

# EXISTENCE OF RATIONAL POINTS AS A HOMOTOPY LIMIT PROBLEM

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ABSTRACT. We show that the existence of rational points on smooth varieties over a field can be detected using homotopy fixed points of étale topological types under the Galois action. As our main example we show that the surjectivity statement in Grothendieck's Section Conjecture would follow from the surjectivity of the map from fixed points to continuous homotopy fixed points on the level of connected components. Along the way we define a new model for the continuous étale homotopy fixed point space of a smooth variety over a field under the Galois action.

## 1. INTRODUCTION

Let  $k$  be a field and  $X$  a variety over  $k$ . To find all  $k$ -rational points of  $X$  is an important and often very difficult problem. Many techniques have been developed to either prove the existence or non-existence of rational points. Recently, several topological approaches have been established for example in [2], [15], [16], [24]. In particular, Harpaz-Schlack showed in [11] that certain obstructions to the existence of rational points can be formulated in terms of homotopy fixed points under the Galois action.

In this paper we continue the independent approach in [17] and show that also the existence of rational points can be detected via continuous homotopy fixed points under the Galois action. As the main example and motivation for this approach we briefly recall Grothendieck's section conjecture which is one of the most important open problems on rational points.

Let  $\bar{k}$  be an algebraic closure of  $k$ ,  $G := \text{Gal}(\bar{k}/k)$  and  $X$  be a geometrically connected variety over  $k$  equipped with a geometric point  $x$ . Let  $X_{\bar{k}}$  be the lift of  $X$  to  $\bar{k}$ . Taking étale fundamental groups  $\pi_1(-, x) = \pi_1^{\text{ét}}(-, x)$  induces a short exact sequence of profinite groups

$$(1) \quad 1 \rightarrow \pi_1(X_{\bar{k}}, x) \rightarrow \pi_1(X, x) \rightarrow G \rightarrow 1.$$

If  $a \in X(k)$  is a  $k$ -rational point on  $X$ , then the functoriality of  $\pi_1$  induces a continuous section  $\sigma_a: G_k \rightarrow \pi_1(X, x)$  of (1) which is well-defined up to conjugation by elements in  $\pi_1(X_{\bar{k}}, x)$ . Grothendieck's Section Conjecture predicts that this map has an inverse in the following case (see also [21] for more details on this conjecture).

**Conjecture 1.1.** (Grothendieck [10]) *Let  $k$  be a field which is finitely generated over  $\mathbb{Q}$  and let  $X$  be a smooth, projective curve of genus at least two. The map  $a \mapsto [\sigma_a]$  is a bijection between the set  $X(k)$  of  $k$ -rational points of  $X$  and the set  $S(\pi_1(X/k))$  of  $\pi_1(X_{\bar{k}}, x)$ -conjugacy classes of continuous sections  $G_k \rightarrow \pi_1(X, x)$ .*

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It is well-known that the map  $a \mapsto [\sigma_a]$  is injective. Hence the conjecture is a statement about the existence of rational points. The main result of this paper is that the surjectivity of the map  $a \mapsto [\sigma_a]$  would follow from the solution of a homotopy limit problem in the spirit of the Sullivan Conjecture.

We now outline the main ideas of the paper. Let  $k$  be an arbitrary field with algebraic closure  $\bar{k}$ ,  $G := \text{Gal}(\bar{k}/k)$  and  $X$  be a quasi-projective geometrically connected smooth variety over  $k$ . A  $k$ -rational point  $a : \text{Spec } k \rightarrow X$  induces a map of étale homotopy types  $(\text{Spec } k)_{\text{ét}} \rightarrow X_{\text{ét}}$  which is a section of the map  $X_{\text{ét}} \rightarrow (\text{Spec } k)_{\text{ét}}$  induced by the structure morphism. The pro-space  $(\text{Spec } k)_{\text{ét}}$  is homotopy equivalent to the classifying pro-space  $BG$  of  $G$ . Hence we can consider  $X_{\text{ét}}$  as an object over  $BG$ . Since spaces over  $BG$  are equivalent, in a sense to be made precise later, to spaces with a  $G$ -action, we would like to form the homotopy fixed points  $X_{\text{ét}}^{hG}$  of  $X_{\text{ét}}$ . The étale homotopy type functor then induces a natural map from the set  $X(k)$  of  $k$ -rational points to the set of connected components  $\pi_0(X_{\text{ét}}^{hG})$ . The non-existence of homotopy fixed points of  $X_{\text{ét}}$  would therefore be an obstruction to the existence of rational points. We will show that this idea can also be used to detect rational points.

In order to make this precise we have to specify a suitable model for the étale homotopy type. We will use the rigid Čech étale type over  $k$  introduced by Friedlander in [5]. We denote the resulting pro-space by  $\mathcal{X} := (X/k)_{\text{rét}}$ . It is weakly equivalent to the usual étale topological type of [1] and [6]. The pro-space  $(\text{Spec } k/k)_{\text{rét}}$  is isomorphic to the classifying pro-space  $BG$ . Since  $G$  is a profinite group, we can consider  $BG$  as a profinite space, i.e. an object in the category  $\hat{\mathcal{S}}$  of simplicial profinite sets. Moreover, since  $X$  is smooth and connected, a result of Artin-Mazur [1] shows that all its étale homotopy groups are profinite groups. This leads to the construction of a fibrant profinite model  $\mathcal{X}_{\text{pf}}$  of the étale topological type of  $X$  in the category of profinite spaces over  $BG$ . Via this model we define the continuous étale homotopy fixed points  $\mathcal{X}_{\text{pf}}^{hG}$  of  $X$  over  $k$ .

The new model  $\mathcal{X}_{\text{pf}}^{hG}$  for the homotopy fixed point space of  $X$  is one of the main technical ingredients of the paper and is a key improvement compared to previous approaches as in [11] where only a set of connected components of a potential  $X^{hG}$  is defined.

We continue the outline of ideas. Taking the rigid Čech type of the base change  $X_{\bar{k}}$  yields a pro-space which we denote by  $\bar{\mathcal{X}}$ . A nice feature of the rigid Čech type over  $k$  is that the 0-simplices of  $\bar{\mathcal{X}}$  are given by the constant pro-set  $X(\bar{k})$  of  $\bar{k}$ -valued geometric points. Moreover,  $\bar{\mathcal{X}}$  inherits a natural action by the absolute Galois group  $G$ . (One should note that this action is only defined on the whole pro-space and not on each individual space.) This induces an action of  $G$  on the limit of the underlying diagram of  $\bar{\mathcal{X}}$ . We denote by  $\bar{\mathcal{X}}^G$  the  $G$ -fixed points of the limit of the inverse system underlying the pro-space  $\bar{\mathcal{X}}$ . The set of 0-simplices of  $\bar{\mathcal{X}}^G$  is then a subset of the set of rational points  $X(k)$  of  $X$ . In particular, we obtain a surjective map of sets

$$X(k) \rightarrow \pi_0(\bar{\mathcal{X}}^G)$$

from  $X(k)$  to the set of connected components of  $\bar{\mathcal{X}}^G$ . Moreover, there is a canonical map of simplicial sets

$$\eta: \bar{\mathcal{X}}^G \rightarrow \bar{\mathcal{X}}^{hG}$$

where we write  $\bar{\mathcal{X}}^{hG}$  for the continuous homotopy fixed point space

$$\bar{\mathcal{X}}^{hG} := \mathcal{X}_{\text{pf}}^{hG}$$

of  $X$  over  $k$ . Overall we have the following diagram

$$(2) \quad \begin{array}{ccc} X(k) & \xrightarrow{\quad} & \pi_0(\bar{\mathcal{X}}^{hG}) \\ & \searrow & \nearrow \scriptstyle \pi_0(\eta) \\ & & \pi_0(\bar{\mathcal{X}}^G) \end{array}$$

of natural maps of sets. Hence if  $\pi_0(\eta)$  is surjective, it would follow that each connected component of the homotopy fixed point space  $\bar{\mathcal{X}}^{hG}$  corresponds to a rational point of  $X$ .

Let us return to the special case of a variety  $X$  as in Conjecture 1.1. It is an example of a  $K(\pi, 1)$ -variety, i.e. its étale topological type is weakly equivalent to an Eilenberg-MacLane space of the type  $K(\pi, 1)$  (this is a well-known fact a proof of which may be found in [21]). For such a variety, there is a natural bijection of sets

$$\pi_0(\bar{\mathcal{X}}^{hG}) \cong S(\pi_1(X/k))$$

where we recall that  $S(\pi_1(X/k))$  denotes the set of conjugacy classes of continuous sections of (1). One should note that for this bijection it is crucial that we are able to define *continuous* homotopy fixed points. As a consequence of the previous discussion we can formulate our main result.

**Theorem 1.2.** *Let  $k$  and  $X$  be as in Conjecture 1.1. Then the map  $a \mapsto [\sigma_a]$  is surjective if the map of sets*

$$\pi_0(\eta): \pi_0(\bar{\mathcal{X}}^G) \rightarrow \pi_0(\bar{\mathcal{X}}^{hG})$$

*is surjective.*

The question whether the comparison map from fixed points to homotopy fixed points, such as  $\eta$ , is a weak equivalence is a special case of a homotopy limit problem (see [23]). Unfortunately, to solve a homotopy limit problem is in general a very difficult task. For example, the comparison of fixed and homotopy fixed points under the action of finite  $p$ -groups was known as the Sullivan conjecture which has been proved in different variations in the famous works of Miller [13], Carlsson [4] and Lannes [12]. Nevertheless, we are optimistic that in special cases of arithmetic interest there will be enough information on the Galois action to deduce information about  $\pi_0(\eta)$  in diagram (2).

The content of the paper is organized as follows. In the second section, we provide a framework for continuous homotopy fixed points of pro-spaces with an action by a profinite group. In the third section, we discuss rigid Čech types of algebraic varieties over a field and define a new profinite model for them. In the last section, we define Galois homotopy fixed points of varieties and construct the map  $\eta$  which we need for diagram (2). In the final paragraph we show Theorem 1.2.

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## 2. MODELS IN PROFINITE HOMOTOPY

**2.1. Notations.** Let  $\mathcal{S}$  be the category of simplicial sets whose objects we also call spaces and let  $\mathcal{S}_*$  be the category of pointed spaces. We denote by  $\hat{\mathcal{S}}$  the category of profinite spaces, i.e. simplicial objects in the category of profinite sets with continuous maps as morphisms. Let  $\hat{\mathcal{S}}_*$  be the associated category of pointed profinite spaces. We consider  $\hat{\mathcal{S}}$  and  $\hat{\mathcal{S}}_*$  with the simplicial model structures described in [17] and [19], and denote the corresponding homotopy categories by  $\hat{\mathcal{H}}$  and  $\hat{\mathcal{H}}_*$ , respectively. (The reader should note that Morel had already introduced the category  $\hat{\mathcal{S}}$  and equipped it with a  $\mathbb{Z}/p$ -model structure in [14]).

**Example 2.1.** Important examples of profinite spaces are classifying spaces for profinite groups. For a profinite group  $G$ , the simplicial set  $BG$  given in degree  $n$  by the product of  $n$  copies of the profinite group  $G$  is in a natural way an object of  $\hat{\mathcal{S}}$ . Moreover, it comes equipped with the profinite space  $EG$  over  $BG$ , given in degree  $n$  by the product of  $n+1$  copies of  $G$  with a free  $G$ -action in each dimension.

If  $B$  is a profinite space, we denote by  $\hat{\mathcal{S}}/B$  the category of profinite spaces  $X$  together with a map  $X \rightarrow B$  in  $\hat{\mathcal{S}}$ . This category of profinite spaces over  $B$  inherits a model structure from  $\hat{\mathcal{S}}$  via the forgetful functor.

If  $X$  and  $Y$  are objects in  $\hat{\mathcal{S}}/B$ , we denote by  $\text{Map}_{\hat{\mathcal{S}}/B}(X, Y)$  the simplicial set whose set of  $n$ -simplices is given as the set maps in  $\hat{\mathcal{S}}/B$

$$\text{Map}_{\hat{\mathcal{S}}/B}(X, Y)_n = \text{Hom}_{\hat{\mathcal{S}}/B}(X \times \Delta[n], Y)$$

is given as the set maps in  $\hat{\mathcal{S}}/B$  where  $\Delta[n]$  denotes the standard simplicial  $n$ -simplex. This defines a functor

$$\text{Map}_{\hat{\mathcal{S}}/B}(-, -): (\hat{\mathcal{S}}/B)^{\text{op}} \times \hat{\mathcal{S}}/B \rightarrow \mathcal{S}.$$

**Remark 2.2.** Since the model structure on  $\hat{\mathcal{S}}/B$  is simplicial (see [17] and [19, §2.2]), this functor is homotopy invariant in the following sense. Let  $Z$  be an object in  $\hat{\mathcal{S}}/B$  and  $f: X \rightarrow Y$  a map between fibrant objects in  $\hat{\mathcal{S}}/B$ . If  $f$  is a weak equivalence in  $\hat{\mathcal{S}}/B$ , then the map  $\text{Map}_{\hat{\mathcal{S}}/B}(Z, f)$  is a homotopy equivalence of fibrant simplicial sets.

**2.2. Profinite models for spaces.** Our first step in the construction of Galois homotopy fixed point spaces is to show that a space with finite homotopy groups has a concrete model in the category  $\hat{\mathcal{S}}$ .

**Definition 2.3.** *A connected simplicial set  $X$  is called  $\pi$ -finite if all its homotopy groups are finite.*

The following theorem shows that a  $\pi$ -finite space is homotopy equivalent to a profinite space in the following sense.

**Theorem 2.4.** *Let  $G$  be a finite group and let  $X$  be a connected simplicial set which is  $\pi$ -finite together with a map  $X \rightarrow BG$ . Then there is a profinite space  $FX$  over  $BG$  which is a fibrant object in  $\hat{\mathcal{S}}/BG$  and a map  $\varphi_X: X \rightarrow FX$  over  $BG$  which is a weak equivalence of underlying simplicial sets. In particular, it induces an isomorphism  $\pi_* X \cong \pi_* FX$  of homotopy groups of the underlying simplicial sets. The assignment  $X \mapsto FX$  is functorial such that the following holds. If  $f: X \rightarrow Y$  is a map between connected simplicial sets over  $BG$  which are  $\pi$ -finite,*

then the induced map  $F(f): FX \rightarrow FY$  between profinite spaces over  $BG$  makes the following diagram of underlying simplicial sets commute

$$(3) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi_X \downarrow & & \downarrow \varphi_Y \\ FX & \xrightarrow{F(f)} & FY. \end{array}$$

*Proof.* Let  $G$  and  $X$  be as in the theorem. In a first step, we apply the profinite completion functor  $\mathcal{S} \rightarrow \hat{\mathcal{S}}$ ,  $X \mapsto \hat{X}$ . It is defined by taking in each dimension  $n$  the set-theoretic profinite completion  $\hat{X}_n$  of the set  $X_n$ . Since  $BG$  is a simplicial finite set, the universal property of profinite completion yields an induced map  $\hat{X} \rightarrow BG$  in  $\hat{\mathcal{S}}$ .

Let  $Z \mapsto R_{BG}Z$  be a fixed functorial fibrant replacement in  $\hat{\mathcal{S}}/BG$ . Such a functor exists since the model structure on  $\hat{\mathcal{S}}$  is simplicial and fibrantly generated and so is the induced model structure on the overcategory  $\hat{\mathcal{S}}/BG$  (see [17, Theorem 2.3]). We define  $FX$  to be the fibrant replacement  $R_{BG}\hat{X}$  of the profinite completion of  $X$  considered as a profinite space over  $BG$ . The map  $\varphi_X: X \rightarrow FX$  is the composition of the canonical maps  $X \rightarrow \hat{X}$  and  $\hat{X} \rightarrow R_{BG}\hat{X}$  over  $BG$ .

It remains to show that  $\varphi_X$  is a weak equivalence of underlying simplicial sets. If  $Z$  is a profinite space, let  $|Z|$  denote its underlying simplicial set and let  $Z \mapsto RZ$  be a functorial fibrant replacement in  $\hat{\mathcal{S}}$ . Recall that the homotopy groups of  $Z$  are equal to the homotopy groups of the Kan complex  $|RZ|$ . Since  $X$  is  $\pi$ -finite, the map  $X \rightarrow |R\hat{X}|$  is a weak equivalence of simplicial sets. This follows from Sullivan's work [22] on profinite completion. We refer to [18, §3.7 and Corollary 3.15] for a translation to the context of profinite spaces. Finally, since a map is a weak equivalence in  $\hat{\mathcal{S}}/BG$  if its underlying map in  $\hat{\mathcal{S}}$  is a weak equivalence, the maps  $\hat{X} \rightarrow R\hat{X}$  and  $\hat{X} \rightarrow R_{BG}\hat{X}$  are isomorphisms in  $\hat{\mathcal{H}}$ . This implies that the map  $X \rightarrow FX$  is a weak equivalence of underlying simplicial sets as well. The stated functoriality of  $F$  follows from the universal property of profinite completion and from the fact that profinite completion and the applied fibrant replacement are functorial.  $\square$

**Remark 2.5.** 1. We think of  $FX$  as a model of  $X$  in the category  $\hat{\mathcal{S}}/BG$ .  
2. Finding a profinite model for a space can be easily simplified in the following special case. Let  $G$  be a profinite group and let  $(X, x)$  be a connected pointed simplicial set over  $BG$  whose only nontrivial homotopy group is the profinite fundamental group  $\pi_1(X, x) =: \pi$ . Then the profinite classifying space  $B\pi \in \hat{\mathcal{S}}_*/BG$  is equipped with a pointed map  $X \rightarrow B\pi$  over  $BG$  which is a weak equivalence of underlying simplicial sets. We can consider  $B\pi$  as a profinite model for  $X$  in  $\hat{\mathcal{S}}/BG$ .

**2.3. Continuous homotopy fixed points.** Let  $G$  be a profinite group. Recall that we denote by  $X \mapsto R_{BG}X$  a fixed functorial fibrant replacement in  $\hat{\mathcal{S}}/BG$ .

**Definition 2.6.** For  $X \in \hat{\mathcal{S}}/BG$ , we define the space  $X^{hG}$  to be

$$X^{hG} := \text{Map}_{\hat{\mathcal{S}}/BG}(BG, R_{BG}X).$$

We call  $X^{hG}$  the homotopy fixed point space of  $X$ .

This notation and terminology is justified by the following observation. Let  $\hat{\mathcal{S}}_G$  be the category of profinite  $G$ -spaces, i.e. simplicial objects in the category of profinite sets with a continuous  $G$ -action. By taking homotopy orbits, we obtain a functor

$$\hat{\mathcal{S}}_G \rightarrow \hat{\mathcal{S}}/BG, Y \mapsto (Y \times_G EG \rightarrow BG)$$

from  $\hat{\mathcal{S}}_G$  to the category of profinite spaces over  $BG$ . This functor is right adjoint to the functor

$$\hat{\mathcal{S}}/BG \rightarrow \hat{\mathcal{S}}_G, X \mapsto X \times_{BG} EG.$$

Moreover,  $Y \mapsto (Y \times_G EG \rightarrow BG)$  sends fibrant profinite  $G$ -spaces to fibrations over  $BG$ . Let  $Y \mapsto R_G Y$  be a fixed fibrant replacement in  $\hat{\mathcal{S}}_G$ . Then, for a profinite  $G$ -space  $Y$ , we have a natural isomorphism

$$\mathrm{Map}_{\hat{\mathcal{S}}_G}(EG, R_G Y) \cong \mathrm{Map}_{\hat{\mathcal{S}}/BG}(BG, R_G Y \times_G EG).$$

The mapping space on the left is the (continuous) homotopy fixed point space of the profinite  $G$ -space  $Y$  (see also [17] and [20]).

**Remark 2.7.** The crucial point in the construction of  $X^{hG}$  is that we do take the topology of  $G$  into account by considering continuous mapping spaces in  $\hat{\mathcal{S}}/BG$ . Moreover, the functor  $X \mapsto X^{hG}, \hat{\mathcal{S}}/BG \rightarrow \mathcal{S}$  is homotopy invariant and does not depend on the choice of fibrant replacement in  $\hat{\mathcal{S}}/BG$ . This follows from the fact that  $BG$  is cofibrant in  $\hat{\mathcal{S}}/BG$  and that the model structure on  $\hat{\mathcal{S}}/BG$  is simplicial (see also [17] and [19, §2.2]).

**2.4. Homotopy fixed points and sections.** For our main arithmetic application, we need to relate homotopy fixed point spaces to the following set of sections. Let  $\bar{\pi}$  be a profinite group and let

$$(4) \quad 1 \rightarrow \bar{\pi} \rightarrow \pi \rightarrow G \rightarrow 1$$

be a fixed extension of  $G$  by  $\bar{\pi}$ . We denote the set of  $\bar{\pi}$ -conjugacy classes of continuous sections of (4) by  $S(\pi)$ .

The homotopy fixed points of the classifying space  $B\pi$  are related to the set  $S(\pi)$  in the following way.

**Proposition 2.8.** *There is a natural bijection*

$$\pi_0(\mathrm{Map}_{\hat{\mathcal{S}}/BG}(BG, B\pi)) \cong S(\pi).$$

*Proof.* The set of connected components of  $\mathrm{Map}_{\hat{\mathcal{S}}/BG}(BG, B\pi)$  is in bijection with the set of homotopy classes of maps

$$\mathrm{Hom}_{\hat{\mathcal{H}}/BG}(BG, B\pi).$$

The universal property of classifying spaces implies that the latter set is in bijection with the set of continuous outer homomorphisms from  $G$  to  $\pi$  over  $G$ . The latter set is in bijection with  $S(\pi)$ .  $\square$

**Remark 2.9.** One should note that, if the groups  $\pi$  and  $G$  are infinite profinite groups, it is crucial for the assertion in Proposition 2.8 that we use mapping spaces in  $\hat{\mathcal{S}}/BG$ , since we are interested in the set of *continuous* sections of (4).

**2.5. Profinite models for pro-spaces.** Our next goal is to apply the constructions of the previous sections to pro-spaces. Since the étale topological type of a variety is given as a pro-object in the category of spaces, we need this generalization for the arithmetic applications of the next section.

For a category  $\mathcal{C}$ , let  $\text{pro-}\mathcal{C}$  be the category of pro-objects of  $\mathcal{C}$ , i.e. the category of filtered diagrams in  $\mathcal{C}$  with morphism sets defined by

$$\text{Hom}_{\text{pro-}\mathcal{C}}(\{X(i)\}, \{Y(j)\}) := \lim_j \text{colim}_i \text{Hom}_{\mathcal{C}}(X(i), Y(j)).$$

Let  $\mathcal{X} = \{\mathcal{X}(i)\}_I$  and  $\mathcal{Y} = \{\mathcal{Y}(j)\}_J$  be pro-objects of  $\mathcal{C}$ . Assume we have a functor  $\alpha: J \rightarrow I$  between the indexing categories and a natural transformation  $T: \mathcal{X} \circ \alpha \rightarrow \mathcal{Y}$ . This datum defines an element

$$(T(j))_{j \in J} \in \lim_j \text{Hom}_{\mathcal{C}}(\mathcal{X}(\alpha(j)), \mathcal{Y}(j))$$

which we consider as a morphism in  $\text{pro-}\mathcal{C}$  via the natural map

$$\lim_j \text{Hom}_{\mathcal{C}}(\mathcal{X}(\alpha(j)), \mathcal{Y}(j)) \rightarrow \text{Hom}_{\text{pro-}\mathcal{C}}(\mathcal{X}, \mathcal{Y}).$$

Such a morphism of pro-objects is called a *strict* morphism.

If  $\mathcal{C}$  is a simplicial category, then the mapping space of two pro-objects is defined by

$$\text{Map}_{\text{pro-}\mathcal{C}}(\{\mathcal{X}(i)\}, \{\mathcal{Y}(j)\}) := \lim_j \text{colim}_i \text{Map}_{\mathcal{C}}(\mathcal{X}(i), \mathcal{Y}(j)).$$

We are interested in the following special situation. Let  $G = \lim_k G(k)$  be a profinite group given as the inverse limit of finite groups  $G(k)$  indexed over the cofiltering category  $K$ . Let  $\{\mathcal{X}(i)\}_I$  be a pro-object in the category of spaces. We assume that every  $\mathcal{X}(i)$  is a connected  $\pi$ -finite space in the sense of Definition 2.3. Assume that we are given a strict morphism  $\{\mathcal{X}_i\}_I \rightarrow \{BG(k)\}_K$  of pro-objects in  $\hat{\mathcal{S}}$ . By definition of a strict morphism, this means that we have a functor  $\alpha: K \rightarrow I$  and natural maps  $\mathcal{X}(\alpha(k)) \rightarrow BG(k)$  in  $\hat{\mathcal{S}}$  for every  $k \in K$ . (For those  $i \in I$  for which there might be no  $k \in K$  with  $\alpha(k) = i$ , we consider  $\mathcal{X}(i)$  to be a space over the trivial classifying space  $B\{e\} = *$ .)

For each  $i \in I$ ,  $\mathcal{X}(i)$  satisfies the assumptions of Theorem 2.4 and we can apply the functor  $\mathcal{X}(i) \mapsto F\mathcal{X}(i)$ . We obtain a pro-object  $\{F\mathcal{X}(i)\}_I$  in the category of pointed profinite spaces together with a strict morphism

$$\{F\mathcal{X}(i)\}_I \rightarrow \{BG(k)\}_K$$

of pro-objects in  $\hat{\mathcal{S}}_*$ . Since taking homotopy limits is functorial with respect to strict morphisms, we get an induced map in  $\hat{\mathcal{S}}_*$

$$\varphi: \text{holim}_i F\mathcal{X}(i) \rightarrow \text{holim}_k BG(k)$$

which, by abuse of notations, is also denoted by  $\varphi$ . Since filtered homotopy inverse limits preserve fibrations,  $\varphi$  is a fibration in  $\hat{\mathcal{S}}_*$ . (We refer the reader to [19, §2.5] for homotopy limits in  $\hat{\mathcal{S}}_*$ .)

**Lemma 2.10.** *For each  $n \geq 1$ , the homotopy group  $\pi_n(\text{holim}_i F\mathcal{X}(i))$  is naturally isomorphic in the category of profinite groups to the profinite group  $\{\pi_n(\mathcal{X}(i))\}_I$ .*

*Proof.* By Theorem 2.4, we have natural isomorphisms  $\pi_n(\mathcal{X}(i)) \cong \pi_n(F\mathcal{X}(i))$  for every  $i \in I$ . Since the category of profinite groups is canonically equivalent to the pro-category of finite groups, it suffices to show that the homotopy group

$\pi_n(\operatorname{holim}_i F\mathcal{X}(i))$  is isomorphic to the profinite group  $\lim_i \pi_n(\mathcal{X}(i))$ . But this follows as in [19, Lemma 2.14] from the Bousfield-Kan spectral sequence for homotopy limits.  $\square$

The previous lemma justifies the following terminology.

**Definition 2.11.** We call  $\mathcal{X}_{\text{pf}} := \operatorname{holim}_i F\mathcal{X}(i) \in \hat{\mathcal{S}}$  together with the map  $\varphi$  to  $BG$  in  $\hat{\mathcal{S}}$  a profinite model over  $BG$  of the pro-space  $\mathcal{X} = \{\mathcal{X}(i)\}_I$ .

We define the continuous  $G$ -homotopy fixed points of  $\mathcal{X}$  to be the space

$$\mathcal{X}_{\text{pf}}^{hG} := \operatorname{Map}_{\mathcal{S}/BG}(BG, \mathcal{X}_{\text{pf}}).$$

**Remark 2.12.** The canonical map from limits to homotopy limits induces a natural map of underlying spaces

$$(5) \quad \lim_i \mathcal{X}(i) \rightarrow \operatorname{holim}_i F\mathcal{X}(i).$$

**Remark 2.13.** In the above situation, let us assume that  $\mathcal{X} = \{\mathcal{X}(i)\}_I$  be a pro-space such that each  $\mathcal{X}(i)$  is a connected  $\pi$ -finite space such that  $\{\pi_1(\mathcal{X}(i))\}_I$  is the only nontrivial pro-homotopy group of  $\mathcal{X}$  (even though not every individual space  $\mathcal{X}(i)$  is a  $B\pi_1$ -space). Then, by Lemma 2.10, we could also define a profinite model for  $\mathcal{X}$  via the pro-space  $\{B\pi_1(\mathcal{X}(i))\}_I$ . The limit  $\lim_i B\pi_1(\mathcal{X}(i))$  is isomorphic to the simplicial profinite set  $B(\lim_i \pi_1(\mathcal{X}(i)))$  which in degree  $n$  is given by the  $n$ -fold product of copies of the profinite group  $\lim_i \pi_1(\mathcal{X}(i))$ . The canonical map

$$\lim_i B(\pi_1(\mathcal{X}(i))) \rightarrow \operatorname{holim}_i B(\pi_1(\mathcal{X}(i)))$$

is then a weak equivalence of profinite spaces. Hence in this case,

$$B(\lim_i \pi_1(\mathcal{X}(i))) = \lim_i B(\pi_1(\mathcal{X}(i))) \rightarrow \lim_k BG(k) = BG$$

would serve just as well as a profinite model of the pro-space  $\{\mathcal{X}(i)\}_I$ .

**2.6. Group actions on pro-spaces.** Finally, the Galois action on the étale topological type of a variety leads us to the following notion of a group action on a pro-space.

Let  $G$  be a profinite group which we assume to be indexed by the cofiltered system of its open normal subgroups and let  $\bar{\mathcal{X}} = \{\bar{\mathcal{X}}(j)\}_J$  be a pro-object of  $\mathcal{S}$ . We assume that  $G$  acts on  $\bar{\mathcal{X}}$  by strict automorphisms, i.e., every element  $g \in G$  induces a strict automorphism of  $\bar{\mathcal{X}}$  given by an induced functor  $g: J \rightarrow J$  and a natural transformation  $T(g): \bar{\mathcal{X}}(g(j)) \rightarrow \bar{\mathcal{X}}(j)$ . Then  $G$  acts on the mapping space

$$\operatorname{Map}_{\text{pro-}\mathcal{S}}(*, \bar{\mathcal{X}}) = \lim_j \bar{\mathcal{X}}(j).$$

A map  $f: EG \rightarrow \bar{\mathcal{X}}$  in  $\text{pro-}\mathcal{S}$  is given by a compatible collection of maps  $\{f_j: EG \rightarrow \bar{\mathcal{X}}(j)\}_j$  where each  $f_j$  is an element of  $\operatorname{colim}_U \operatorname{Hom}(EG/U, \bar{\mathcal{X}}(j))$ . Considering  $EG$  as a pro-space  $\{EG/U\}_U$  where  $U$  runs through the open normal subgroups of  $G$ , then  $G$  acts trivially on the indexing category of  $EG$  and by multiplication on each space  $EG/U$ . Since  $G$  acts trivially on the indexing category of  $EG$ , we can define the action of  $G$  on the map  $f$  as follows. An element  $g \in G$  sends  $f$  to the morphism

$$gf = \{gf_j: EG \rightarrow \bar{\mathcal{X}}(j)\}_j$$



in  $\text{pro-}\mathcal{S}$  given by the composite

$$(6) \quad gf_j : EG \xrightarrow{g^{-1}} EG \xrightarrow{f_{g(j)}} \bar{\mathcal{X}}(g(j)) \xrightarrow{T(g)} \bar{\mathcal{X}}(j).$$

Moreover, this induces an action of  $G$  on the mapping space

$$\text{Map}_{\text{pro-}\mathcal{S}}(EG, \bar{\mathcal{X}}).$$

Let  $\text{Map}_{\text{pro-}\mathcal{S}}(EG, \bar{\mathcal{X}})^G$  denote the fixed points. Then the canonical  $G$ -equivariant map  $EG \rightarrow *$  in  $\text{pro-}\mathcal{S}$  induces a canonical map of spaces

$$\text{Map}_{\text{pro-}\mathcal{S}}(*, \bar{\mathcal{X}})^G \rightarrow \text{Map}_{\text{pro-}\mathcal{S}}(EG, \bar{\mathcal{X}})^G.$$

Now given a  $G$ -equivariant map  $f : EG \rightarrow \bar{\mathcal{X}}$  of pro-spaces, i.e., an element in the  $G$ -fixed point set of  $\text{Hom}_{\text{pro-}\mathcal{S}}(EG, \bar{\mathcal{X}})$ , we obtain an associated  $G$ -equivariant map  $EG \rightarrow \bar{\mathcal{X}} \times EG$ . Taking the quotient by the  $G$ -action on both sides induces a map  $EG/G = BG \rightarrow EG \times_G \bar{\mathcal{X}} = (EG \times \bar{\mathcal{X}})/G$  of pro-spaces which is a section of the canonical map  $EG \times_G \bar{\mathcal{X}} \rightarrow BG$  in  $\text{pro-}\mathcal{S}$ . Hence there is a canonical map of spaces

$$\text{Map}_{\text{pro-}\mathcal{S}}(EG, \bar{\mathcal{X}})^G \rightarrow \text{Map}_{\text{pro-}\mathcal{S}/BG}(BG, EG \times_G \bar{\mathcal{X}}).$$

Thus, overall we have constructed a map of spaces

$$\text{Map}_{\text{pro-}\mathcal{S}}(*, \bar{\mathcal{X}})^G \rightarrow \text{Map}_{\text{pro-}\mathcal{S}}(EG, \bar{\mathcal{X}})^G \rightarrow \text{Map}_{\text{pro-}\mathcal{S}/BG}(BG, EG \times_G \bar{\mathcal{X}}).$$

Now let  $p : \bar{\mathcal{X}} \rightarrow \mathcal{X}$  be a strict morphism of pro-objects from  $\bar{\mathcal{X}}$  to a pro-object  $\mathcal{X}$  over  $BG$  which satisfies the hypotheses of Section 2.5. Then  $p$  induces a natural map

$$\text{Map}_{\text{pro-}\mathcal{S}/BG}(BG, EG \times_G \bar{\mathcal{X}}) \rightarrow \text{Map}_{\text{pro-}\mathcal{S}/BG}(BG, \mathcal{X})$$

After taking a profinite model  $\mathcal{X}$  as in the previous section we obtain a map of spaces

$$\text{Map}_{\text{pro-}\mathcal{S}/BG}(BG, \mathcal{X}) \rightarrow \text{Map}_{BG}(BG, \mathcal{X}_{\text{pf}}) = \mathcal{X}_{\text{pf}}^{hG}.$$

Hence overall we obtain a canonical map of spaces

$$\eta : (\lim_j \bar{\mathcal{X}}(j))^G \rightarrow \mathcal{X}_{\text{pf}}^{hG}.$$

**Remark 2.14.** In the case that  $\mathcal{X}$  has the homotopy type of the homotopy orbit  $EG \times_G \bar{\mathcal{X}}$  of  $\bar{\mathcal{X}}$ , we may consider  $\mathcal{X}_{\text{pf}}^{hG}$  as the continuous homotopy fixed points of  $\bar{\mathcal{X}}$  and also write

$$\bar{\mathcal{X}}^{hG} := \mathcal{X}_{\text{pf}}^{hG}$$

for this space. Moreover, we then consider  $\eta$  as a map from fixed points to the homotopy fixed points of  $\bar{\mathcal{X}}$  under  $G$ . The main example for this situation is the Galois action on the étale topological type of a smooth variety over a field.

### 3. ÉTALE TOPOLOGICAL TYPES

We will now turn to the cases of arithmetic geometric origin in which we apply the ideas of the previous sections. The first step is to choose a specific model for the étale topological type of a variety. Instead of using the étale type of schemes defined by Friedlander in [6], we consider the rigid Čech étale topological type over a field. It has been first defined and applied by Friedlander in [5].

**3.1. Rigid Čech types over a field.** We briefly recall the definition of the rigid Čech type of a variety over a field from [5, §3]. We start with the notion of a rigid covering. Let  $k$  be a field,  $\bar{k}$  an algebraic closure of  $k$  and let  $X$  be a scheme of finite type over  $k$ . We denote by  $X(\bar{k})$  the set of geometric points of  $X$  with values in  $\bar{k}$  covering the structure morphism  $p: X \rightarrow \text{Spec } k$ . A rigid covering  $\alpha: U \rightarrow X$  of  $X$  over  $k$  is a disjoint union of pointed, étale, separated maps

$$\coprod_{x \in X(\bar{k})} (\alpha_x: U_x, u_x \rightarrow X, x)$$

where each  $U_x$  is connected and  $u_x$  is a geometric point of  $U_x$  such that  $\alpha_x \circ u_x = x$ . If  $Y$  is another scheme of finite type over  $k$  and  $f: X \rightarrow Y$  is a morphism of schemes, then a morphism of rigid coverings  $\phi: (\alpha: U \rightarrow X) \rightarrow (\beta: V \rightarrow Y)$  over  $f$  is a morphism of schemes  $\phi: U \rightarrow V$  over  $f$  such that  $\phi \circ u_x = v_{f(x)}$  for all  $x \in X(\bar{k})$ .

If  $\alpha: U \rightarrow X$  and  $\beta: V \rightarrow Y$  are rigid coverings of  $X$  and  $Y$  over  $k$ , then the rigid product  $U \times_k^R V \rightarrow X \times_k Y$  is defined to be the closed and open immersion of  $U \times_k V \rightarrow X \times_k Y$  given as the disjoint union indexed by geometric points  $x \times y$  of  $X \times_k Y$  of

$$\alpha_x \times \beta_x: (U_x \times_k V_y)_0 \rightarrow X \times_k Y$$

where  $(U_x \times_k V_y)_0$  is the connected component of  $U_x \times_k V_y$  containing the distinguished geometric point  $u_x \times v_y$ .

If  $f: X \rightarrow Y$  is a map of schemes and  $V \rightarrow Y$  a rigid covering of  $Y$ , then the pullback  $f^*(V \rightarrow Y) = U \rightarrow X$  is the disjoint union of pointed maps

$$(V_{f(x)} \times_Y X)_x \rightarrow X$$

where  $(V_{f(x)} \times_Y X)_x$  is the connected component of  $V_{f(x)} \times_Y X$  containing the geometric point  $f(x) \times x$ .

The category of rigid coverings of  $X$  over  $k$  is denoted by  $RC(X/k)$ . The fact that each connected component  $U_x$  is equipped with a geometric point implies that there is at most one map between any two objects of  $RC(X/k)$ . For, a map of connected, separated étale schemes over  $X$  is determined by the image of any geometric point (see [6, Proposition 4.1]). Together with the construction of rigid products this shows that  $RC(X/k)$  is cofiltering.

For a rigid covering  $U \rightarrow X$ , we denote by  $N_X(U) = \text{cosk}_0^X(U)$  its Čech nerve, i.e. the simplicial scheme given in degree  $n$  by the  $(n+1)$ -fold fiber product of  $U$  with itself over  $X$  in the category of schemes. Since  $X$  is locally noetherian, the connected component functor  $\pi$  is well-defined. In [5, §3], Friedlander defines the rigid Čech étale topological type of  $X$  over  $k$  to be the pro-simplicial set

$$(X/k)_{\text{rét}}: RC(X/k) \rightarrow \mathcal{S}$$

given by sending  $U \rightarrow X$  in  $RC(X/k)$  to the simplicial set  $\pi(N_X(U))$  of connected components of the Čech nerve. For a map  $f: X \rightarrow Y$  of schemes of finite type over  $k$ , there is a strict morphism

$$f_{\text{rét}}: (X/k)_{\text{rét}} \rightarrow (Y/k)_{\text{rét}}$$

of pro-simplicial sets induced by the pullback functor  $f^*: RC(Y/k) \rightarrow RC(X/k)$ . This makes the assignment

$$X \mapsto (X/k)_{\text{rét}}$$

into a functor from the category of schemes of finite type over  $k$  to the category of pro-simplicial sets.

The following proposition shows that if  $X$  is quasi-projective, then  $(X/k)_{\text{rét}}$  has the same homotopy type as the usual étale topological type. The proof follows from a combination of Friedlander's arguments in [5, Proposition 3.2 and a remark on page 102], and [6, Proposition 8.2].

**Proposition 3.1.** *Let  $X$  be a quasi-projective scheme of finite type over a field  $k$ . Then there is a zig-zag of canonical weak equivalences in  $\text{pro} - \mathcal{S}$  between  $(X/k)_{\text{rét}}$  and the étale topological type  $X_{\text{ét}}$  of [6, §4].*

**Remark 3.2.** The set of 0-simplices of  $\pi(N_X(U))$  for any rigid cover  $U \rightarrow X$  in  $RC(X/k)$  is the set  $X(\bar{k})$  of geometric points with values in  $\bar{k}$ . Hence the pro-set of vertices of  $(X/k)_{\text{rét}}$  is just the constant functor sending each rigid covering  $U = \coprod_{x \in X(\bar{k})} U_x \rightarrow X$  to  $X(\bar{k})$ . This makes  $(X/k)_{\text{rét}}$  a very convenient object for our purposes.

**Lemma 3.3.** *Let  $k$  be a field with absolute Galois group  $G$ . The rigid étale Čech type of  $k$  is isomorphic in  $\text{pro} - \mathcal{S}$  to the pro-classifying space  $BG$ , i.e. there is an isomorphism*

$$(\text{Spec } k/k)_{\text{rét}} \cong BG.$$

*Proof.* Let  $L/k$  be a finite Galois extension of  $k$  contained in a fixed separable closure  $\bar{k}$ . The associated Čech nerve  $N_k(L)$  consists in degree  $n$  of the fiber product over  $\text{Spec } k$  of  $n + 1$  copies of  $\text{Spec } L$ . The set of connected components in each degree is hence just given by the product of  $n$  copies of the finite Galois group  $\text{Gal}(L/k)$  of the extension  $L/k$ . Hence the simplicial set of connected components of  $N_k(L)$  is naturally isomorphic to  $B\text{Gal}(L/k)$ . Since the rigid covers given by finite Galois extensions  $L \subset \bar{k}$  are cofinal among all rigid covers defining  $(\text{Spec } k/k)_{\text{rét}}$ , this proves the assertion.  $\square$

The following two examples of morphisms will be most important for us.

**Example 3.4.** Let  $X$  be a geometrically connected variety over  $k$ . The map  $p_{\text{rét}} : (X/k)_{\text{rét}} \rightarrow (\text{Spec } k/k)_{\text{rét}}$  induced by the structure map  $p : X \rightarrow \text{Spec } k$  has the following shape. As we have mentioned in the previous proof, a rigid cover of  $\text{Spec } k$  is given by a finite Galois extension  $L/k$  inside the chosen algebraic closure  $\bar{k}$ . The pullback functor  $p^* : RC(k/k) \rightarrow RC(X/k)$  sends the finite Galois extension  $L/k$  to the rigid cover  $U_L \rightarrow X$

$$(U_L \rightarrow X) := \coprod_{x \in X(\bar{k})} X_L, x_L \rightarrow X, x \in RC(X/k).$$

given by the disjoint union of the (connected) finite Galois covers  $X_L = X \times_k L \rightarrow X$  indexed by the geometric points  $x \in X(\bar{k})$ . The component  $X_L$  is equipped with the canonical lift  $x_L$  of  $x$  induced by the map  $\text{Spec } \bar{k} \rightarrow \text{Spec } L$ . The canonical isomorphism  $X_L \times_X X_L = X \times_k (L \times_k L)$  induces a functorial map of simplicial sets

$$\pi(N_X(U_L)) \rightarrow \pi(N_k(L)).$$

This determines the strict map  $p_{\text{rét}}$  as an element in the set

$$\varinjlim_{L/k} \text{Hom}_{\mathcal{S}}(\pi(N_X(U_L)), \pi(N_k(L))).$$

In particular, since  $\pi(N_k(L))$  is isomorphic to  $B\text{Gal}(L/k)$ , we see that each simplicial set  $\pi(N_X(U_L))$  is equipped with a map to the classifying space  $B\text{Gal}(L/k)$

of the finite group  $\text{Gal}(\bar{k}/k)$ . Since  $X$  is geometrically connected over  $k$ , this map induces a surjective homomorphism of fundamental groups.

**Example 3.5.** Let  $X$  be a geometrically connected variety over  $k$ . Every element  $g \in \text{Gal}(\bar{k}/k)$  defines a morphism  $X_{\bar{k}} \rightarrow X_{\bar{k}}$  of  $X_{\bar{k}} = X \otimes_k \bar{k}$ . The induced map  $g_{\text{rét}} : (X_{\bar{k}}/k)_{\text{rét}} \rightarrow (X_{\bar{k}}/k)_{\text{rét}}$  of rigid étale types is induced by the functor  $g^* : RC(X_{\bar{k}}/k) \rightarrow RC(X_{\bar{k}}/k)$  sending the rigid cover

$$\coprod_{x \in X(\bar{k})} U_x \rightarrow X_{\bar{k}}$$

to the rigid cover

$$\coprod_{x \in X(\bar{k})} (U_{g(x)} \times_{X_{\bar{k}}} X_{\bar{k}})_x \rightarrow X_{\bar{k}}$$

where  $U_{g(x)} \times_{X_{\bar{k}}} X_{\bar{k}}$  is the fiber product of the diagram

$$\begin{array}{ccc} U_{g(x)} \times_{X_{\bar{k}}} X_{\bar{k}} & \longrightarrow & U_{g(x)} \\ \downarrow & & \downarrow \\ X_{\bar{k}} & \xrightarrow{g} & X_{\bar{k}} \end{array}$$

and  $(U_{g(x)} \times_{X_{\bar{k}}} X_{\bar{k}})_x$  is the connected component containing  $x$ . Hence on 0-simplices, the map  $g_{\text{rét}}$  is given by sending the connected component  $(U_{g(x)} \times_{X_{\bar{k}}} X_{\bar{k}})_x$  indexed by  $x$  to the component  $U_{g(x)}$  indexed by  $g(x)$ .

We conclude that, after identifying the pro-set of 0-simplices with the set of geometric points  $X(\bar{k})$  over  $\bar{k}$ , the map  $g_{\text{rét}}$  is just given by the natural action of  $g$  on  $X(\bar{k})$ . Moreover, a 0-simplex of  $(X_{\bar{k}}/k)_{\text{rét}}$  which is fixed under the action of all elements  $g \in \text{Gal}(\bar{k}/k)$  must be indexed by a rational point of  $X$ .

**3.2. Profinite models for étale types.** Let  $k$  be a field with algebraic closure  $\bar{k}$  and absolute Galois group  $G := \text{Gal}(\bar{k}/k)$ . Let  $X$  be a geometrically connected smooth variety over  $k$ . In the following we denote the rigid Čech type  $(X/k)_{\text{rét}}$  of  $X$  over  $k$  by  $\mathcal{X}$  and write  $I$  for the indexing category  $RC(X/k)$ , i.e.  $\mathcal{X} = \{\mathcal{X}(i)\}_I$ .

By Lemma 3.3, we can identify pro-spaces over  $(\text{Spec } k/k)_{\text{rét}}$  with pro-spaces over  $BG$ . Hence we can consider  $\mathcal{X}$  as a pro-space over  $BG$ . By our assumption on  $X$  and by [1, Theorem 11.1], every  $\mathcal{X}(i)$  is a connected  $\pi$ -finite space in the sense of Definition 2.3. Moreover, by Example 3.4, we know that each  $\mathcal{X}(i)$  is equipped with a map to the classifying space  $B\Gamma$  for some finite quotient group  $\Gamma$  of  $G$ .

Hence, as described in Section 2.5, we can associate to  $\mathcal{X}$  a functorial profinite model  $\mathcal{X}_{\text{pf}}$  over  $BG$ . By Lemma 2.10 and Proposition 3.1, we obtain the following result.

**Theorem 3.6.** *For  $k$  and  $X$  as above, the fibrant profinite space  $\mathcal{X}_{\text{pf}}$  over  $BG$  has the same homotopy type as the étale topological type of  $X$ , i.e. the levelwise map of underlying pro-spaces  $\{\mathcal{X}(i)\}_I \rightarrow \{F\mathcal{X}(i)\}_I$  induces an isomorphism of profinite groups  $\pi_n(\mathcal{X}_{\text{pf}}) \cong \pi_n^{\text{ét}}(X)$  for all  $n \geq 1$ .*

#### 4. RATIONAL POINTS AND GALOIS HOMOTOPY FIXED POINTS

We can now give a new definition of continuous homotopy fixed points of a smooth variety over a field under the natural Galois action. A previous definition

has been given in [17]. In [11], Harpaz and Schlank provide a definition only for the set of connected components of a potential homotopy fixed point space.

**4.1. Galois homotopy fixed point spaces.** Let  $k$  be a field with algebraic closure  $\bar{k}$  and absolute Galois group  $G := \text{Gal}(\bar{k}/k)$ . Let  $X$  be a geometrically connected smooth variety over  $k$  and  $X_{\bar{k}}$  be its lift to  $\bar{k}$ . We denote  $(X/k)_{\text{rét}}$  by  $\mathcal{X} = \{\mathcal{X}(i)\}_I$  and write  $\bar{\mathcal{X}} = \{\bar{\mathcal{X}}(i)\}_I$  for the rigid Čech type  $(X_{\bar{k}}/k)_{\text{rét}}$  of  $X_{\bar{k}}$ . The pro-space  $\bar{\mathcal{X}}$  is equipped with a natural action of  $G$  of the form described in Section 2.6.

Let  $x: \text{Spec } \bar{k} \rightarrow X$  be any geometric point of  $X$ . It turns  $\mathcal{X}$  into a pro-object in  $\mathcal{S}_*$ . In particular, we can form the profinite model  $\mathcal{X}_{\text{pf}}$  over  $BG$  of  $\mathcal{X}$  described in Section 3.2. Essentially the same proof as for [17, Theorem 3.5] shows that the profinite model of  $\mathcal{X}$  over  $BG$  has the homotopy type of the  $G$ -homotopy orbits of  $\bar{\mathcal{X}}$ .

**Definition 4.1.** *We define*

$$\bar{\mathcal{X}}^{hG} := \mathcal{X}_{\text{pf}}^{hG} = \text{Map}_{\mathcal{S}/BG}(BG, \mathcal{X}_{\text{pf}})$$

to be the continuous homotopy fixed point space of  $\bar{\mathcal{X}} = (X_{\bar{k}}/k)_{\text{rét}}$ .

The canonical morphism  $X_{\bar{k}} \rightarrow X$  induces a morphism of pro-objects of pointed spaces  $\bar{\mathcal{X}} \rightarrow \mathcal{X}$ . As explained in Section 2.6, this yields a canonical map

$$(7) \quad \eta: \bar{\mathcal{X}}^G \rightarrow \bar{\mathcal{X}}^{hG}$$

from the  $G$ -fixed points  $\bar{\mathcal{X}}^G = (\lim_{\bar{i}} \bar{\mathcal{X}}(\bar{i}))^G$  to the continuous homotopy fixed points space of  $\bar{\mathcal{X}}$ .

**Remark 4.2.** One should note that the action of  $G$  on  $(X_{\bar{k}}/k)_{\text{rét}}$  is only defined on the whole pro-object and not on each space. Hence  $(X_{\bar{k}}/k)_{\text{rét}}$  is in general not a pro-object of simplicial  $G$ -sets. But after forming the mapping space, i.e. after taking the limit of the underlying filtered diagram, we obtain a simplicial object in the category of  $G$ -sets as described in Section 2.6.

**4.2. Rational points and homotopy fixed points.** We keep the notations of the previous section. By functoriality of rigid Čech types, every rational point of  $X$  induces a map of pro-spaces

$$(\text{Spec } k/k)_{\text{rét}} \rightarrow \mathcal{X}$$

compatible with the induced structure map  $\mathcal{X} \rightarrow (\text{Spec } k/k)_{\text{rét}}$ . After taking profinite models, we get a well-defined map of sets

$$(8) \quad X(k) \rightarrow \text{Hom}_{\mathcal{H}/BG}(BG, \mathcal{X}_{\text{pf}}) \cong \pi_0(\text{Map}_{\mathcal{S}/BG}(BG, \mathcal{X}_{\text{pf}})).$$

Since the mapping space on the right of (8) is by definition the continuous homotopy fixed point space of  $\mathcal{X}_{\text{pf}}$  (which we also denote by  $\bar{\mathcal{X}}^{hG}$ ), we obtain a natural map of sets

$$X(k) \rightarrow \pi_0(\bar{\mathcal{X}}^{hG}).$$

Moreover, we know from Remark 3.2 that the pro-set of 0-simplices of  $\bar{\mathcal{X}}$  is canonically isomorphic to the set  $X(\bar{k})$  of  $\bar{k}$ -valued geometric points of  $X$ . By Example 3.5, the action of the absolute Galois group  $G$  of  $k$  on the pro-set of 0-simplices of  $\bar{\mathcal{X}}$  is just given by the natural action of  $G$  on  $X(\bar{k})$ . Since each fixed point under this  $G$ -action has to be indexed by a rational point, we see that

the set of  $G$ -fixed points the 0-simplices of  $\bar{\mathcal{X}}$  is a subset of the  $k$ -rational points  $X(k) = X(\bar{k})^G$  of  $X$ . Hence we obtain a canonical surjective map of sets

$$X(k) \rightarrow \pi_0(\bar{\mathcal{X}}^G).$$

Together with the map  $\eta$  we obtain a map of sets

$$X(k) \rightarrow \pi_0(\bar{\mathcal{X}}^G) \xrightarrow{\pi_0(\eta)} \pi_0(\bar{\mathcal{X}}^{hG}).$$

Hence, since the map  $X(k) \rightarrow \pi_0(\bar{\mathcal{X}}^G)$  is surjective, it is possible to detect rational points on the smooth  $k$ -variety  $X$  by studying the map  $\eta: \bar{\mathcal{X}}^G \rightarrow \bar{\mathcal{X}}^{hG}$  which we consider as a fixed points to homotopy fixed points map.

**Remark 4.3.** It is important that we are able to consider *continuous* homotopy fixed points under the action of the profinite Galois group. One reason will be given in the final section where we will see that, for a suitable  $X$ , the set  $\pi_0(\text{Map}_{\hat{S}/BG}(BG, \mathcal{X}_{\text{pf}}))$  is in bijection with the set of continuous sections of the short exact sequence (1) of the introduction. Another reason is given by the following argument. The  $E_2$ -terms of a descent spectral sequence for Galois homotopy fixed points should be isomorphic to Galois cohomology and not to ordinary group cohomology. This is in fact the case for our definition of  $\mathcal{X}_{\text{pf}}^{hG}$ . If we are given a rational point  $x$  on  $X$ , there is a spectral sequence of the form

$$E_2^{s,t} = H^s(G; \pi_t(\bar{\mathcal{X}}, x)) \Rightarrow \pi_{t-s}(\mathcal{X}_{\text{pf}}^{hG})$$

where  $H^s(G; \pi_t(\bar{\mathcal{X}}))$  denotes the continuous cohomology of  $G$  with coefficients in the profinite  $G$ -module  $\pi_t(\bar{\mathcal{X}}, x)$  (respectively profinite  $G$ -set for  $t = 0$  and profinite  $G$ -group if  $t = 1$ ). This spectral sequence conditionally converges to  $\pi_{t-s}(\mathcal{X}_{\text{pf}}^{hG})$  for  $s \geq 0$  and  $t \geq s + 1$ . A proof of this statement can be read off from the arguments given in the proofs of [17, Theorem 2.16], [20, Theorem 3.17], and [3, IX 5.4]. Unfortunately, the convergence for  $t - s = 0$  is more complicated (see [3, IX 5.4]).

**4.3. The section conjecture as a homotopy limit problem.** Our main example of a case where the observation of Section 4.2 might be interesting is Grothendieck's section conjecture. Let  $k$  be a number field and  $G = \text{Gal}(\bar{k}/k)$  its absolute Galois group. Let  $X$  be a geometrically connected variety over  $k$ . For any given geometric point  $x$  of  $X$ , there is a natural short exact sequence of étale fundamental groups

$$(9) \quad 1 \rightarrow \pi_1(X_{\bar{k}}, x) \rightarrow \pi_1(X, x) \rightarrow G \rightarrow 1.$$

Let  $a: \text{Spec } k \rightarrow X$  be a rational point on  $X$  and let  $y: \text{Spec } \bar{k} \rightarrow X$  be a geometric point lying above  $a$ . Applying the functor  $\pi_1(-, y)$  to the morphism  $a$  induces a continuous homomorphism of groups

$$\sigma_a: G \rightarrow \pi_1(X, y).$$

Since  $X$  is geometrically connected, there is an étale path from  $y$  to  $x$  which induces an isomorphism  $\lambda: \pi_1(X, y) \rightarrow \pi_1(X, x)$ . Composing  $\sigma_a$  with  $\lambda$  defines a section

$$\lambda \circ \sigma_a: G \rightarrow \pi_1(X, x)$$

of sequence (9). The choice of a different path from  $y$  to  $x$  changes this section by composition with an inner automorphism of  $\pi_1(X_{\bar{k}}, x)$ . Hence a rational point of  $X$  induces a section of (9) which is well-defined up to conjugacy by an element of  $\pi_1(X_{\bar{k}}, x)$ . We denote the conjugacy class of the section induced by the rational

point  $a$  by  $[\sigma_a]$  and denote the set of all  $\pi_1(X_{\bar{k}}, x)$ -conjugacy classes of sections of (9) by  $S(\pi_1(X/k))$ . With these notations, there is a map of sets

$$(10) \quad X(k) \rightarrow S(\pi_1(X/k)), \quad a \mapsto [\sigma_a].$$

Grothendieck's section conjecture states that map (10) is a bijection if  $X$  is a smooth projective curve of genus at least two. It is known that the map is injective. The harder and still open question is whether it is surjective.

We would like to shed some light on map (10) from an étale homotopy-theoretical point of view. Denoting the rigid Čech type  $(X/k)_{\text{rét}}$  of  $X$  again by  $\mathcal{X} = \{\mathcal{X}(i)\}_I$ , we know that each  $\mathcal{X}(i)$  is a pointed connected  $\pi$ -finite space. Since  $X$  is smooth, the pro-system of these finite fundamental groups is just the profinite étale fundamental group  $\pi_1(X) = \pi_1^{\text{ét}}(X, x)$  of  $X$ . This follows from the finiteness result of [1, Theorem 11.1] and [6, Corollary 5.8]. The crucial and well-known observation is that  $X$  is a  $K(\pi, 1)$ -variety over  $k$  (see for example [21]), i.e., there is a weak equivalence of pro-spaces

$$(11) \quad \mathcal{X} \simeq B\pi_1(X).$$

We denote the rigid Čech type  $(X_{\bar{k}}/k)_{\text{rét}}$  again by  $\bar{\mathcal{X}}$ . Then we have the canonical map of sets

$$X(k) \rightarrow \pi_0(\bar{\mathcal{X}}^G) \xrightarrow{\pi_0(\eta)} \pi_0(\bar{\mathcal{X}}^{hG})$$

described in the previous section.

Furthermore, we deduce from Proposition 2.8 that there is a natural bijection of sets

$$\pi_0(\bar{\mathcal{X}}^{hG}) \cong S(\pi_1(X/k)).$$

It follows from this result that map (10) is surjective if the map

$$X(k) \rightarrow \pi_0(\bar{\mathcal{X}}^G)$$

is surjective. Since the map  $X(k) \rightarrow \pi_0(\bar{\mathcal{X}}^G)$  is surjective, we get the following criterion.

**Theorem 4.4.** *Let  $k$  be a number field and let  $X$  be a smooth, projective curve of genus  $g \geq 2$ . Then the map (10),  $a \mapsto [\sigma_a]$ , is surjective if the map*

$$\pi_0(\bar{\mathcal{X}}^G) \xrightarrow{\pi_0(\eta)} \pi_0(\bar{\mathcal{X}}^{hG})$$

*is surjective.*

**Remark 4.5.** It follows from equivalence (11) that we could even take the profinite space  $\text{holim}_i B\pi_1(\mathcal{X}(i))$ , or equivalently, as we explained in Remark 2.13,  $\lim_i B\pi_1(\mathcal{X}(i)) = B\pi_1(X)$ , as a fibrant profinite model  $\mathcal{X}_{\text{pf}}$  of  $\mathcal{X}$  in  $\hat{\mathcal{S}}/BG$ .

**Remark 4.6.** The category  $\hat{\mathcal{S}}$  of profinite spaces has first been studied by Morel in [14] where a model structure was constructed in which the weak equivalences are the maps that induce an isomorphism in continuous  $\mathbb{Z}/p$ -cohomology. Since it seems more likely that techniques from the proofs of the Sullivan conjecture ([13], [4], [12]) can be translated first to the pro- $p$ -case, one may consider it to be a more accessible problem to decide one of the following related questions. Is a  $p$ -completed version of  $\eta$  a weak equivalence? Does  $\eta$  induce an isomorphism on mod  $p$ -homology?

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