?

?

$$MU^n_{\mathscr{D}}(p)(X) \quad \approx \quad MU^n(p)(X)$$

additive isomorphism

$\approx$ $\approx$

zig-zag of weak equivalences

$MU_{\mathscr{D}}(p) \simeq MU_{\text{hs}}(p)$

Pontryagin-Thom construction plus...

$$MU^n_{\text{hs}}(p)(X) = \pi_0(MU_{\text{hs}}(p)_n(X))$$

Thom space of universal bundle

$$MU_{\text{hs}}(p)_n \longrightarrow \text{Map}^{\text{sm}}(\Delta^\bullet \times - \,, \text{Th}(\gamma_{m,l}))$$

htpy cart.

$$F^p\mathscr{A}(\Delta^\bullet \times - \,; V_\star) \longrightarrow \mathscr{A}(\Delta^\bullet \times - \,; V_\star)$$

Mathai-Quillen Thom form on universal bundle

Hodge filtered pushforward:

Chern-Simons form
$dCS_K = K(\nabla_g) - K(\nabla'_g)$

$g: X \longrightarrow Y$

relation if

$\sigma_g - \sigma'_g = CS_K(\nabla_g, \nabla'_g)$

proper holomorphic
map of codim q

+ orientation

$(N_g, \nabla_g, \sigma_g) \sim (N_g, \nabla'_g, \sigma'_g)$

currential
version

$$g_*: MU^n(p)(X) \longrightarrow MU^{n+2q}(p+q)(Y)$$

$$(f: Z \to X, N_f, \nabla_f, h) \longmapsto \Big(g \circ f, N_f \oplus f^* N_g, \nabla_f \oplus f^* \nabla_g,$$

$$g_*\big(K(\nabla_g) \wedge h + \sigma_g \wedge (f_* K(\nabla_f) - dh)\big)\Big)$$

in $\mathscr{A}^{-1}(X; V_\star)$ such that
$K(\nabla_g) - d\sigma$ in $F^0\mathscr{A}^0(X; V_\star)$

Bott orientation:

proper holomorphic map of codim q

$g\colon X \longrightarrow Y \qquad g_*\colon MU^n(p)(X) \longrightarrow MU^{n+2q}(p+q)(Y)$

N_g not a holomorphic bundle

+ canonical orientation

holomorphic bundle on X $\quad E$

$g^*[T_Y, D_Y, 0] - [T_X, D_X, 0]$ is a virtual orientation

Karoubi

Bott connection D

locally matrix of 1-forms in $F^1\mathscr{A}^1(X)$ ⟶ $K(D)$ in $F^0\mathscr{A}^0(X; V_\star)$ ⟶ $(E, D, 0)$ is an orientation!

Secondary cobordism invariants:

X compact Kähler manifold

assuming $f_*(1_Y) = 0$

$f\colon Y \longrightarrow X$ proper holomorphic codim p

Bott orientation

Abel-Jacobi invariant

$$AJ(f) \longrightarrow f_*^{\mathscr{D}}(1_Y) \longrightarrow f_*(1_Y)$$

$$0 \longrightarrow \mathrm{Jac}^{2p-1}_{MU}(X) \longrightarrow MU^{2p}(p)(X) \longrightarrow \mathrm{Hdg}^{2p}_{MU}(X) \longrightarrow 0$$

home for "secondary invariants"

Abel-Jacobi cobordism invariant:

X compact Kähler manifold

assuming $f_*(1_Y) = 0$

$f\colon Y \longrightarrow X$ proper holomorphic

Abel-Jacobi invariant

$AJ(f)$

$\mathrm{Jac}^{2p-1}_{MU}(X)$

bordism data

W N_b (normal bundle) ∇_b (connection) σ_f (Chern-Simons form)

$$W \xrightarrow{\;b\;} \mathbb{R} \times X$$

"measures" the failure of the normal bundle N_f to be holomorphic

$$AJ(f) = \int_Y \sigma_f \wedge f^*\omega + \int_{W_{[0,1]}} K(\nabla_b) \wedge f^*\omega$$

in $F^{n-p+1}\mathscr{A}^{2n-2p+1}(X; V'_\star)$

Thank you!