Real projective groups are formal

Motivic and Equivariant Topology
Swansea University
15 May 2023

Gereon Quick
NTNU

This is joint work with
Ambrus Pál

Milnor conjecture: k field with $\mathrm{char}(k) \neq 2$

Milnor K-theory

$T(k^\times)/(a \otimes (1-a), a \neq 0,1)$

continuous Galois cohomology

Voevodsky

$$K_*^M(k)/2 \xrightarrow{\cong} H^*(k, \mathbb{F}_2)$$

quadratic algebra

- generators in degree 1
- relations in degree 2

strong restriction on which $\mathbb{F}_2$-algebras can occur as the Galois cohomology of a field

Question: What other restrictions are there?

n-fold Massey product:

$(C^*, \delta, \cup)$ = a differential graded k-algebra

$a_1, \ldots, a_n$ classes with $a_i \in H^{d_i}$

with $i < j, (i,j) \neq (1, n+1)$

$\{a_{ij} \in C^{d_{ij}}\}$ is a defining system if

- a_{ij} represents a_i
- $\delta(a_{ij}) = \sum_{k=i+1}^{j-1} \bar{a}_{ik} \cup a_{kj}$

$\bar{a}_{ik} = (-1)^{d_{ik}} a_{ik}$

- The n-fold Massey product $\langle a_1, \ldots, a_n \rangle$ is the set

$$\left\{ \langle a_1, \ldots, a_n \rangle_{\{a_{ij}\}} := \sum_{k=2}^{n} \bar{a}_{1k} \cup a_{kn+1} \in H^{d_{n+1}} \right\}$$

for all defining systems

- $\langle a_1, \ldots, a_n \rangle$ is defined if a defining system exists.
- $\langle a_1, \ldots, a_n \rangle$ vanishes if it contains zero.

Formal dg-algebras: $(C^*, \delta, \cup)$ a differential graded k-algebra is called **formal** if

morphisms of dgas + qisos

T^*

$\simeq$ $\simeq$

cohomology of C^*
dga with trivial differential

$$C^* \xleftarrow{\ \simeq\ } \cdots \xrightarrow{\ } H^*$$

quasi-isomorphism

Massey product set $\langle a_1, \ldots, a_n \rangle$ ⟷ bijection ⟷ Massey product set $\langle b_1, \ldots, b_n \rangle$

$\delta = 0$ implies $\langle b_1, \ldots, b_n \rangle$ vanishes

Formality implies strong Massey vanishing.

all Massey products in all degrees vanish whenever they are defined

Massey product vanishing:

- $\langle a_1, \ldots, a_n \rangle$ is defined if nonempty.
- $\langle a_1, \ldots, a_n \rangle$ vanishes if it contains 0.

Hopkins and Wickelgren $C^* = C^*(\Gamma(k), \mathbb{F}_2)$

absolute Galois group

for k a global field of char $\neq 2$

- triple Massey products of elements in $H^1(\Gamma(k), \mathbb{F}_2)$ vanish whenever they are defined.

for every field k

Mináč and Tân

Massey vanishing conjecture of Mináč-Tân:

for every field k, all $n \geq 3$, all primes p

Conjecture: n-fold Massey products of elements in $H^1(\Gamma(k), \mathbb{F}_p)$ vanish whenever they are defined.

- Matzri, Efrat-Matzri, Mináč-Tân: all fields, all primes, $n = 3$. ✓

  new restrictions for type of groups that can be absolute Galois groups

- Guillot-Mináč-Topaz-Wittenberg: all number fields, $p = 2$, $n = 4$. ✓
- Harpaz-Wittenberg: all number fields, all primes, all $n \geq 3$. ✓
- Merkurjev-Scavia: all fields, $p = 2$, $n = 4$. ✓
- Quadrelli: Efrat's Elementary Type Conjecture for pro-p-groups implies Massey vanishing. ✓

Hopkins-Wickelgren formality:

for k a global field of char $\neq 2$

Massey vanishing conjecture suggests

Question: Is $C^*(\Gamma(k), \mathbb{F}_2)$ formal?

Question: Is $C^*(\Gamma(k), \mathbb{F}_p)$ formal for all fields and all primes?

- Positselski: $C^*(\Gamma(k), \mathbb{F}_p)$ can be not formal.
  - local fields of characteristic $\neq p$ which contain a primitive pth root of unity

but satisfy triple Massey vanishing

- Merkurjev-Scavia: $C^*(\Gamma(k), \mathbb{F}_p)$ may be not formal.

- for every k_0 of characteristic $\neq 2$ there is an extension k/k_0 such that

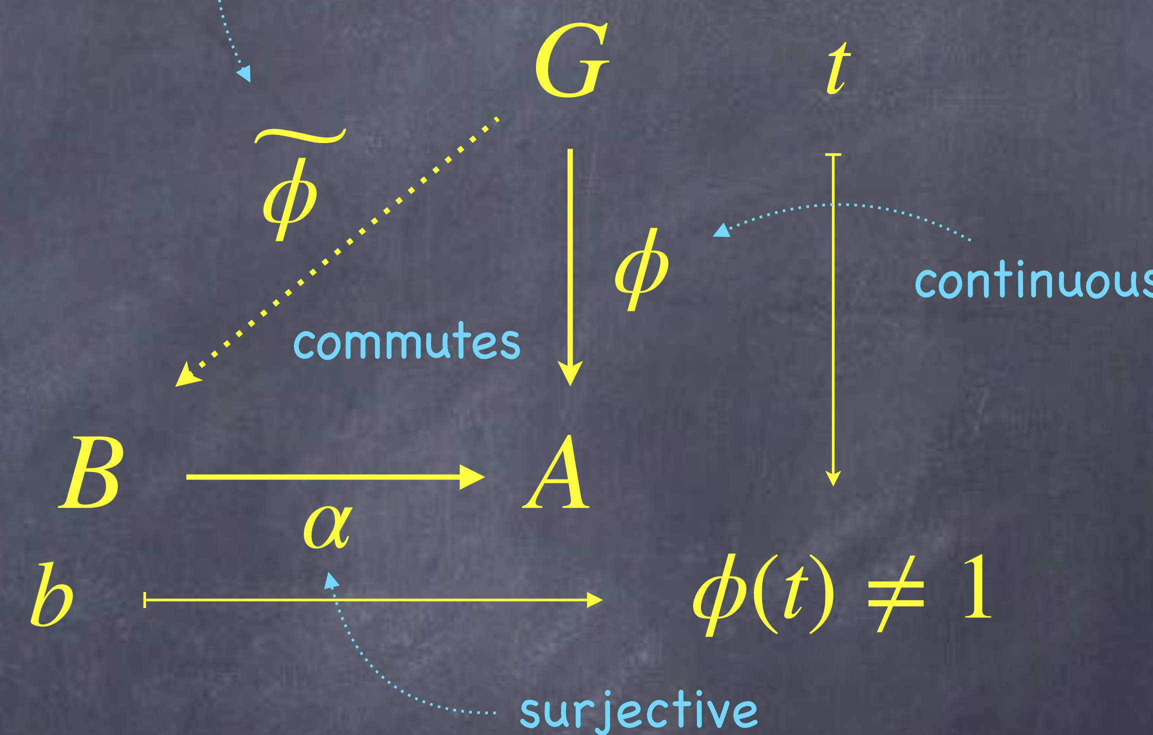
Real projective groups: G a profinite group, A, B finite groups

real embedding problem

solution

for every involution

G is real projective if it has an open subgroup without 2-torsion and every real embedding problem has a solution.



k a field with absolute Galois group $\Gamma(k)$

there exists an involution

Examples: • $\Gamma(k)$ real projective, if $\mathrm{cd}(k(\mathbf{i})) \leq 1$

• k pseudo real closed field, then $\Gamma(k)$ is real projective

$\Gamma(k)$ is real projective

Haran-Jarden

k has virtual cohomological dimension ≤ 1

$k(\mathbf{i})$ pseudo algebraically closed, i.e., every geometrically irreducible $k(\mathbf{i})$-variety has a $k(\mathbf{i})$-rational point

Our main results (Pal + Q.): G a real projective group

Formality of $C^*(G, \mathbb{F}_p)$ for p odd follows from $H^i(G, \mathbb{F}_p) = 0$ for $i \geq 2$.

- The dga $C^*(G, \mathbb{F}_2)$ is formal and satisfies strong Massey vanishing.

k a field of virtual cohomological dimension ≤ 1

- k is formal and satisfies strong Massey vanishing.

first case for field with infinite cohomological dimension

- $H^*(\Gamma(k), \mathbb{F}_2)$ is Koszul.

Kadeishvili's theorem: k field

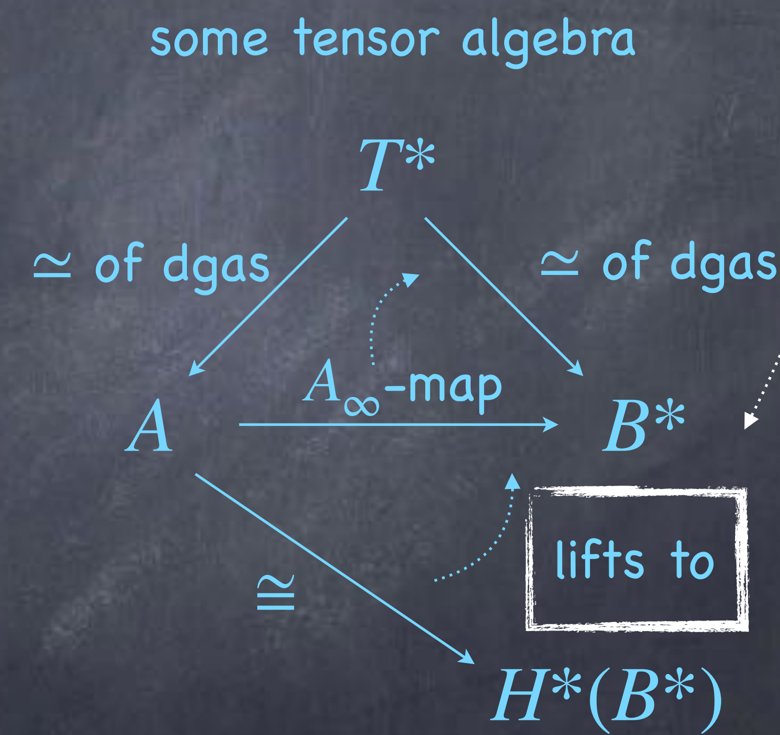
Hochschild cohomology

A pos. graded k-algebra, $A_0 = k$

If $\mathrm{HH}^{n,2-n}(A, A) = 0$ for all $n \geq 3$,

then A is intrinsically formal.

every dga with cohomology algebra equal A is formal



apply this for $A = H^*(G, \mathbb{F}_2)$ = ?

and get $C^*(G, \mathbb{F}_2)$ is formal.

Scheiderer's theorem: G a real projective group

$\mathscr{X}(G)$ = set of conjugacy classes of involutions

connected sum of quadratic algebras

- $(A \sqcap B)_0 = \mathbb{F}_2$
- $(A \sqcap B)_i = A_i \oplus B_i$, and $A_+ \cdot B_+ = 0 = B_+ \cdot A_+$

$$H^*(G, \mathbb{F}_2) \quad \cong \quad B_* \sqcap V_*$$

B = ring of continuous functions $\mathscr{X}(G) \to \mathbb{F}_2$

dual algebra

- $V_0 = \mathbb{F}_2, V_{i \geq 2} = 0$

graded Boolean algebra

- Boolean ring B: $x^2 = x$ for all x
- $B_* = \bigoplus_{n \geq 0} B_n$: $B_0 = \mathbb{F}_2$ and $B_n = B$ for $n \geq 1$ and use multiplication in B

Koszul algebras: $A = T(V)/(R)$ quadratic algebra

$k = \mathbb{F}_2$

tensor algebra of vector space V

$\tau : T(V) \to A$,
$R = \ker(\tau) \cap (V \otimes V)$

- Koszul complex: $(K(A), d)$ $\quad K_0^0(A) = \mathbb{F}_2 \quad K_1^1(A) = V \quad K_2^2(A) = R$

$$K_i(A) = A \otimes K_i^i(A) \otimes A \qquad K_i^i(A) = \bigcap_{0 \le j \le i-2} V^{\otimes j} \otimes R \otimes V^{\otimes i-j-2} \subset V^{\otimes i},\ i \ge 3$$

- A is a **Koszul algebra** if the multiplication map $\mu : K(A) \to A$ is a quasi-isomorphism.

shift grading by s:
$M[s]_n = M_{n+s}$

the complex $\underline{\mathrm{Hom}}_{\mathbb{F}_2}(K_*^*(A), M[s], \partial)$ computes $\mathrm{HH}^{*,s}(A, M)$

graded homomorphisms

$$\partial^{n-1}(f)(x_1 \otimes \cdots \otimes x_n)$$

$$= x_1 \otimes f(x_2 \otimes \cdots \otimes x_n) + f(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n$$

Cohomology algebra is Koszul:

graded Boolean algebra

- Boolean ring B: $x^2 = x$ for all x
- $B_* = \oplus_{n \geq 0} B_n$: $B_0 = \mathbb{F}_2$, $B_n = B$ for $n \geq 1$, multiply in B

dual algebra

- $V_0 = \mathbb{F}_2, V_{i \geq 2} = 0$

$$A = B_* \sqcap V_*$$

- B_* is a Koszul algebra.

if locally finite, then $B_* = \mathbb{F}_2[x_1] \sqcap \ldots \sqcap \mathbb{F}_2[x_n]$

- V_* is a Koszul algebra.

$K(A)$ = bar resolution

connected sums and colimits preserve Koszulity

Theorem (Pal-Q.): A is a Koszul algebra.

Hochschild vanishing theorem:

graded Boolean algebra

dual algebra

- Boolean ring B: $x^2 = x$ for all x
- $B_* = \bigoplus_{n \geq 0} B_n$: $B_0 = \mathbb{F}_2$, $B_n = B$ for $n \geq 1$, multiply in B

- $V_0 = \mathbb{F}_2, V_{i \geq 2} = 0$

$$A = B_* \sqcap V_*$$

Theorem (Pal-Q.): $\mathrm{HH}^{n,2-n}(A, A) = 0$ for all $n \geq 3$.

Idea of proof:

- Step 1: prove assertion for $B'_* \subset B_*$ locally finite

  explicit combinatorial computation: every cocycle is a coboundary

- Step 2: colimit over all locally finite subalgebras

  spectral sequence and show higher lim-terms vanish

Hochschild vanishing theorem:

$R_B = \bigoplus_{i \neq j} \langle x_i \otimes x_j \rangle$

Simple case: $A = B_* = \mathbb{F}_2[x_1] \sqcap \cdots \sqcap \mathbb{F}_2[x_n]$

with $x_i \cdot x_j = 0$ if $i \neq j$

$$d^k : \mathrm{Hom}_{\mathbb{F}_2}(K_k^k(A), A_2) \to \mathrm{Hom}_{\mathbb{F}_2}(K_{k+1}^{k+1}(A), A_2)$$

$$d^k(f) : x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_{j_1} \otimes f(x_{j_2} \otimes \cdots \otimes x_{j_{k+1}}) + f(x_{j_1} \otimes \cdots \otimes x_{j_k}) \otimes x_{j_{k+1}}$$

with $x_{j_t} \neq x_{j_{t+1}}$

$d^k(f) = 0$ if

$f(x_{j_1} \otimes \cdots \otimes x_{j_k}) = x_{j_1}^2 + x_{j_k}^2$ for $j_1 \neq j_k$

$f(x_{j_1} \otimes \cdots \otimes x_{j_k}) = x_{j_1}^2$ for $j_1 = j_k$

- $\dim_{\mathbb{F}_2} \ker(d^k) = n \cdot (n-1)^{k-1}$

- $\dim_{\mathbb{F}_2} K_k^k(B) = n \cdot (n-1)^{k-1}$
- $\dim_{\mathbb{F}_2} \mathrm{Hom}_{\mathbb{F}_2}(K_k^k(B), B_2) = n^2 \cdot (n-1)^{k-1}$

- $\dim_{\mathbb{F}_2} \mathrm{im}(d^{k-1}) = \dim_{\mathbb{F}_2} \mathrm{Hom}_{\mathbb{F}_2}(K_k^k(B), B_2) - \dim_{\mathbb{F}_2} \ker(d^k)$

$$= n^2 \cdot (n-1)^{k-1} - n \cdot (n-1)^{k-1}$$

$$= n \cdot (n-1) \cdot (n-1)^{k-1}$$

$$= n \cdot (n-1)^{k-1}$$

$$\mathrm{im}(d^{k-1}) = \ker(d^k)$$

- Unfortunately, taking the sum with V_* makes life much more complicated...

Thank you!