A_3-formality and Massey products for Demushkin groups

Higher Homotopy Algebras in Topology III

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This is joint work with Ambrus Pál

for simplicity

k field with $\mathrm{char}(k) \neq p$ containing primitive pth root of unity

Norm Residue Theorem:

Voevodsky, Rost, ...

Milnor K-theory
$T(k^{\times})/(u \otimes (1-u), u \neq 0,1)$

continuous cohomology of absolute Galois group of k with trivial action on $\mathbb{F}_p$

$$K^M_\bullet(k)/p \xrightarrow{\cong} H^\bullet(k, \mathbb{F}_p)$$

quadratic algebra

- generators in degree 1
- relations in degree 2

strong restriction on which $\mathbb{F}_p$-algebras can occur as the Galois cohomology of a field

Which quadratic algebras occur as $H^\bullet(k, \mathbb{F}_p)$?

Additional properties? $H^\bullet(k, \mathbb{F}_p)$ quadratic algebra

- if the inclusion $K(A) \hookrightarrow B(A)$ of Koszul complex into bar complex is a quasi-isom.

- Conjecture of Positselski-Vishik, Voevodsky: If k contains a primitive pth root of unity, then $H^\bullet(k, \mathbb{F}_p)$ is Koszul.

Is $K^M_\bullet(k)/p$ Koszul?

- Positselski: local and global fields ✓
- Mináč-Panini-Quadrelli-Tân: finite fields, pseudo algebraically closed fields, elementary type pro p-groups, Pythagorean fields if $p = 2$, ... ✓

Additional properties? $H^\bullet(k, \mathbb{F}_p)$ quadratic algebra

- Is $H^\bullet(k, \mathbb{F}_p)$ a Koszul algebra?

- Can $H^\bullet(k, \mathbb{F}_p)$ be described in "elementary terms"?

motivates
but does not imply

$\mathscr{C}^\bullet$ is quasi-isom as a dga to $(H^\bullet(\mathscr{C}^\bullet), \delta = 0)$

continuous Galois cochains

- Is $\mathscr{C}^\bullet(k, \mathbb{F}_p)$ a formal dg-algebra?

- Can the dga $\mathscr{C}^\bullet(k, \mathbb{F}_p)$ be described in "elementary terms" as well?

Massey products provide an obstruction to formality

Additional properties? $H^\bullet(k, \mathbb{F}_p)$ quadratic algebra

- Is $\mathscr{C}^\bullet(k, \mathbb{F}_p)$ a formal dg-algebra?

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Massey products provide an obstruction to formality

- many non-vanishing Massey products in arithm. & alg. geometry: e.g. Ekedahl, Morishita, Sharifi, Gärtner, Bleher-Chinburg-Gillibert, Deninger,...

elements in $H^1(k, \mathbb{F}_2)$

- Hopkins and Wickelgren: $\langle a_1, a_2, a_3 \rangle \neq \varnothing \iff 0 \in \langle a_1, a_2, a_3 \rangle$

k a local or global field of $\mathrm{char}(k) \neq 2$

is defined, i.e., $a_1 \cup a_2 = 0 = a_2 \cup a_3$

Massey product vanishes

- have cocycles a_{12}, a_{23}, a_{34} representing a_1, a_2, a_3 and cochains a_{13}, a_{24} such that $\delta a_{13} = a_{12} \cup a_{23}$ and $\delta a_{24} = a_{23} \cup a_{34}$

- exist cochain a_{14} st $\delta a_{14} = a_{12} \cup a_{24} + a_{13} \cup a_{34}$

Massey vanishing conjecture of Mináč-Tân:

for every field k, all $n \geq 3$, all primes p

Conjecture: For $a_1, \ldots, a_n \in H^1(k, \mathbb{F}_p)$,

$$\langle a_1, \ldots, a_n \rangle \neq \varnothing \iff 0 \in \langle a_1, \ldots, a_n \rangle.$$

✓

- Efrat-Matzri, Mináč-Tân: all fields, all primes, $n = 3$

  - new examples of profinite groups which are not absolute Galois groups

  - Example: S = free pro-p group on generators $x_1, \ldots, x_5$

commutator: $[x, y] = x^{-1}y^{-1}xy$

Mináč-Tân: and relation $1 = [x_4, x_5][[x_2, x_3]x_1]$

Then $G = S/\langle r \rangle$ is not the maximal pro-p quotient of an absolute Galois group

Massey vanishing conjecture of Mináč-Tân:

for every field k, all $n \geq 3$, all primes p

Conjecture: For $a_1, \ldots, a_n \in H^1(k, \mathbb{F}_p)$:

$$\langle a_1, \ldots, a_n \rangle \neq \varnothing \iff 0 \in \langle a_1, \ldots, a_n \rangle.$$

✓

- Efrat-Matzri, Mináč-Tân: all fields, all primes, $n = 3$
- Merkurjev-Scavia: all fields, $p = 2$, $n = 4$
- Harpaz-Wittenberg: all number fields, all primes, all $n \geq 3$

and before Guillot—Mináč—Topaz—Wittenberg for $p = 2$, $n = 4$

- Pál-Szabó: fields with $\mathrm{vcd} \leq 1$
- Quadrelli: elementary type pro-p-groups

all primes, all $n \geq 3$

strong Massey vanishing conjecture

$$0 \in \langle a_1, \ldots, a_n \rangle \iff a_i \cup a_{i+1} = 0$$

for all $i = 1, \ldots, n-1$

Hopkins-Wickelgren formality:

Massey vanishing conjecture
and Koszulity suggest

for every field k and
all primes p?

Question: Is $\mathscr{C}^\bullet(k, \mathbb{F}_p)$ formal?

i.e., quasi-isomorphic to $(H^\bullet(k, \mathbb{F}_p), \delta = 0)$ as dgas

- Yes, for example, for pseudo-algebraically closed fields

The answer is **no** in general

- Positselski: $\mathbb{Q}_\ell$, for odd $\ell \neq p$, is not formal at p
- Harpaz-Wittenberg: $\mathbb{Q}$ is not formal for $p = 2$

- formality implies all Massey products are defined and vanish when neighbouring cup products are zero

- exist $a_1, a_2, a_3, a_4 \in H^1(\mathbb{Q}, \mathbb{F}_2)$

with $a_1 \cup a_2 = a_2 \cup a_3 = a_3 \cup a_4 = 0$

but $\langle a_1, a_2, a_3, a_4 \rangle$ not defined

cochains which witness vanishing of cup products

cochains witnessing vanishing of triple Massey products

a_{12} a_{13} a_{14}
a_{23} a_{24} a_{25}
a_{34} a_{35}
a_{45}

- $\langle a_1, a_2, a_3, a_4 \rangle$ is defined if both $\langle a_1, a_2, a_3 \rangle$ and $\langle a_2, a_3, a_4 \rangle$ vanish …

with the same choice of a_{24}

Hopkins-Wickelgren formality:

Massey vanishing conjecture and Koszulity suggest

for every field k and all primes p?

Question: Is $\mathscr{C}^\bullet(k, \mathbb{F}_p)$ formal?

i.e., quasi-isomorphic to $(H^\bullet(k, \mathbb{F}_p), \delta = 0)$ as dgas

- Positselski: $\mathbb{Q}_\ell$, for odd $\ell \neq p$, is not formal at p
- Harpaz-Wittenberg: $\mathbb{Q}$ is not formal for $p = 2$
- Merkurjev-Scavia: more examples of fields of characteristic $\neq p$ which are not formal at p

However, there are also important positive cases...

- **Theorem (Pál-Q.):** If k has virtual cohomological dimension ≤ 1, then $H^\bullet(k, \mathbb{F}_p)$ is intrinsically formal and Koszul.

every dga $\mathscr{C}^\bullet$ over $\mathbb{F}_p$ with $H^\bullet \cong H^\bullet(G, \mathbb{F}_p)$ is formal

A_3-formality and Massey products:

we assume all dgas are over a field $\mathbb{F}$

Theorem (Kadeishvili; Merkulov): $\mathscr{C}^\bullet$ a dga with cohomology $H^\bullet$.

Then there is an A_∞-algebra $(H^\bullet, \{m_n\}_{n\geq 1})$ such that $m_1 = 0$ and a quasi-isomorphism of A_∞-algebras $H^\bullet \xrightarrow{\simeq} \mathscr{C}^\bullet$.

minimal model for $\mathscr{C}^\bullet$

Recall: A dga $\mathscr{C}^\bullet$ is called A_∞-formal if its minimal model satisfies $m_n = 0$ for all $n \geq 3$.

Definition: We say that the dga $\mathscr{C}^\bullet$ is A_3-formal if its minimal model satisfies $m_3 = 0$.

e.g. Buijs—Moreno-Fernández—Murillo

all triple Massey products in $\mathscr{C}^\bullet$ vanish when defined and if $a_1 \cup a_2 = a_2 \cup a_3 = a_3 \cup a_4 = 0$ then $\langle a_1, a_2, a_3, a_4 \rangle$ is defined $\Longleftarrow$ $\mathscr{C}^\bullet$ is A_3-formal $\Longleftarrow$ $\mathscr{C}^\bullet$ is A_∞-formal

- provides a stronger obstruction than triple Massey products which vanishes for certain classes of profinite groups...

Demushkin groups: p a prime number, G a pro-p group

Definition: G is called a Demushkin group if

- $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty$,
- $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1$,
- the cup-product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

Classification: Demushkin, Serre, Labute

- the only finite Demushkin group is $\mathbb{Z}/2\mathbb{Z}$

- set $d := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ and $q = 0$ if $G^{ab} \cong \mathbb{Z}_p^d$ and $q = p^f$ if $G^{ab} \cong \mathbb{Z}/p^f\mathbb{Z} \times \mathbb{Z}_p^{d-1}$

Theorem (Demushkin): Assume $q \neq 2$. Then G is isomorphic to the pro-p group with generators $x_1, \ldots, x_d$, where d is even, subject to the single relation $1 = x_1^q[x_1, x_2][x_3, x_4]\cdots[x_{d-1}, x_d]$.

commutator: $[x, y] = x^{-1}y^{-1}xy$

- Serre and Labute: For $q = 2$ there are three further types of presentations, one with d odd.

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Examples: Demushkin: $G = \langle x_1, \ldots, x_d \rangle / (x_1^q [x_1, x_2] \cdots [x_{d-1}, x_d])$

- pro-p completion of the fundamental group of compact orientable surface of genus g • $d = 2g$ • $q = 0$

- $K/\mathbb{Q}_\ell$ of finite degree n, G = maximal pro-p Galois extension of K

  Then G is Demushkin with • $d = n + 2$ • $q = p^f$ maximal power such that a primitive qth root of unity is in K

  assume $\ell \neq p$

- Demushkin groups are the building blocks for groups of elementary type in number theory

Efrat's Elementary Type Conjecture

A_3-formality for Demushkin groups:

Theorem (Pál-Q.): Let $d \geq 2$ be even and let G be a pro-p Demushkin group with invariant q.

- If $q = 0$ or $q \geq 4$, then $\mathscr{C}^\bullet(G, \mathbb{F}_p)$ is A_3-formal.
- If $q = p = 3$, then $\mathscr{C}^\bullet(G, \mathbb{F}_p)$ is not A_3-formal.

Examples:

- $G = \mathbb{Z}_p \times \mathbb{Z}_p$ with $q = 0$ (this G is even formal)
- $G = \mathbb{Z}_p \rtimes \mathbb{Z}_p$ with $\mathbb{Z}_p$ acting via a homomorphism $\theta \colon \mathbb{Z}_p \to 1 + q\mathbb{Z}_p$

  e.g. max pro-p quotient of $\mathrm{Gal}(\mathbb{Q}_\ell)$, $(\ell, p) = 1$, q = max p-power st $q \mid \ell - 1$
- $G = \langle x_1, x_2, x_3, x_4 \rangle / (x_1^3 [x_1, x_2,][x_3, x_4])$ with $q = 3$

  Efrat, Koenigsmann:
  max pro-p quotient of $\mathbb{Q}_3(\zeta_3)$

The canonical class:

Assume that $\mathscr{C}^\bullet$ is a dga over $\mathbb{F}_p$ with $H^\bullet$ a Koszul algebra, i.e. $K(A) \hookrightarrow B(A)$ is a quasi-iso.

- $H^\bullet = T(H^1)/(R)$ with $R \subset H^1 \otimes H^1$
- $K_3(H^\bullet) = (H^1 \otimes R) \cap (R \otimes H^1) \subset (H^1)^{\otimes 3}$

There are $\mathbb{F}_p$-linear maps

- $f_1 : H^1 \to \operatorname{Ker} \mathscr{C}^1$ which induces $\mathrm{id}_{H^\bullet}$ on cohomology
- $f_2 : R \to \mathscr{C}^1$ st $\delta f_2(x, y) = -f_1(x) \cup f_1(y)$

Then we can define the $\mathbb{F}_p$-linear map

- $\kappa_3 : K_3(H^\bullet) \to \operatorname{Ker}(\delta) \subset \mathscr{C}^2$

by $\kappa_3(x, y, z) = -f_1(x) \cup f_2(y, z) - f_2(x, y) \cup f_1(z)$

can check that formula always yields a cocycle

and we can check that the induced map $[\kappa_3] : K_3(H^\bullet) \to H^2$ is a cocycle for $\partial : \mathrm{Hom}_{\mathbb{F}_p}(K_\bullet(H^\bullet), H^\bullet[-1])$ which computes Hochschild cohomology $\mathrm{HH}^{\bullet,-1}(H^\bullet)$

Theorem (Kadeishvili): $\mathscr{C}^\bullet$ is A_3-formal if and only if $[\kappa_3] \in \mathrm{HH}^{3,-1}(H^\bullet)$ is zero.

canonical class of $\mathscr{C}^\bullet$

also considered by Benson-Krause-Schwede for different purposes

Idea of the proof: p a prime number, G a pro-p Demushkin group

- compute the canonical class γ_G of $\mathscr{C}^\bullet(G, \mathbb{F}_p)$ in $\mathrm{HH}^{3,-1}(H^\bullet(G, \mathbb{F}_p))$

    - construct γ_G using the Koszul complex of $H^\bullet(G, \mathbb{F}_p)$

      Koszul algebra by Mináč-Pasini-Quadrelli-Tân

    - compute γ_G using Dwyer's Theorem relating Massey products and unitary representations of G

Dwyer's Theorem: p a prime, G a pro-p group, $a_1, a_2, a_3 \in H^1(G, \mathbb{F}_p)$

set of defining systems M for $\langle a_1, a_2, a_3 \rangle$

bijection ⟷

set of continuous group homomorphisms $\bar{\rho}_M : G \to \bar{U}_4(\mathbb{F}_p)$ such that $(i, i+1)$-component = $-a_i$

cont. homomorphisms $G \to \mathbb{F}_p$

cont. maps $G \to \mathbb{F}_p$

$M = \{a_{12}, a_{23}, a_{34}, a_{13}, a_{24}\}$ ⟷ $\bar{\rho}_M = \begin{pmatrix} 1 & -a_{12} & -a_{13} & * \\ & 1 & -a_{23} & -a_{24} \\ & & 1 & -a_{34} \\ & & & 1 \end{pmatrix}$

(and $\langle a_1, a_2, a_3 \rangle_M = a_{12} \cup a_{24} + a_{13} \cup a_{34}$)

and $0 = \langle a_1, a_2, a_3 \rangle_M$ if and only if $\bar{\rho}_M$ lifts to ρ_M

$\rho_M = \begin{pmatrix} 1 & -a_{12} & -a_{13} & -a_{14} \\ & 1 & -a_{23} & -a_{24} \\ & & 1 & -a_{34} \\ & & & 1 \end{pmatrix}$

$G \xrightarrow{\bar{\rho}_M} \bar{U}_4(\mathbb{F}_p)$, $G \xrightarrow{\rho_M} U_4(\mathbb{F}_p)$ (commutes)

$U_4(\mathbb{F}_p) \longrightarrow \bar{U}_4(\mathbb{F}_p) := U_4(\mathbb{F}_p)/Z_4(\mathbb{F}_p)$

group of unitary (4×4)-matrices

centre in $U_4(\mathbb{F}_p)$

Sketch of proof: p a prime number, G a pro-p Demushkin group

- Assume that G is generated by x_1, x_2 with relation $1 = x_1^q[x_1, x_2]$ and $q \neq 2$.
- Let $\chi_i\colon G \to \mathbb{F}_p$ be the generator of $H^1(G, \mathbb{F}_p) = \mathrm{Hom}(G, \mathbb{F}_p)$ st $\chi_i(x_j) = -\delta_{ij}$.

then $H^\bullet(G, \mathbb{F}_p)$ is an exterior algebra on two generators

to have fewer signs later

such that $H^\bullet = T(H^1)/(R)$

- Then $R = \{\chi_1 \otimes \chi_1, \chi_2 \otimes \chi_2, \chi_1 \otimes \chi_2 + \chi_2 \otimes \chi_1\}$ since $\chi_i \cup \chi_i = 0$ and $\chi_1 \cup \chi_2 = -\chi_2 \cup \chi_1$.

can choose identity on $\mathrm{Hom}(G, \mathbb{F}_p)$

- need to construct $f_1\colon H^1 \to \mathrm{Ker}\,\mathscr{C}^1$, $f_2\colon R \to \mathscr{C}^1$, and then $\kappa_3\colon K_3(H^\bullet) \to \mathrm{Ker}(\delta) \subset \mathscr{C}^2$,
$\kappa_3(x, y, z) = -f_1(x) \cup f_2(y, z) - f_2(x, y) \cup f_1(z)$

can be done via Dwyer's Theorem

here of dimension 4; in general of dimension $d^3 - 2d$

- $K_3(H^\bullet) = (H^1 \otimes R) \cap (R \otimes H^1)$

$$= \{\chi_1 \otimes \chi_1 \otimes \chi_1, \chi_2 \otimes \chi_2 \otimes \chi_2, \ \chi_1 \otimes \chi_1 \otimes \chi_2 + \chi_1 \otimes \chi_2 \otimes \chi_1 + \chi_2 \otimes \chi_1 \otimes \chi_1,$$
$$\chi_2 \otimes \chi_2 \otimes \chi_1 + \chi_2 \otimes \chi_1 \otimes \chi_2 + \chi_1 \otimes \chi_2 \otimes \chi_2\}$$

pure tensors leading to triple Massey products

do not correspond directly to triple Massey products

Sketch of proof: p a prime number, G a pro-p Demushkin group, $q \neq 2$

- want to check whether $\kappa_3(\chi_1 \otimes \chi_1 \otimes \chi_1) = -\chi_1 \cup f_2(\chi_1 \otimes \chi_1) - f_2(\chi_1 \otimes \chi_1) \cup f_1(\chi_1)$ is a coboundary in $\mathscr{C}^2(G, \mathbb{F}_p)$

this is a cocycle in a triple Massey product

such that $\delta f_2(\chi_1 \otimes \chi_1) = -\chi_1 \cup \chi_1$

- can use Dwyer's Theorem: $\bar{\rho}: G \to \bar{U}_4(\mathbb{F}_p)$ with $\bar{\rho} = \begin{pmatrix} 1 & -\chi_1 & f_2(\chi_1 \otimes \chi_1) & * \\ & 1 & -\chi_1 & f_2(\chi_1 \otimes \chi_1) \\ & & 1 & -\chi_1 \\ & & & 1 \end{pmatrix}$

- define $\bar{\rho}$ by $x_1 \mapsto A := \begin{pmatrix} 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$ and $x_2 \mapsto I_4$ in $\bar{U}_4(\mathbb{F}_p)$

Can we lift $\bar{\rho}$ to a cont. homomorphism $\rho: G \to U_4(\mathbb{F}_p)$?

- need to check relation $A^q[A, I_4] = I_4$: ✓ always in $\bar{U}_4(\mathbb{F}_p)$

$$A^q = \begin{pmatrix} 1 & q & \binom{q}{2} & \binom{q}{3} \\ & 1 & q & \binom{q}{2} \\ & & 1 & q \\ & & & 1 \end{pmatrix} \overset{?}{=} I_4$$

$\binom{q}{3} \neq 0$ in $\mathbb{F}_3$ if $q = 3$

✓ in $U_4(\mathbb{F}_p)$ if and only if $q = 0$ or $q = p^f \geq 4$

- check Hochschild coboundaries vanish on $\chi_1 \otimes \chi_1 \otimes \chi_1$

- obstruction for $q = 3$ for every d

- computations are more complicated when $d \geq 4$ ...

Thank you!