NON-REALIZABILITY OF A TRIPLE MASSEY PRODUCT

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Abstract. We show that an often used example of a cohomology algebra with non-vanishing triple Massey product is intrinsically A_3-formal and therefore, in fact, cannot be realized as the cohomology of a differential graded algebra with non-vanishing triple Massey product. We prove this result by computing the graded Hochschild cohomology group which contains the potential obstruction to the vanishing.

1. Introduction

Let $\mathcal{C}^\bullet$ be a differential graded $\mathbb{F}_2$-algebra (DGA) with differential δ and cohomology algebra $H^\bullet$. Let a, b, c be cohomology classes such that $a \cup b = 0$ and $b \cup c = 0$. We recall that the Massey product $\langle a, b, c\rangle$ is defined as follows. Let A, B, C be cocycles representing a, b, c, respectively, and let E_{ab} and E_{bc} be cochains such that $\delta E_{ab} = A \cup B$ and $\delta E_{bc} = B \cup C$. The set $M := \{A, B, C, E_{ab}, E_{bc}\}$ is called a defining system for the triple Massey product of a, b, and c. The cochain $A \cup E_{bc} + E_{ab} \cup C$ is a cocycle. We write $\langle a, b, c\rangle_M \in H^\bullet$ for the corresponding cohomology class. The *triple Massey product* $\langle a, b, c\rangle$ is the set of all cohomology classes $\langle a, b, c\rangle_M$ for all such defining systems M. The class $\langle a, b, c\rangle_M$ depends on the choice of the defining system M. The image in the quotient $H^\bullet/(a \cup H^\bullet + H^\bullet \cup c)$, however, is uniquely determined by a, b, and c. We say that $\langle a, b, c\rangle$ *vanishes* if its corresponding class in $H^\bullet/(a \cup H^\bullet + H^\bullet \cup c)$ is zero. Massey products play an important role in the classification of DGAs with a given cohomology ring.

To construct the simplest commutative graded algebra which may be realized as the cohomology of a DGA with a non-vanishing Massey product, one may consider $\mathbb{F}_2[a, b, c]/(ab, bc)$ with a, b, c elements in degree one. We have $ab = 0$ and $bc = 0$ by construction, and hence $\langle a, b, c\rangle$ is defined. We may then expect that it may be possible to find a DGA such that $\langle a, b, c\rangle$ does not vanish. In this note, however, we show that the Massey product $\langle a, b, c\rangle$ always vanishes. More precisely, our main result is the following. Recall that a differential graded algebra is called A_3-*formal* if the minimal A_∞-model of $\mathcal{C}^\bullet$ has a trivial homotopy associator m_3 (see e.g. [7]).

Theorem 1.1. *Let $\mathcal{C}^\bullet$ be a differential graded algebra over $\mathbb{F}_2$ with cohomology algebra isomorphic to $\mathbb{F}_2[a, b, c]/(ab, bc)$. Then $\mathcal{C}^\bullet$ is A_3-formal. Thus, all triple Massey products for $\mathcal{C}^\bullet$ vanish. In particular, the triple Massey product $\langle a, b, c\rangle$ vanishes for $\mathcal{C}^\bullet$.*

Our interest in the realizability of triple Massey products grew out of the work of Hopkins–Wickelgren in [3] on triple Massey products in Galois cohomology. The latter has inspired a lot of research in recent years, see for example [9] and [10].

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Remark 1.2. One may consider other $\mathbb{F}_2$-algebras and ask whether they realize a non-trivial Massey product. Since the definition of the Massey product does not require distinct elements, we may first consider the algebra $\mathbb{F}_2[a]/(a^2)$ with just one generator. However, $\mathbb{F}_2[a]/(a^2)$ is zero in degree two, and hence $\langle a, a, a\rangle$ must vanish. The algebra $\mathbb{F}_2[a, b]/(ab)$ is a Boolean graded algebra in the sense of [11, Definition 6.1]. More generally, any algebra of the form $\mathbb{F}_2[a_1, \dots, a_n]/I$ where I is the ideal generated by all products $a_i a_j$ for $i \neq j$ is a Boolean graded algebra. The algebra $\mathbb{F}_2[a, b]/(a^2, ab)$ is a connected sum of a dual and a Boolean graded algebra in the sense of [11, Section 6]. All these algebras are intrinsically A_∞-formal by [11, Theorem 7.13] and do not allow for non-vanishing Massey products.

We now outline the proof of Theorem 1.1 and thereby describe the content of the paper. For the whole manuscript, we assume that all algebras and vector spaces are over $\mathbb{F}_2$. In Section 2 we recall the Hochschild cohomology of graded algebras and construct the Hochschild cohomology class $[m_3] \in \mathrm{HH}^{3,-1}(H^\bullet(\mathcal{C}^\bullet))$ associated to a differential graded algebra $\mathcal{C}^\bullet$. We note that $[m_3]$ equals the canonical class of $\mathcal{C}^\bullet$ introduced by Benson–Krause–Schwede in [1] as an obstruction for the realizability of modules over Tate cohomology. We then show that $\mathcal{C}^\bullet$ is A_3-formal if and only if $[m_3]$ is zero. For the latter, we assume familiarity with some basic theory of A_∞-algebras. In Section 3 we recall the definition of Koszul algebras and show that $\mathbb{F}_2[a, b, c]/(ab, bc)$ is Koszul. Knowing that an algebra is Koszul simplifies the task to compute its Hochschild cohomology significantly. In Section 4 we prove Theorem 4.1 which states that $\mathrm{HH}^{3,-1}(\mathbb{F}_2[a, b, c]/(ab, bc)) = 0$ by computing the image and the kernel of the differential in the Hochschild complex. Theorem 4.1 then implies Theorem 1.1.

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2. Hochschild cohomology and Massey products

Let A be a graded $\mathbb{F}_2$-algebra. We recall that the bar resolution $B(A)$ of A is the non-negative chain complex of free graded A-bimodules given by $B_n(A) := A^{\otimes n+2}$ for $n \geqslant 0$. The differential $d\colon B_n(A) \to B_{n-1}(A)$ is given by

$$a_0 \otimes \cdots \otimes a_{n+1} \mapsto \sum_{i=0}^{n} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

We write $A^e = A \otimes A^{\mathrm{op}}$. Note that $A^{\otimes n+2} \cong A^e \otimes A^{\otimes n}$ as a graded A-bimodule, hence $B(A)$ indeed consists of free modules.

Proposition 2.1. *The bar resolution $B(A)$ is a free resolution of A as a graded A-bimodule.*

Proof. It suffices to show that the extended complex $\widetilde{B}(A)$ is acyclic, where $\widetilde{B}(A)$ is extended from $B(A)$ by adjoining $\widetilde{B}_{-1}(A) := A$ in degree -1 via the multiplication map $\mu\colon A \otimes A \to A$. We claim that the map $h\colon \widetilde{B}(A) \to \widetilde{B}(A)$ of degree 1 given by

$$a_0 \otimes \cdots \otimes a_{n+1} \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$$

is a contracting homotopy i.e., $dh + hd = 1$. Indeed, we compute directly that

$$dh(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_{n+1} - hd(a_0 \otimes \cdots \otimes a_n). \qquad \square$$

Definition 2.2. Let M and N be graded left A-modules. We define $\underline{\mathrm{Hom}}_A(M,N)$ as the graded $\mathbb{F}_2$-vector space with degree s component given by A-linear graded maps $f : M \to N[s]$, where $N[s]$ is the graded A-module given by $N[s]^n = N^{s+n}$.

Definition 2.3. Let M be a graded A-bimodule. We define the Hochschild cohomology $\mathrm{HH}^{n,*}(A,M)$ as the nth cohomology of the cochain complex

$$\underline{\mathrm{Hom}}_{A^e}(B(A),M)$$

of graded $\mathbb{F}_2$-vector spaces. When $M = A$ we will write $\mathrm{HH}(A) := \mathrm{HH}(A,A)$.

We note that the groups $\mathrm{HH}^{*,*}(A,M)$ are equipped with a cohomological grading, and an internal grading induced by the grading of A and M. We can describe $\mathrm{HH}^{n,s}(A,M)$ more concretely as follows. Using the natural contracting isomorphism

$$\underline{\mathrm{Hom}}_{A^e}(A^e \otimes A^{\otimes n}, M) \cong \underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes n}, M)$$

we see that $\mathrm{HH}^{n,*}(A,M)$ is isomorphic to the nth cohomology of the complex

$$\cdots \to \underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes n-1}, M) \xrightarrow{\partial} \underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes n}, M) \xrightarrow{\partial} \underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes n+1}, M) \to \cdots, \tag{1}$$

where the differentials are given by

$$\begin{aligned}\partial(f)(a_1 \otimes \cdots \otimes a_{n+1}) = {} & a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ & + \sum_{i=1}^{n} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ & + f(a_1 \otimes \cdots \otimes a_n) a_{n+1}.\end{aligned}$$

Remark 2.4. By Proposition 2.1, we see that $\mathrm{HH}(A,M)$ computes the graded Ext modules $\underline{\mathrm{Ext}}_{A^e}(A,M)$. In particular, we can compute $\mathrm{HH}(A,M)$ using any free resolution of A as a graded A-bimodule.

We now assume that the reader is familiar with A_∞-algebras. For an introduction to the theory of A_∞-algebras and references with all details we refer to [7]. Let $(\mathcal{C}^\bullet, \delta, \cup)$ be a differential graded algebra (DGA) over $\mathbb{F}_2$ with cohomology algebra $H^\bullet$. By the work of Kadeishvili [4, 5] (see also [6], [7], and [8]), one can equip $H^\bullet$ with the structure of an A_∞-algebra $(H^\bullet, \{m_n\}_{n\geqslant 1})$ such that $m_1 = 0$ together with a quasi-isomorphism of A_∞-algebras $(H^\bullet, \{m_n\}) \xrightarrow{\simeq} (\mathcal{C}^\bullet, \delta, \cup)$. The A_∞-algebra $(H^\bullet, \{m_n\})$ is called a *minimal model* of $(\mathcal{C}^\bullet, \delta, \cup)$. A DGA is called A_∞*-formal* if its minimal model can be chosen such that $m_n = 0$ for all $n \geqslant 3$. We now consider the following weaker notion.

Definition 2.5. Let $\mathcal{C}^\bullet$ be a DGA. We say that $\mathcal{C}^\bullet$ is A_3*-formal* if its minimal model can be chosen such that $m_3 = 0$.

We recall the following special case from [2, Theorem C]. Let $a, b, c \in H^\bullet$ be cohomology classes such that $a \cup b = 0$ and $b \cup c = 0$. Then $m_3(a \otimes b \otimes c) \in \langle a, b, c\rangle$. This implies the following well-known fact.

Proposition 2.6. *Let $\mathcal{C}^\bullet$ be a DGA. Assume that $\mathcal{C}^\bullet$ is A_3-formal. Then the triple Massey product $\langle a, b, c\rangle$ contains zero.* □

We note that m_3 is a homomorphism $m_3 \colon (H^\bullet)^{\otimes 3} \to H^\bullet[-1]$ of graded algebras, and we can construct m_3 as follows (see for example [1, Section 5], [8]). We choose an $\mathbb{F}_2$-linear graded map $f_1 \colon H^\bullet \to \mathrm{Ker}\,\delta$ which induces the identity on $H^\bullet$. Since f_1

is multiplicative on cohomology, we can find a graded $\mathbb{F}_2$-linear map $f_2\colon H^\bullet \otimes H^\bullet \to \mathcal{C}^\bullet$ of degree -1 satisfying

$$\delta(f_2(a \otimes b)) = f_1(a \cup b) + f_1(a) \cup f_1(b).$$

Now we define a graded $\mathbb{F}_2$-linear map $\Phi_3\colon (H^\bullet)^{\otimes 3} \to \mathcal{C}^\bullet[-1]$ by

$$\Phi_3(a \otimes b \otimes c) := f_1(a) f_2(b \otimes c) + f_2(a \otimes b) f_1(c) + f_2((ab) \otimes c + a \otimes (bc)) \tag{2}$$

for all homogeneous elements $a, b, c \in H^\bullet$ where we write xy for the product $x \cup y$ to shorten the notation. We check that Φ_3 has image in the cocycles of $\mathcal{C}^\bullet$, and hence Φ_3 induces a graded map $[\Phi_3]\colon (H^\bullet)^{\otimes 3} \to H^\bullet[-1]$. We set $m_3 := [\Phi_3]$. By [1, Proposition 5.4], m_3 is a cocycle in the complex (1). By [1, Corollary 5.7], the corresponding Hochschild cohomology class $[m_3] \in \mathrm{HH}^{3,-1}(H^\bullet)$ is independent of the choice of f_1 and f_2, and it is called the *canonical class* of $\mathcal{C}^\bullet$ following Benson–Krause–Schwede who studied this class as an obstruction to the realizability of modules over Tate cohomology in [1]. The following result is a modified version of Kadeishvili's theorem [5] (see also [13, Theorem 4.7]).

Theorem 2.7. *Let $\mathcal{C}^\bullet$ be a DGA with canonical class $[m_3] \in \mathrm{HH}^{3,-1}(H^\bullet)$. Then $\mathcal{C}^\bullet$ is A_3-formal if and only if $[m_3] = 0$.*

Proof. If $\mathcal{C}^\bullet$ is A_3-formal then m_3 is a trivial cocycle, and the class of m_3 vanishes in $\mathrm{HH}^{3,-1}(H^\bullet)$. Now we assume that $[m_3] = 0$ in $\mathrm{HH}^{3,-1}(H^\bullet)$. We may assume that Φ_3 and hence m_3 is constructed using maps f_1, f_2 as in (2). Then there exists an $\mathbb{F}_2$-linear map $\eta\colon (H^\bullet)^{\otimes 2} \to (\operatorname{Ker}\delta)[-1]$ such that $\partial^2[\eta] = m_3$ as maps $(H^\bullet)^{\otimes 3} \to H^\bullet[-1]$. We set $\tilde{f}_2 = f_2 + \eta$. We note that $\tilde{f}_2$ satisfies

$$\delta \tilde{f}_2(a \otimes b) = \delta(f_2(a \otimes b) + \eta(a \otimes b)) = f_1(a \cup b) + f_1(a) \cup f_1(b)$$

since $\delta \circ \eta = 0$. We then define the map $\widetilde{\Phi}_3$ by replacing f_2 with $\tilde{f}_2$, i.e., we define

$$\widetilde{\Phi}_3(a \otimes b \otimes c) := f_1(a) \tilde{f}_2(b \otimes c) + \tilde{f}_2(a \otimes b) f_1(c) + \tilde{f}_2((ab) \otimes c + a \otimes (bc))$$

for all homogeneous elements $a, b, c \in H^\bullet$ where we again write xy for $x \cup y$ to shorten the notation. We then have

$$(\Phi_3 - \widetilde{\Phi}_3)(a \otimes b \otimes c) = f_1(a)\eta(b \otimes c) + \eta(a \otimes b) f_1(c) + \eta((ab) \otimes c + a \otimes (bc)).$$

By definition of ∂^2 and the assumption on η, this implies $\widetilde{\Phi}_3 = \Phi_3 - \partial^2\eta = 0$ as maps $(H^\bullet)^{\otimes 3} \to H^\bullet[-1]$. $\square$

As a direct consequence we get:

Corollary 2.8. *Let A be a graded algebra with $\mathrm{HH}^{3,-1}(A) = 0$. Then every DGA $\mathcal{C}^\bullet$ whose cohomology algebra is isomorphic to A is A_3-formal.* $\square$

By Proposition 2.6 and Corollary 2.8, in order to show Theorem 1.1 it will suffice to show $\mathrm{HH}^{3,-1}(\mathbb{F}_2[a,b,c]/(ab,bc)) = 0$. This is what we now set out to prove.

3. Koszul algebras

For a vector space V, let $T(V)$ denote its graded tensor algebra over $\mathbb{F}_2$. We recall that a graded $\mathbb{F}_2$-algebra $A = \bigoplus_{n \geqslant 0} A^n$ is called *quadratic* if the map $T(A^1) \to A$ is surjective with kernel generated by elements in $A^1 \otimes A^1$. We see that any quadratic algebra A is canonically isomorphic to $T(V)/(R)$, for a vector space of generators V and a subspace of relations $R \subseteq V \otimes V$. For any quadratic algebra we can define the following chain complex of free graded A-bimodules.

Definition 3.1. Let $A = T(V)/(R)$ be a quadratic algebra. For $n \geqslant 0$ and $1 \leqslant i \leqslant n-1$, let

$$X_i^n = V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1} \subseteq V^{\otimes n} \tag{3}$$

and

$$K'_n = \bigcap_{i=1}^{n-1} X_i^n \subseteq V^{\otimes n}.$$

Here we interpret the empty intersection as the whole space, i.e., $K'_0 = \mathbb{F}_2$ and $K'_1 = V$. The Koszul complex $K(A^e, A)$ of A is defined as the nonnegative chain complex of graded A-bimodules with

$$K_n(A^e, A) = A \otimes K'_n \otimes A,$$

and differential d_n induced by the one in the bar resolution $B(A)$, i.e.,

$$d_n : a \otimes v_1 \otimes \cdots \otimes v_n \otimes b \mapsto av_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes b + a \otimes v_1 \otimes \cdots \otimes v_n b.$$

Definition 3.2. A quadratic algebra A is called *Koszul* if its Koszul complex $K(A^e, A)$ is a resolution of A as a graded A-bimodule, i.e., if $H_n(K(A^e, A)) = 0$ for $n > 0$ and $H_0(A^e, A) = A$.

We will now show that $A = \mathbb{F}_2[a, b, c]/(ab, bc)$ is Koszul. Consider the $\mathbb{F}_2$-vector spaces $V := \mathrm{Span}_{\mathbb{F}_2}\{a, b, c\}$ and

$$R := \mathrm{Span}_{\mathbb{F}_2}\{a \otimes b, b \otimes a, c \otimes b, b \otimes c, a \otimes c + c \otimes a\} \subseteq V \otimes V,$$

chosen such that we can identify A with $T(V)/(R)$, and in particular we see that A is quadratic. Let $e := a \otimes c + c \otimes a$. In the space X_i^n defined in (3) we consider the set $\mathcal{B}^n$ consisting of strings $x = x_1 \cdots x_k$ of symbols from the set $\{a, b, c, e\}$ such that $|x_1| + \cdots + |x_k| = n$ and ca does not occur as a substring of x. Here $|x_i|$ denotes the degree of the symbol x_i, so $|a| = |b| = |c| = 1$ and $|e| = 2$. We will identify the strings in $\mathcal{B}^n$ with tensors in $V^{\otimes n}$. As an example we see that

$$\mathcal{B}^2 = \{a \otimes a, a \otimes b, a \otimes c, b \otimes a, b \otimes b, b \otimes c, c \otimes b, c \otimes c, e\}.$$

We also introduce the subsets $\mathcal{B}_i^n$ for $1 \leqslant i \leqslant n-1$ consisting of the strings in $\mathcal{B}^n$ where the $(i, i+1)$-part is in R, where we count e with multiplicity 2. More precisely, for a string $x \in \mathcal{B}^n$ and an integer i, $1 \leqslant i \leqslant n$, we obtain a symbol $x_{(i)}$ by the following process. We first modify the string x into a string x' of length n by doubling every occurrence of e in x, and we then set $x_{(i)} = x'_i$. We now define

$$\mathcal{B}_i^n := \{x \in \mathcal{B}^n \mid x_{(i)} x_{(i+1)} \in \{ab, ba, cb, bc, eb, be, ee\}\}.$$

For instance, for the string $x = bebe$ we get that $x' = beebee$ such that, e.g., $x_{(4)} = b$, and we see that $x \in \mathcal{B}_i^6$ for all $1 \leqslant i \leqslant 5$.

Lemma 3.3. *For each n, the set $\mathcal{B}^n$ is a basis for $V^{\otimes n}$.*

Proof. From the basis $\{a, b, c\}$ of V we obtain a standard basis of $V^{\otimes n}$ consisting of the pure tensors in the symbols $\{a, b, c\}$. We obtain $\mathcal{B}^n$ from this standard basis using only elementary column operations, hence showing that $\mathcal{B}^n$ is also a basis. Starting from the standard basis, we can replace each occurrence of $c \otimes a$ with e by adding a suitable linear combination of standard pure tensors. More formally, we will do this using induction.

For $i \geqslant 0$, let $\mathcal{B}^{n,i}$ be the set of strings $x_1 \cdots x_k$ in the symbols $\{a, b, c, e\}$ satisfying the conditions:

- $|x_1| + \cdots + |x_k| = n$,
- there is at most i occurrences of e,
- the substring ca only occurs after all the e's,
- if there is less than i occurrences of e, there are no occurrences of ca.

We see that $\mathcal{B}^{n,0}$ is the standard basis, while $\mathcal{B}^{n,i} = \mathcal{B}^n$ for large enough i (e.g., for $i \geqslant n/2$). We will show that each $\mathcal{B}^{n,i+1}$ can be obtained from $\mathcal{B}^{n,i}$ using only elementary column operations, where we have identified the strings with tensors in $V^{\otimes n}$. We obtain $\mathcal{B}^{n,i+1}$ from $\mathcal{B}^{n,i}$ in the following manner. If $x \in \mathcal{B}^{n,i}$ have no occurrence of ca, then we already have $x \in \mathcal{B}^{n,i+1}$, so we do nothing. Otherwise, there are precisely i occurrences of e and at least one occurrence of ca in x. Consider the string x' obtained by replacing the first occurrence of ca with ac. We see that $x' \in \mathcal{B}^{n,i}$ and we add x' to x to obtain the tensor $x' + x \in \mathcal{B}^{n,i+1}$. All tensors in $\mathcal{B}^{n,i+1}$ can be obtained in precisely one of these two ways, hence we obtain $\mathcal{B}^{n,i+1}$ from $\mathcal{B}^{n,i}$ as wanted. □

Now we can prove the main result of this section.

Proposition 3.4. *The quadratic algebra $A = \mathbb{F}_2[a, b, c]/(ab, bc)$ is Koszul.*

Proof. We will show that, for each $n \geqslant 0$, the set $\mathcal{B}^n$ is a basis for $V^{\otimes n}$ which distributes the subspaces $X_1^n, \ldots, X_{n-1}^n$, i.e., for each X_i^n the subset $\mathcal{B}_i^n \subseteq \mathcal{B}^n$ forms a basis for X_i^n. By [12, Chapter 2, Theorem 4.1], this implies that A is Koszul. By definition, $X_i^n = V^{\otimes i-1} \otimes R \otimes V^{n-i-1}$, hence from the standard basis of V and the basis $\mathcal{R} = \{a \otimes b, b \otimes a, c \otimes b, b \otimes c, e\}$ of R, we obtain a basis of X_i^n consisting of strings x in the symbols $\{a, b, c, e\}$ with at most one e and $x_{(i)}x_{(i+1)} \in \mathcal{R}$, where $x_{(i)}$ is the notation introduced to define $\mathcal{B}_i^n$. Using a similar induction argument as above, we replace each occurrence of ca with e in this basis using only elementary column operations to obtain a new basis for X_i^n. This gives precisely the set $\mathcal{B}_i^n \subseteq \mathcal{B}^n$, which shows that $\mathcal{B}^n$ distributes $X_1^n, \ldots, X_{n-1}^n$. □

Remark 3.5. As a consequence of the proof of Proposition 3.4 we see that $\mathcal{B}'_n := \bigcap_{i=1}^{n-1} \mathcal{B}_i^n$ is a basis for $K'_n = \bigcap_{i=1}^{n-1} X_i^n$. This basis $\mathcal{B}'_n$ can be explicitly described as the set of strings $x_1 \cdots x_k$ in the symbols $\{a, b, c, e\}$ satisfying that $|x_1| + \cdots + |x_k|$ and the symbols in the string alternates between b and one from the set $\{a, c, e\}$. For example, we get

$$\mathcal{B}'_3 = \{aba, abc, cba, cbc, bab, bcb, eb, be\} \tag{4}$$

and

$$\mathcal{B}'_4 = \{abab, abcb, bcba, bcbc, baba, babc, cbab, cbcb, abe, eba, cbe, ebc, beb\}. \tag{5}$$

4. Proof of the main result

By Proposition 2.6 and Corollary 2.8, Theorem 1.1 will follow from the following:

Theorem 4.1. *We have* $\mathrm{HH}^{3,-1}(\mathbb{F}_2[a, b, c]/(ab, bc)) = 0$.

Proof. Since A is Koszul, the natural inclusion $K(A^e, A), A) \hookrightarrow B(A)$ is a quasi-isomorphism. Hence we can compute $\mathrm{HH}(A)$ as the cohomology of the complex $\underline{\mathrm{Hom}}_{A^e}(K(A^e, A), A)$. We first observe that we have

$$K(A^e, A)_n = A \otimes K'_n \otimes A \cong A^e \otimes K'_n,$$

as graded A-bimodules. Using the contracting isomorphism

$$\underline{\mathrm{Hom}}_{A^e}(A^e \otimes K'_n, A) \cong \underline{\mathrm{Hom}}_{\mathbb{F}_2}(K'_n, A)$$

of graded vector spaces we see that $\mathrm{HH}(A)$ can be computed as the cohomology of the following complex of graded vector spaces:

$$\cdots \to \underline{\mathrm{Hom}}_{\mathbb{F}_2}(K'_{n-1}, A) \xrightarrow{\partial^{n-1}} \underline{\mathrm{Hom}}_{\mathbb{F}_2}(K'_n, A) \xrightarrow{\partial^n} \underline{\mathrm{Hom}}_{\mathbb{F}_2}(K'_{n+1}, A) \to \cdots$$

where the differential ∂^n is given by

$$\partial^n(f)(v_1 \otimes \cdots \otimes v_{n+1}) = v_1 f(v_2 \otimes \cdots \otimes v_{n+1}) + f(v_1 \otimes \cdots \otimes v_n)v_{n+1}.$$

To compute $\mathrm{HH}^{3,-1}(A) = 0$, we need to show that

$$\mathrm{Hom}_{\mathbb{F}_2}(K'_2, A^1) \xrightarrow{\partial^2} \mathrm{Hom}_{\mathbb{F}_2}(K'_3, A^2) \xrightarrow{\partial^3} \mathrm{Hom}_{\mathbb{F}_2}(K'_4, A^3)$$

is exact in the middle, i.e., $\mathrm{Im}\,\partial^2 = \mathrm{Ker}\,\partial^3$.

First we will describe $\mathrm{Ker}\,\partial^3$. To do so, we use the following notation for elements in the basis $\mathcal{B}'_4$ of K'_4. Since a and c play symmetrical roles in A, we will introduce the notation $(a|c)$ to mean that each of a and c can be used in the expression. For example, for a map $f \in \mathrm{Hom}_{\mathbb{F}_2}(K'_3, A^2)$, the equation $f((a|c) \otimes b) = 0$ would mean that we have two equations $f(a \otimes b) = 0$ and $f(c \otimes b) = 0$. If there are several instances of $(a|c)$ in the expression, each instance can be replaced by a or c independently of each other.

Lemma 4.2. *A map $f \in \mathrm{Hom}_{\mathbb{F}_2}(K'_3, A^2)$ lies in $\mathrm{Ker}\,\partial^3$ if and only if it satisfies the following relations:*

$$f(b \otimes (a|c) \otimes b) \in \mathrm{Span}_{\mathbb{F}_2}\{b^2\}, \tag{i}$$

$$f((a|c) \otimes b \otimes (a|c)) \in \mathrm{Span}_{\mathbb{F}_2}\{a^2, ac, c^2\}, \tag{ii}$$

$$a(f(c \otimes b \otimes a) + f(e \otimes b)) + cf(a \otimes b \otimes a) = 0, \tag{iii}$$

$$c(f(a \otimes b \otimes c) + f(e \otimes b)) + af(c \otimes b \otimes c) = 0, \tag{iv}$$

$$a(f(a \otimes b \otimes c) + f(b \otimes e)) + cf(a \otimes b \otimes a) = 0, \tag{v}$$

$$c(f(c \otimes b \otimes a) + f(b \otimes e)) + af(c \otimes b \otimes c) = 0, \tag{vi}$$

$$f(e \otimes b) + f(b \otimes e) \in \mathrm{Span}_{\mathbb{F}_2}\{a^2, ac, c^2\}. \tag{vii}$$

Proof. A map $f \in \mathrm{Hom}_{\mathbb{F}_2}(K'_3, A^2)$ is in the kernel of ∂^3 if and only if $\partial^3(f)$ vanishes on all elements of the basis $\mathcal{B}'_4$. We now evaluate $\partial^3(f)$ on $\mathcal{B}'_4$ as described in (5). First, since $(a|c)b = 0$ in A^2, the equation

$$\partial^3(f)((a|c) \otimes b \otimes (a|c) \otimes b) = (a|c)f(b \otimes (a|c) \otimes b) + f((a|c) \otimes b \otimes (a|c))b = 0$$

implies $f(b \otimes (a|c) \otimes b) \in \mathrm{Span}_{\mathbb{F}_2}\{b^2\}$, and $f((a|c) \otimes b \otimes (a|c)) \in \mathrm{Span}_{\mathbb{F}_2}\{a^2, ac, c^2\}$. The equation $\partial^3(f)(b \otimes (a|c) \otimes b \otimes (a|c)) = 0$ gives the same relations. This shows that (i) and (ii) are necessary and sufficient. Second, (iii) and (iv) are imposed by the equations

$$\partial^3(f)(e \otimes b \otimes a) = af(c \otimes b \otimes a) + cf(a \otimes b \otimes a) + f(e \otimes b)a = 0$$

and

$$\partial^3(f)(e \otimes b \otimes c) = af(c \otimes b \otimes c) + cf(a \otimes b \otimes c) + f(e \otimes b)c = 0.$$

Similarly, (v) and (vi) are imposed by the equations

$$\partial^3(f)(a \otimes b \otimes e) = af(b \otimes e) + f(a \otimes b \otimes a)c + f(a \otimes b \otimes c)a = 0$$

and

$$\partial^3(f)(c \otimes b \otimes e) = cf(b \otimes e) + f(c \otimes b \otimes a)c + f(c \otimes b \otimes c)a = 0.$$

Finally, the condition

$$\partial^3(f)(b \otimes e \otimes b) = bf(e \otimes b) + f(b \otimes e)b = 0$$

gives the relation $f(e \otimes b) + f(b \otimes e) \in \mathrm{Span}_{\mathbb{F}_2}\{a^2, ac, c^2\}$ which is (vii). □

Notation 4.3. For $v \in \mathcal{B}'_n$ and $x \in A^i$, we write $F_n(v; x)$ for the map in $\mathrm{Hom}_{\mathbb{F}_2}(K'_n, A^i)$ sending v to x and other basis vectors in $\mathcal{B}'_n$ to zero.

Lemma 4.4. *The set of maps*

$$\begin{aligned}
\mathcal{S}_3 := \Big\{ \partial^2(F_2(b \otimes a; b)) &= F_3(b \otimes a \otimes b; b^2), \\
\partial^2(F_2(b \otimes c; b)) &= F_3(b \otimes c \otimes b; b^2), \\
\partial^2(F_2(e; b)) &= F_3(b \otimes e; b^2) + F_3(e \otimes b; b^2), \\
\partial^2(F_2(c \otimes b; a)) &= F_3(e \otimes b; a^2) + F_3(c \otimes b \otimes a; a^2) + F_3(c \otimes b \otimes c; ac), \\
\partial^2(F_2(b \otimes c; a)) &= F_3(b \otimes e; a^2) + F_3(a \otimes b \otimes c; a^2) + F_3(c \otimes b \otimes c; ac), \\
\partial^2(F_2(a \otimes b; a)) &= F_3(e \otimes b; ac) + F_3(a \otimes b \otimes c; ac) + F_3(a \otimes b \otimes a; a^2), \\
\partial^2(F_2(a \otimes b; c)) &= F_3(e \otimes b; c^2) + F_3(a \otimes b \otimes c; c^2) + F_3(a \otimes b \otimes a; ac), \\
\partial^2(F_2(b \otimes a; c)) &= F_3(b \otimes e; c^2) + F_3(c \otimes b \otimes a; c^2) + F_3(a \otimes b \otimes a; ac), \\
\partial^2(F_2(c \otimes b; c)) &= F_3(e \otimes b; ac) + F_3(c \otimes b \otimes a; ac) + F_3(c \otimes b \otimes c; c^2), \\
\partial^2(F_2(b \otimes c; c)) &= F_3(b \otimes e; ac) + F_3(a \otimes b \otimes c; ac) + F_3(c \otimes b \otimes c; c^2) \Big\}
\end{aligned}$$

is a basis of $\mathrm{Ker}\, \partial^3$.

Proof. We consider the set $\mathcal{S}'_3$ of $\mathbb{F}_2$-linear maps $K'_3 \to A^2$ defined by

$$\begin{aligned}
\mathcal{S}'_3 := \Big\{ &F_3(b \otimes a \otimes b; b^2), F_3(b \otimes c \otimes b; b^2), F_3(b \otimes e; b^2), F_3(e \otimes b; a^2), F_3(b \otimes e; a^2), \\
&F_3(a \otimes b \otimes a; a^2), F_3(e \otimes b; c^2), F_3(b \otimes e; c^2), F_3(c \otimes b \otimes a; ac), F_3(b \otimes e; ac) \Big\}.
\end{aligned}$$

We observe that each element of $\mathcal{S}'_3$ occurs exactly once as a term in one of the maps in $\mathcal{S}_3$. Since $\mathcal{A}_2 := \{a^2, b^2, c^2, ac\} \subseteq A^2$ is a basis of A^2, it follows that the set $\mathcal{S}_3$ is linearly independent. It remains to show that $\mathcal{S}_3$ generates $\mathrm{Ker}\, \partial^3$. Let $f \in \ker \partial^3$ be an arbitrary element. For $v \in \mathcal{B}'_3$ and $x \in \mathcal{A}_2$, let $\varphi(v; x) \in \mathbb{F}_2$ be the coefficient such that, for all $v \in \mathcal{B}'_3$,

$$f(v) = \sum_{x \in \mathcal{A}_2} \varphi(v; x)x.$$

Again, since each element of $\mathcal{S}'_3$ occurs exactly once as a term in one of the maps in $\mathcal{S}_3$, we can assume by adding a linear combination of the elements of $\mathcal{S}_3$ to f that the coefficients

$$\begin{aligned}
&\varphi(b \otimes a \otimes b; b^2), \varphi(b \otimes c \otimes b; b^2), \varphi(b \otimes e; b^2), \varphi(e \otimes b; a^2), \varphi(b \otimes e; a^2), \\
&\varphi(a \otimes b \otimes a; a^2), \varphi(e \otimes b; c^2), \varphi(b \otimes e; c^2), \varphi(c \otimes b \otimes a; ac), \varphi(b \otimes e; ac)
\end{aligned}$$

are all zero. Now it suffices to show that then f must be the zero map. Since $f \in \ker \partial^3$, f must satisfy the relations in Lemma 4.2. We see that (i) implies that $\varphi(b \otimes (a|c) \otimes b; x) = 0$ for $x \neq b^2$. Hence we have $f(b \otimes (a|c) \otimes b) = 0$. We also see that (vii) implies that $\varphi(b\otimes e; b^2) = \varphi(e\otimes b; b^2) = 0$. We therefore have $f(b\otimes e) = 0$. Equation (v) implies that $\varphi(a\otimes b\otimes a; c^2) = 0$, and (ii) implies that $\varphi(a\otimes b\otimes a; b^2) = 0$. This shows that either $f(a\otimes b\otimes a) = ac$ or $f(a\otimes b\otimes a) = 0$. If $f(a\otimes b\otimes a) = ac$, then (v) implies that $f(a\otimes b\otimes c) = c^2$. Now (iv) forces $f(e\otimes b) = c^2$, which contradicts $\varphi(e\otimes b; c^2) = 0$. We thus have $f(a\otimes b\otimes a) = 0$. From (v) and (ii), we then get $f(a\otimes b\otimes c) = 0$. From (iii) we deduce that $\varphi(e\otimes b; ac) = \varphi(c\otimes b\otimes a; ac) = 0$, and hence $f(e\otimes b) = 0$. It now follows from (iii) and (ii) that $f(c\otimes b\otimes a) = 0$. Finally, from (vi) and (ii) we get $f(c\otimes b\otimes c) = 0$. This shows that f vanishes on all elements of the basis $\mathcal{B}'_3$, and hence f is the zero map. □

Since the elements in the set $\mathcal{S}_3$ belong to the subset $\operatorname{Im}\partial^2$ of $\operatorname{Ker}\partial^3$, Lemma 4.4 implies $\operatorname{Im}\partial^2 = \operatorname{Ker}\partial^3$. This finishes the proof of Theorem 4.1. □

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