REAL PROJECTIVE GROUPS ARE FORMAL

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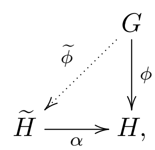
ABSTRACT. We prove that the mod 2 cohomology algebras of real projective groups are intrinsically formal. As a consequence we derive the Hopkins–Wickelgren formality and the strong Massey vanishing conjecture for a large class of fields.

1. Introduction

Let G be a profinite group. An *embedding problem for G* is a solid diagram:



where H and $\widetilde{H}$ are finite groups, the solid arrows are continuous homomorphisms and α is surjective. A *solution* of this embedding problem is a continuous homomorphism $\widetilde{\phi}\colon G \to \widetilde{H}$ which makes the diagram commutative. We say that the embedding problem above is *real* if for every involution $t \in G$ with $\phi(t) \neq 1$ there is an involution $\tilde{h} \in \widetilde{H}$ with $\alpha(\tilde{h}) = \phi(t)$, i.e., if involutions do not provide an obstruction for the existence of a solution.

Definition 1.1. Following Haran and Jarden [11] we say that a profinite group G is *real projective* if G has an open subgroup without 2-torsion, and if every real embedding problem for G has a solution.

For a prime number p and a profinite group G, let $C^*(G, \mathbb{F}_p)$ denote the differential graded $\mathbb{F}_p$-algebra of continuous cochains of G with values in $\mathbb{F}_p$. The aim of this paper is to show that the differential graded algebra $C^*(G, \mathbb{F}_p)$ of a real projective group G has the following strong property.

Definition 1.2. Let $\mathbb{F}$ be a field and let C^* be a differential graded $\mathbb{F}$-algebra with cohomology H^*. Then C^* is called *formal* if there is a sequence $C^* \leftarrow T^* \rightarrow H^*$ of quasi-isomorphisms of differential graded algebras between C^* and H^* where we consider H^* as a differential graded $\mathbb{F}$-algebra with trivial differential. Moreover, a graded $\mathbb{F}$-algebra A is called *intrinsically formal* if every differential graded algebra C^* with $H^*(C^*) \cong A$ is formal.

The main result of this paper is the following:

Theorem 1.3. *Let G be a real projective profinite group. Then the graded algebra $H^*(G, \mathbb{F}_2)$ is intrinsically formal. In particular, the differential graded $\mathbb{F}_2$-algebra $C^*(G, \mathbb{F}_2)$ is formal.*

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Remark 1.4. For an odd prime p and G real projective, the $\mathbb{F}_p$-algebra $H^*(G,\mathbb{F}_p)$ is concentrated in degrees 0 and 1, i.e., $H^i(G,\mathbb{F}_p)=0$ for $i\geq 2$. The intrinsic formality of $H^*(G,\mathbb{F}_p)$ then follows easily from the analog of Theorem 4.7 for an odd prime p (the result holds for augmented graded algebras over any field, see [36, Theorem 4.7, page 85]) and the argument of the proof of Lemma 6.5 below. Hence, together with Theorem 1.3, we can conclude that $H^*(G,\mathbb{F}_p)$ is intrinsically formal and therefore $C^*(G,\mathbb{F}_p)$ is formal for all primes p.

Our proof of Theorem 1.3 relies on a theorem due to Kadeishvili which states that a positively graded algebra A over a field is intrinsically formal if the Hochschild cohomology groups $\mathrm{HH}^{n,2-n}(A,A)$ vanish for all $n\geq 3$. In order to show this vanishing for $A=H^*(G,\mathbb{F}_2)$ we deduce from Scheiderer's work in [34] the description of $H^*(G,\mathbb{F}_2)$ for a real projective group G as the following type of quadratic algebra: First, let B be a Boolean ring, i.e., $x^2=x$ for every $x\in B$, and let $B_*=\bigoplus_{j\geq 0}B_j$ be its associated Boolean graded algebra, i.e., the $\mathbb{F}_2$-algebra with $B_0=\mathbb{F}_2$ and, for every positive integer j, $B_j=B$. Second, for an $\mathbb{F}_2$-vector space V, let $V_*=\bigoplus_{j\geq 0}V_j$ denote the graded algebra with $V_0=\mathbb{F}_2$, $V_1=V$ and $V_j=0$ for $j\geq 2$. We refer to Section 6 and Definitions 6.1 and 6.4 for more details. We show that $H^*(G,\mathbb{F}_2)$ is a quadratic algebra given as the connected sum of quadratic algebras $B_*\sqcap V_*$ (see Definition 5.2) where V is a suitable vector space and B is the Boolean ring of continuous functions from the space of conjugacy classes of involutions in G to $\mathbb{F}_2$. Then we calculate the Hochschild cohomology groups of such an algebra by an explicit description of the cocycles and coboundaries in the case of a locally finite Boolean graded algebra in Theorem 7.13. Finally, we use a spectral sequence argument and the vanishing of the higher derived limits of certain non-vanishing Hochschild cohomology groups to deduce the general case. Our main technical result is the following, see Theorem 9.14:

Theorem 1.5. *Let B be a Boolean ring with $|B|\geq 8$, and let V be an $\mathbb{F}_2$-vector space. Then*
$$\mathrm{HH}^{k,m}(B_*\sqcap V_*,B_*\sqcap V_*)=0$$
for every $m<0$ and $k\geq 0$ such that $k\neq 1-m$.

Remark 1.6. In fact, we thereby prove that for every profinite group G such that $H^*(G,\mathbb{F}_2)$ is of the form $B_*\sqcap V_*$ as in Theorem 1.5, the graded algebra $H^*(G,\mathbb{F}_2)$ is intrinsically formal. However, real projective groups are the main examples of groups with this type of cohomology algebra that we are aware of.

One of the main consequences of formality of a differential graded associative algebra is the vanishing of Massey products. We will formulate a stronger form of the latter next.

Definition 1.7. Let $\mathbb{F}$ be a field and let $(C^*,\cup,\delta)$ be a differential graded associative $\mathbb{F}$-algebra with differential δ. For an element $a\in C^d$ we write $\bar{a}:=(-1)^{1+d}a$. For an integer $n\geq 2$, let $a_1,a_2,\ldots,a_n$ be a set of cohomology classes with $a_i\in H^{d_i}$. A *defining system* for the n-fold Massey product of $a_1,a_2,\ldots,a_n$ is a set $\{a_{i,j}\}$ of elements of $C^{d_{i,j}}$ for $1\leq i<j\leq n+1$ and $(i,j)\neq(1,n+1)$ such that
$$\delta(a_{i,j})=\sum_{k=i+1}^{j-1}\bar{a}_{i,k}\cup a_{k,j}$$

and $a_1, a_2, \ldots, a_n$ is represented by $a_{1,2}, a_{2,3}, \ldots, a_{n,n+1}$. The degree $d_{i,j}$ of $a_{i,j}$ satisfies

$$d_{i,j} = \sum_{s=i}^{j-1} d_s - (j-1-i) \text{ for } i+1 \leq j \leq n+1.$$

We say that the *n-fold Massey product* of $a_1, a_2, \ldots, a_n$ is *defined* if there exists a defining system. The n-fold Massey product $\langle a_1, a_2, \ldots, a_n \rangle_{\{a_{i,j}\}}$ of $a_1, a_2, \ldots, a_n$ with respect to the defining system $\{a_{i,j}\}$ is the cohomology class of

$$\sum_{k=2}^{n} \bar{a}_{1,k} \cup a_{k,n+1}$$

in $H^{d_{n+1}}$ where $d_{n+1} = \sum_{i=1}^{n} d_i - n + 2$. Let $\langle a_1, a_2, \ldots, a_n \rangle$ denote the subset of $H^{d_{n+1}}$ consisting of the n-fold Massey products of $a_1, a_2, \ldots, a_n$ with respect to all defining systems. We say that the n-fold Massey product of $a_1, a_2, \ldots, a_n$ *vanishes* if $\langle a_1, a_2, \ldots, a_n \rangle$ contains zero.
We say that C^* *satisfies strong Massey vanishing* if for very $a_1, a_2, \ldots, a_n$ as above the following assertion holds: if

$$a_1 \cup a_2 = a_2 \cup a_3 = \cdots = a_{n-1} \cup a_n = 0, \tag{1}$$

then the n-fold Massey product of $a_1, a_2, \ldots, a_n$ vanishes. When condition (1) above is satisfied we say that *all neighbouring cup products vanish for the n-tuple* $a_1, a_2, \ldots, a_n$.

For real projective groups, strong Massey vanishing for cohomology classes in degree one was proven in [27, Theorem 1.7]. For pro p-groups of elementary type, strong Massey vanishing for cohomology classes in degree one was proven in [33, Theorem 1.2]. (See also the comments below Theorem 1.10.) As we explain in section 3, Theorem 1.3 implies the following result:

Theorem 1.8. *Let G be a real projective profinite group, and let p be a prime number. Then $C^*(G, \mathbb{F}_p)$ satisfies strong Massey vanishing for cohomology classes in arbitrary degrees.*

Remark 1.9. For an odd prime p, $C^*(G, \mathbb{F}_p)$ satisfies strong Massey vanishing since $H^i(G, \mathbb{F}_p) = 0$ for $i \geq 2$. Hence the difficult case is $p = 2$. We emphasise that Theorem 1.8 proves the strong Massey vanishing conjecture for real projective groups for cohomology classes in arbitrary degrees. We thereby provide the first nontrivial class of profinite groups for which strong Massey vanishing holds beyond degree one.

Our interest in the formality of profinite groups originates from the arithmetic of fields. Let F be any field whose characteristic is not two, and let $\Gamma(F)$ denote its absolute Galois group. In [13, Question 1.4 on page 1306] Hopkins and Wickelgren raised the question whether $C^*(\Gamma(F), \mathbb{F}_2)$ is formal. We know that the answer is no in general. In fact, Positselski proved in [29, Section 9.11] that, for an odd prime number p and a finite extension F of $\mathbb{Q}_p$ which contains a primitive p-root of unity, the differential graded algebra $C^*(\Gamma(F), \mathbb{F}_2)$ is not formal (see also [31, §6]). We note that there is a counterexample to the strong Massey vanishing for fourfold products over $\mathbb{Q}$, due to Wittenberg–Harpaz, in [9, Example A.15]. This implies that the absolute Galois group of $\mathbb{Q}$ is not formal. In the more recent paper [20]

Merkurjev–Scavia provide a counterexample over a field containing the algebraic closure of $\mathbb{Q}$, so even a more restricted prediction of Positselski on the formality of fields is false, see [20, Question 1.5 and Theorem 1.6] and [21]. However, we still expect that many fields have formal absolute Galois groups. Our result is actually the first affirmative contribution to this conjecture as we will explain next.

Recall that a field F has virtual cohomological dimension ≤ 1 if there is a finite extension K/F with $\mathrm{cd}(\Gamma(K)) \leq 1$. As we recall in section 2, it is a consequence of classical Artin–Scheier theory and Haran's work that the absolute Galois group of a field with virtual cohomological dimension ≤ 1 is real projective. Hence our results for real projective groups imply the following:

Theorem 1.10. *Let F be a field with virtual cohomological dimension ≤ 1, and let p be a prime number. Then $H^*(F, \mathbb{F}_p)$ is intrinsically formal, and hence $C^*(\Gamma(F), \mathbb{F}_p)$ is formal and satisfies strong Massey vanishing.*

This theorem is also a new contribution to the Massey vanishing conjecture by Mináč and Tân [22, Conjecture 1.1], a very active field of research (see e.g. [7], [9], [13], [20], [23], [27], [33]). In fact, our theorem shows Massey vanishing in arbitrary cohomology degrees for fields with virtual cohomological dimension ≤ 1. This is a much stronger result than in the previously known cases where only the degree one Massey products were considered.

Another consequence of our methods is a positive case of a conjecture by Positselski and Voevodsky on the Koszulity of Galois cohomology [30, §0.1, page 128]. In [25], other positive cases of this conjecture were proven. Recall that a positively graded algebra A over $\mathbb{F}_p$ is called *Koszul* if the groups $H_{i,j}(A) = \mathrm{Tor}^A_{i,j}(\mathbb{F}_p, \mathbb{F}_p)$ vanish for all $i \neq j$, where the first grading i is the homological grading and the second grading j is the internal grading which is induced from the grading of A. Based on Scheiderer's structure theorem for real projective groups we prove:

Theorem 1.11. *Let F be a field with virtual cohomological dimension ≤ 1, and let p be a prime number. Then the cohomology algebra $H^*(F, \mathbb{F}_p)$ is Koszul.*

Contents. In Section 2 we review the work of Haran–Jarden who showed that every real projective group arises as an absolute Galois group of a pseudo real closed field and deduce the structure of the mod 2 cohomology algebra of real projective groups from Scheiderer's local-global principle for real projective groups. In Section 3 we show that formality implies strong Massey vanishing for differential graded algebras. In Section 4 we review a Kadeishvili's sufficient condition for intrinsic formality in terms of graded Hochschild cohomology. In Section 5 we recall the Koszul complex of quadratic algebras and the notion of Koszul algebras. In Section 6 we introduce Boolean and dual graded algebras and show that their connected sum is Koszul. In Section 7 we prove our main technical result on the graded Hochschild cohomology of the connected sum of dual and Boolean graded algebras, first for finite Boolean subrings. In Section 8 we recall some facts needed on Mittag-Leffler functors. In Section 9 we extend the arguments from Section 7 to the case of an infinite Boolean ring and thereby prove the main technical result Theorem 1.5. In Section 10 we combine the results of the previous sections to prove Theorem 1.3 and its consequences. In Appendix A we provide proofs of the assertions on Boolean rings we use in Sections 6, 7 and 9. In Appendix B we prove the existence of the spectral sequence that we use in Section 9.

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2. Fields whose absolute Galois group is real projective and Scheiderer's theorem on their Galois cohomology

In this section, we first review the work of Haran–Jarden who showed that every real projective group arises as an absolute Galois group of a pseudo real closed field. Then we deduce from Scheiderer's local-global principle the structure of the mod 2 cohomology algebra of real projective groups.

For a profinite group G, let $\operatorname{cd}(G)$ denote its cohomological dimension as defined in [37]. Haran gave the following very useful characterisation of real projective groups [10, Proposition 2.2 on page 226]:

Theorem 2.1 (Haran)**.** *A profinite group G is real projective if and only if G has an open subgroup G_0 of index ≤ 2 with $\operatorname{cd}(G_0) \leq 1$, and every involution $t \in G$ is self-centralising, that is, we have $C_G(t) = \{1, t\}$.*

Haran's theorem immediately implies that every closed subgroup of a real projective group G is also real projective, and in particular every torsion element of G has order dividing 2.

For a field F, let $\Gamma(F)$ denote its absolute Galois group, and we write $\operatorname{cd}(F) = \operatorname{cd}(\Gamma(F)$. Recall that a field F has virtual cohomological dimension ≤ 1 if there is a finite extension K/F with $\operatorname{cd}(K) \leq 1$. Since the only torsion elements in the absolute Galois group of F are the involutions coming from the orderings of F, it is equivalent (by a theorem in [38]) to require $\operatorname{cd}(K) \leq 1$ for any fixed finite separable extension K of F without orderings, for example for $K = F(\mathbf{i})$, where $\mathbf{i} = \sqrt{-1}$. In particular, if F itself cannot be ordered (which is equivalent to -1 being a sum of squares in F), this condition is equivalent to $\operatorname{cd}(F) \leq 1$.

Examples 2.2. Examples of fields F which can be ordered with $\operatorname{cd}(F(\mathbf{i})) \leq 1$ include real closed fields, function fields in one variable over any real closed ground field, the field of Laurent series in one variable over any real closed ground field, and the field $\mathbb{Q}^{ab} \cap \mathbb{R}$ which is the subfield of $\mathbb{R}$ generated by the numbers $\cos(\frac{2\pi}{n})$, where $n \in \mathbb{N}$.

The following is a well-known consequence of Artin–Schreier theory.

Lemma 2.3. *Let Γ be an absolute Galois group and let $x \in \Gamma$ be an involution. Then the centraliser $C_\Gamma(x)$ is the group $C = \{1, x\}$ generated by x.*

Proof. Let K be a field whose absolute Galois group is isomorphic to Γ and let $\overline{K}$ be the separable closure of K. Let $R \subset \overline{K}$ be the fixed field of C. Then the absolute Galois group of R is $\operatorname{Gal}(\overline{K}/R) = C \cong \mathbb{Z}/2$, so R is real closed by [19, Proposition 2.4, page 452]. In particular, R has characteristic zero, so the same holds for all subfields of $\overline{K}$. Moreover, the latter is algebraically closed. Since $C_\Gamma(x)$ is the pre-image of 1 under the map $y \mapsto x^{-1}y^{-1}xy$, it is closed. Let $F \subset \overline{K}$ be the fixed field of $C_\Gamma(x)$. Since $C \subseteq C_\Gamma(x)$ we get that F is a subfield of R. Note that $C_\Gamma(x)$ leaves R invariant; indeed if $\alpha \in R$, then $x\alpha = \alpha$, and hence for every $y \in C_\Gamma(x)$ we have $xy\alpha = yx\alpha = y\alpha$, so $y\alpha \in R$. The extension R/F is algebraic, so R is the real closure of F. Therefore, the Galois group $\mathrm{Gal}(R/F)$ is trivial by [19, Theorem 2.9, page 455]. We get that $C_\Gamma(x)$ lies in the subgroup of Γ fixing R, so by the profinite version of the Galois correspondence $C_\Gamma(x) \subseteq C$. □

This shows that Theorem 2.1 has the following consequence:

Corollary 2.4. *The absolute Galois group of a field F is real projective if and only if F satisfies* $\mathrm{cd}(F(\mathbf{i})) \leq 1$. □

Definition 2.5. Let F be a field. The *real spectrum* $\mathrm{Spr}(F)$ of F is the set of all orderings of F. We consider $\mathrm{Spr}(F)$ as a topological space by equipping it with the *Harrison topology*. The latter is defined as the topology on $\mathrm{Spr}(F)$ with sub-basis consisting of the sets $H(a)$ of all orderings on F in which $a \in F$ is positive. By [18, Chapter VIII, §6, Theorem 6.3], the Harrison topology makes $\mathrm{Spr}(F)$ a compact and totally disconnected topological space.

Definition 2.6. For every ordering $<\in \mathrm{Spr}(F)$, let $F_<$ denote the real closure of the ordered field $(F, <)$. We say that the field F is *pseudo real closed* if every absolutely irreducible variety defined over F which has a simple $F_<$-rational point for every $<\in \mathrm{Spr}(F)$ has an F-rational point.

Remark 2.7. By the work of Haran–Jarden [11, Theorem 10.4 on page 487], the absolute Galois group of a pseudo real closed field is real projective. We can also deduce this result from the previous discussion as follows. Recall that a field F is *pseudo algebraically closed* if every geometrically irreducible variety defined over F has an F-rational point. By a theorem of Ax [2], the absolute Galois group Γ of such a field is projective, and in particular it has cohomological dimension at most one. By Prestel's Extension Theorem [32, Theorem 3.1, page 148], every algebraic extension of a pseudo real closed field is pseudo real closed. This implies that if F is pseudo real closed, then $F(\mathbf{i})$ is pseudo algebraically closed since $F(\mathbf{i})$ has no orderings. Hence F has virtual cohomological dimension ≤ 1, and in particular $\Gamma(F)$ is real projective by Haran's Theorem 2.1.

The following result of Haran–Jarden is a strong converse of the remark above (see [11, Theorem 10.4 on page 487]).

Theorem 2.8 (Haran–Jarden)**.** *For every real projective profinite group G, there is a pseudo real closed field F whose absolute Galois group is isomorphic to G.* □

Remark 2.9. Therefore, the Hopkins–Wickelgren formality conjecture for the class of fields of virtual cohomological dimension ≤ 1 is the same as for the class of pseudo real closed fields, and it is a purely group-theoretical problem for the class of real projective profinite groups.

Definition 2.10. Let Γ be a profinite group. Let $\mathcal{Y}(\Gamma)$ denote the subset of elements of order 2 in Γ. We equip $\mathcal{Y}(\Gamma)$ with the subset topology. Let $\mathcal{X}(\Gamma)$ denote the quotient of $\mathcal{Y}(\Gamma)$ by the conjugation action of Γ. We equip $\mathcal{X}(\Gamma)$ with the quotient topology. Note that when Γ is the absolute Galois group of a field F then $\mathcal{X}(\Gamma)$ is just the real spectrum of F equipped with its Harrison topology by classical Artin–Schreier theory.

Let $C(\mathcal{X}(\Gamma), \mathbb{F}_2)$ be the ring of continuous functions from $\mathcal{X}(\Gamma)$ to $\mathbb{F}_2$ where we equip the latter with the discrete topology.

Theorem 2.11 (Scheiderer)**.** *Let Γ be a real projective group and $\mathcal{X}(\Gamma)$ be as above. Then, for each $n \geq 0$, there is a natural homomorphism*

$$\pi_n \colon H^n(\Gamma, \mathbb{F}_2) \to C(\mathcal{X}(\Gamma), \mathbb{F}_2) \tag{2}$$

which is an isomorphism for $n > 1$ and surjective for $n = 1$.

Proof. In [34] Scheiderer constructs his theory of real étale cohomology in two cases: for schemes over which 2 is invertible, and for profinite groups. In the case we are interested in, real projective groups, the two approaches are actually equivalent by Theorem 2.8. The assertion then follows from [35, Proposition 2.15].

We will briefly indicate how to use Theorem 2.8 and the results of [34] to deduce the result. By Theorem 2.8, we can find a field F such that $\Gamma = \Gamma(F)$. By [34, Theorem 6.6, page 61] applied to the spectrum of F and the constant discrete Γ-module $A = \mathbb{F}_2$, there is a long exact sequence of the form

$$\cdots \to H^n_b(F, \mathbb{F}_2) \to H^n(\Gamma, \mathbb{F}_2) \xrightarrow{p^n_F} H^0(\mathcal{X}(\Gamma), \mathcal{H}^n(\mathbb{F}_2)) \to H^{n+1}_b(F, \mathbb{F}_2) \to \cdots \tag{3}$$

where $\mathcal{H}^n(\mathbb{F}_2)$ denotes the Γ-equivariant higher direct image sheaf constructed in [34, Section 8] (see especially [34, Remark 8.9, page 95]), and H_b denotes cohomology group with respect to the b-topology on $\mathrm{Spec}(F)$ as defined in [34, Definition 2.3, page 11]. For the identification of sequence (3) with the long exact sequence of [34, Theorem 6.6, page 61], we recall that $\mathcal{X}(\Gamma)$ is the real spectrum of the field F (see [34, 9.2, page 97]). We can then use [34, Remarks 9.5 and 9.6, page 99] to identify the remaining terms. In particular, we can identify the map p^n_F in (3) with the map π_n in (2). By [34, Theorem 7.3, page 70] (see also [34, Corollary 9.8]), the cohomological 2-dimension of F with respect to the b-topology is equal to the virtual cohomological 2-dimension of Γ. Since the latter is 1 by Theorem 2.1 and Corollary 2.4, the group $H^n_b(F, \mathbb{F}_2)$ in sequence (3) vanishes for $n > 1$. Hence, since sequence (3) is exact, the map π_n in (2) is an isomorphism for $n > 1$ and is still surjective for $n = 1$. □

Remark 2.12. We note that there is also a group-theoretic proof of Theorem 2.11 in [34, Section 12]. Since the virtual cohomological 2-dimension of a real projective group is 1, and Γ contains no subgroup isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2$ by Haran's Theorem 2.1, [34, Theorem 12.13, page 151] applies with coefficients the constant Γ-module $A = \mathbb{F}_2$, and the map π_n in (2) is an isomorphism for every $n > 1$. The case $n = 1$ is proven in [34, Remark 12.20.2, page 158] where the argument of the second paragraph applies since every involution is self-centralising. A proof of Theorem 2.11 using Artin–Schreier structures which are discussed in [34, Appendix B] can also be deduced from [34, Proposition B.8.1, page 257].

3. Massey products and formality

In this section we show that formality implies strong Massey vanishing. This is a well known fact which can for example be deduced from the theory of A_∞-algebras. However, for a lack of reference known to the authors to a proof which avoids the A_∞-structures, especially in positive characteristic, and for the convenience of the reader we provide direct and self-contained proofs for the assertions needed. In this section we only assume familiarity with some basic terminology for model categories, and we recall and use a basic fact about weak equivalences in Lemma 3.4.

Notation 3.1. Let $\mathbb{F}$ be a field, and let **DGA** denote the category of associative differential graded algebras over $\mathbb{F}$. We will often refer to the objects in **DGA** just as dg-algebras. We will often denote the multiplication on objects in **DGA** as a cup product $\cup$. Let C^* be a dg-algebra with differential $\delta\colon C^* \to C^{*+1}$, and cohomology $H^* = \mathrm{Ker}(\delta)/\mathrm{Im}(\delta)$. We will often write C instead of C^*.

We consider **DGA** as a model category with the standard model structure, as constructed in [14, Theorem 5, page 58]. It is determined by the following classes of weak equivalences and fibrations: a map is a *weak equivalence* if its underlying map of chain complexes is a quasi-isomorphism; and it is a *fibration* if it is degree-wise surjective. Recall that maps which are both weak equivalences and fibrations are also called *acyclic fibrations*. The *cofibrations* are the maps which have the left lifting property with respect to all acyclic fibrations.

Definition 3.2. Let C be a dg-algebra with cohomology H. We consider H as an object in **DGA** with trivial differential. Then C is called *formal* if there is a diagram

$$C \leftarrow T \rightarrow H \tag{4}$$

of weak equivalences of dg-algebras.

Remark 3.3. We recall that, since all objects are fibrant in the standard model structure on **DGA**, a sequence of quasi-isomorphisms

$$C \leftarrow T_1 \rightarrow T_2 \leftarrow T_3 \rightarrow \ldots \leftarrow T_n \rightarrow B$$

in **DGA** can be reduced to a sequence $C \leftarrow T \rightarrow B$ of just two quasi-isomorphism.

Lemma 3.4. *Let $f\colon A \to B$ be a weak equivalence of dg-algebras. Then there is a commutative diagram of morphisms of dg-algebras*

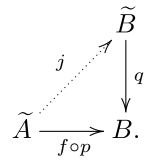
$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{j} & \widetilde{B} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{f} & B \end{array} \tag{5}$$

such that all maps are weak equivalences, the map j is a homotopy equivalence, and p and q are fibrations.

Proof. This is a formal consequence of the model structure on **DGA**. Let

$$A \xrightarrow{i_1} B_1 \xrightarrow{q_1} B$$

be a factorization of f into a cofibration i_1 followed by an acyclic fibration q_1. Since f is a weak equivalence, i_1 is a weak equivalence, too. If B_1 is not cofibrant, we choose a cofibrant replacement $\widetilde{B} \xrightarrow{\widetilde{q}} B_1$ such that $\widetilde{q}$ is an acyclic fibration and $\widetilde{B}$ is cofibrant. The composition $q = \widetilde{q} \circ q_1$ is an acyclic fibration as well. If A is not cofibrant, we choose again a cofibrant replacement $\widetilde{A} \xrightarrow{p} A$ such that p is an acyclic fibration and $\widetilde{A}$ is cofibrant. Now we consider the solid diagram



Since $\widetilde{A}$ is cofibrant and q is an acyclic fibration, there is a dotted lift j which makes the diagram commute. As $f \circ p$ is also a weak equivalence, j must be a weak equivalence. Since $\widetilde{A}$ and $\widetilde{B}$ are cofibrant and fibrant objects (every object in **DGA** is fibrant), j is a homotopy equivalence, in the sense that there is a map $\widetilde{B} \xrightarrow{\widetilde{h}} \widetilde{A}$ which is an inverse to j up to homotopy. This proves the claim. □

The existence of diagram (5) will allow us to lift Massey products also along quasi-isomorphisms of dg-algebras.

Proposition 3.5. *Let $f\colon A \to B$ be a weak equivalence of dg-algebras, and let $a_1, a_2, \dots, a_n$ be classes in $H^*(A)$. Let $b_1, b_2, \dots, b_n$ be their images in $H^*(B)$. Then $f_*\colon H^*(A) \to H^*(B)$ maps the Massey product set $\langle a_1, a_2, \dots, a_n\rangle$ bijectively onto the Massey product set $\langle b_1, b_2, \dots, b_n\rangle$.*

Proof. Let $\{a_{i,i+j}\} \subset A$ for $1 \le i < i+j \le n+1$ be a defining system for the n-fold Massey product of $a_1, \dots, a_n$. Since f commutes with the differentials and cup-products, it sends the defining system $\{a_{i,i+j}\}$ to a defining system $\{b_{i,i+j} := f(a_{i,i+j})\} \subset B$ for the n-fold Massey product of $b_1, \dots, b_n$. Hence f_* sends elements in the Massey product set $\langle a_1, a_2, \dots, a_n\rangle$ to elements in the Massey product set $\langle b_1, b_2, \dots, b_n\rangle$. More concretely, f_* sends the class of the cocycle $\sum_{k=2}^n \bar{a}_{1,k} \cup a_{k,n+1}$ in $\langle a_1, \dots, a_n\rangle$ to the class of the cocycle $\sum_{k=2}^n \bar{b}_{1,k} \cup b_{k,n+1}$ in $\langle b_1, \dots, b_n\rangle$. Since f_* is an isomorphism and therefore injective, it remains to show that every defining system for the n-fold Massey product of $b_1, \dots, b_n$ can be lifted to a defining system for the n-fold Massey product of $a_1, \dots, a_n$.

By Lemma 3.4, in order to prove the claim for f, it suffices to prove the corresponding claims for the maps p, q and j as in diagram (5). The claim for j follows from the following lemma.

Lemma 3.6. *If f is a homotopy equivalence of dg-algebras, then f_* restricts to a bijection between $\langle a_1, a_2, \dots, a_n\rangle$ and $\langle b_1, b_2, \dots, b_n\rangle$.*

Proof. Since f is a homotopy equivalence, we can choose a morphism of dg-algebras $g\colon B \to A$ of f such that there are quasi-isomorphisms $g \circ f \simeq \mathrm{id}_A$ and $f \circ g \simeq \mathrm{id}_B$. In particular, g is itself a quasi-isomorphism of dg-algebras, and g sends the defining system $\{b_{i,i+j}\}$ for the n-fold Massey product of $b_1, \dots, b_n$ to a defining system $\{a'_{i,i+j} := g(b_{i,i+j})\} \subset A$ for the n-fold Massey product of $a_1, \dots, a_n$. The defining system $\{a'_{i,i+j} := g(b_{i,i+j})\}$ is a lift of the defining system $\{b_{i,i+j}\}$. In fact, since $\sum_{k=2}^n \bar{b}_{1,k} \cup b_{k,n+1}$ is a cocycle in A, the quasi-isomorphism $g \circ f \simeq \mathrm{id}_A$ implies that

$$\sum_{k=2}^{n} \bar{a}_{1,k} \cup a_{k,n+1} - \sum_{k=2}^{n} \bar{a}'_{1,k} \cup a'_{k,n+1}$$

is a coboundary in A. Thus, the cohomology classes of the cocycles $\sum_{k=2}^n \bar{a}_{1,k} \cup a_{k,n+1}$ and $\sum_{k=2}^n \bar{a}'_{1,k} \cup a'_{k,n+1}$, respectively, are the same element in the Massey product set $\langle a_1, a_2, \dots, a_n\rangle \subset H^*(A)$. □

To prove the claim of Proposition 3.5 for the acyclic fibrations p and q, we are going to use the following basic statements about quasi-isomorphisms.

Lemma 3.7. *Let $f\colon A \to B$ be an acyclic fibration of dg-algebras. Let $b \in B^n$ be such that $\delta b = 0$. Then there is an $a \in A^n$ such that $f(a) = b$ and $\delta a = 0$.*

Proof. Since f is a quasi-isomorphism, there is an $a' \in A^n$ such that $\delta a' = 0$ and $f(a')$ and b have the same cohomology class. In other words, there is a $c \in B^{n-1}$ such that $f(a') = b - \delta c$. Since f is surjective, there is a $d \in A^{n-1}$ such that $f(d) = c$. Set $a = a' + \delta d$. Then $\delta a = \delta a' + \delta^2 d = 0$, and

$$f(a) = f(a') + f(\delta d) = f(a') + \delta f(d) = b - \delta c + \delta c = b. \qquad \square$$

Lemma 3.8. *Let $f\colon A \to B$ be an acyclic fibration of dg-algebras. Let $d \in A^n$ and $c \in B^{n-1}$ be such that $\delta d = 0$ and $\delta c = f(d)$. Then there is a $\bar{d} \in A^{n-1}$ such that $f(\bar{d}) = c$ and $\delta \bar{d} = d$.*

Proof. Since f is a quasi-isomorphism and the cohomology class of $f(d)$ is trivial, there is a $d' \in A^{n-1}$ such that $\delta d' = d$. Then $\delta c = f(\delta d') = \delta f(d')$, so $\delta(c - f(d')) = 0$. By Lemma 3.7 above there is an $e \in A^{n-1}$ such that $\delta e = 0$ and $f(e) = c - f(d')$. Set $\bar{d} = d' + e$. Then $\delta(\bar{d}) = \delta(d') + \delta(e) = d + 0 = d$, and $f(\bar{d}) = f(d') + f(e) = f(d') + c - f(d') = c$. $\square$

We are now ready to conclude the proof of Proposition 3.5. Since p and q are acyclic fibrations, the proof is reduced to the case when f is an acyclic fibration. Therefore we assume from now on that f is an acyclic fibration. We need to show that any defining system for the n-fold Massey product of $b_1, \ldots, b_n$ can be lifted to a defining system of $a_1, \ldots, a_n$. Let $\{b_{i,i+j}\} \subset B$ for $1 \le i < i+j \le n+1$ be a defining system for the n-fold Massey product of $b_1, \ldots, b_n$. By Lemma 3.7, we can choose cocycles $a_{1,2}, a_{2,3}, \ldots, a_{n,n+1} \in A$ with $f(a_{i,i+1}) = b_{i,i+1}$ and $a_{i,i+1}$ represents the cohomology class a_i for each i. Next, for a given $j \ge 2$, assume that we have constructed elements $a_{i,i+l}$ for all i and all $l < j$ satisfying $f(a_{i,i+l}) = b_{i,i+l}$ and

$$\delta a_{i,i+l} = \sum_{k=i+1}^{i+l-1} \bar{a}_{i,k} \cup a_{k,i+l} \tag{6}$$

where we recall that $d_{i,k}$ denotes the degree of $a_{i,k}$ and $\bar{a}_{i,k} = (-1)^{1+d_{i,k}} a_{i,k}$. We claim that $d := \sum_{k=i+1}^{i+j-1} \bar{a}_{i,k} \cup a_{k,i+j}$ is a cocycle. Applying δ and using $(-1)^{1+d_{i,k}} \cdot (-1)^{d_{i,k}} = -1$ yields

$$\delta(d) = \sum_{k=i+1}^{i+j-1} \left((-1)^{1+d_{i,k}} \delta(a_{i,k}) \cup a_{k,i+j} - a_{i,k} \cup \delta(a_{k,i+j}) \right).$$

Using $\delta(a_{i,i+1}) = 0 = \delta(a_{i+j-1,i+j})$ and the relations given by (6) we get

$$\begin{aligned} \delta(d) = & \sum_{k=i+2}^{i+j-1} (-1)^{1+d_{i,k}} \left(\sum_{s=i+1}^{k-1} (-1)^{1+d_{i,s}} a_{i,s} \cup a_{s,k} \right) \cup a_{k,i+j} \\ & - \sum_{k=i+1}^{i+j-2} a_{i,k} \cup \left(\sum_{r=k+1}^{i+j-1} (-1)^{1+d_{kr}} a_{k,r} \cup a_{r,i+j} \right). \end{aligned}$$

Now we use that, for each triple $i < s < k$, we have $1 + d_{ik} = d_{is} + d_{sk}$ and get

$$\delta(d) = \sum_{k=i+2}^{i+j-1} \left(\sum_{s=i+1}^{k-1} a_{i,s} \cup \bar{a}_{s,k} \right) \cup a_{k,i+j} - \sum_{k=i+1}^{i+j-2} a_{i,k} \cup \left(\sum_{r=k+1}^{i+j-1} \bar{a}_{k,r} \cup a_{r,i+j} \right)$$

$$= \left(\sum_{i+1 \le s < k \le i+j-1} a_{i,s} \cup \bar{a}_{s,k} \cup a_{k,i+j} \right) - \left(\sum_{i+1 \le k < r \le i+j-1} a_{i,k} \cup \bar{a}_{k,r} \cup a_{r,i+j} \right) = 0.$$

Hence we can apply Lemma 3.8 with $d = \sum_{k=i+1}^{i+j-1} (-1)^{1+d_{i,k}} a_{i,k} \cup a_{k,i+j}$ and $c = b_{i,i+j}$ to get an element $a_{i,i+j} \in A$ which satisfies

$$f(a_{i,i+j}) = b_{i,i+j} \text{ and } \delta a_{i,i+j} = \sum_{k=i+1}^{i+j-1} \bar{a}_{i,k} \cup a_{k,i+j}.$$

Continuing this process we get a defining system for the classes $a_1, a_2, \dots, a_n$. □

As a consequence we get that formality implies strong Massey vanishing:

Theorem 3.9. *Every formal differential graded algebra satisfies strong Massey vanishing.*

Proof. Let C be a formal dg-algebra and let H denote the cohomology of C. By Proposition 3.5, it suffices to show that if we consider H as a dg-algebra with trivial differential, then for every n-tuple $a_1, \dots, a_n$ in H of elements whose neighbouring cup products vanish the corresponding Massey product is defined and vanishes in H. Indeed, let $\{a_{i,i+j}\} \in H$ for $1 \le i < j \le n+1$ and $(i,j) \ne (1, n+1)$ be such that $a_{i,i+1} = a_i$ for every index i and $a_{i,j} = 0$, if $|i-j| > 1$. Then $\{a_{i,i+j}\}$ is a defining system for the Massey product of $a_1, \dots, a_n$ whose corresponding Massey product vanishes. □

4. Graded Hochschild cohomology and formality

One way to prove the formality of a dg-algebra is to show that certain Hochschild cohomology groups vanish. We are first going to recall the definition of graded Hochschild cohomology and then discuss the relation to formality. Since we only consider algebras and dg-algebras over the field $\mathbb{F}_2$, we will from now on work over $\mathbb{F}_2$. Unless stated otherwise $\otimes$ will denote $\otimes_{\mathbb{F}_2}$.

Definition 4.1. Let $A = \bigoplus_{i \ge 0} A_i$ be a graded $\mathbb{F}_2$-algebra. A *graded A-module* is an A-module M with a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ into subgroups such that $A_i \cdot M_j \subseteq M_{i+j}$ for every $i \ge 0, j \in \mathbb{Z}$. A homomorphism of graded A-modules from $M = \bigoplus_{i \in \mathbb{Z}} M_i$ to $N = \bigoplus_{i \in \mathbb{Z}} N_i$ is an A-module homomorphism $\phi \colon M \to N$ which respects the gradings, i.e., we have $\phi(M_i) \subseteq N_i$ for every i. Let $\underline{\mathrm{Hom}}_A(M, N)$ denote the set of graded A-module homomorphisms from M to N; it has the structure of an A-module. Graded A-modules with graded A-module homomorphisms form an abelian category. For $s \in \mathbb{Z}$ and a graded A-module M, we write $M[s]$ for the graded A-module given in degree n by $M[s]_n = M_{n+s}$.

Let $A^e = A \otimes A^{\mathrm{op}}$ denote the enveloping algebra of A where A^{op} denotes the opposite algebra of A (see e.g., [41, Section 1.1]). The ring A^e inherits a structure of a graded $\mathbb{F}_2$-algebra from A. We now define the graded Hochschild cohomology groups of a graded $\mathbb{F}_2$-algebra A with coefficients in a graded A^e-module M. First, we recall the bar complex (see e.g., [41, Section 1.1]). For $n \geq 0$, we write $B_n(A) = A^{\otimes(n+2)}$ and define the map $(d_B)_n = d_n \colon B_n(A) \to B_{n-1}(A)$ by

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for $a_0, \ldots, a_{n+1} \in A$. Then the *bar complex* of A is the complex

$$(B(A), d_B) : \quad \cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A^{\otimes 2} \xrightarrow{\mu} A \to 0$$

where μ denotes the multiplication on A. For every $n \geq 0$, $B_n(A)$ may be considered as an A^e-module via

$$(a \otimes a') \cdot (a_0 \otimes \cdots \otimes a_{n+1}) = (aa_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes (a_{n+1}a').$$

We equip $B_n(A)$ with the grading induced by the grading of A, and we consider d_n as a graded homomorphism.

Definition 4.2. Let M be a graded A^e-module. The graded Hochschild cohomology group $\mathrm{HH}^{n,s}(A, M)$ is defined as the nth cohomology of the cochain complex $(\underline{\mathrm{Hom}}_{A^e}(B_*(A), M[s]), d_B^*)$ with differential $d_B^* f = f \circ d_B$.

Remark 4.3. We now recall that graded Hochschild cohomology can be computed using a slightly simpler complex. For every n, there is a natural $\mathbb{F}_2$-linear isomorphism

$$\underline{\mathrm{Hom}}_{A^e}(B_n(A), M) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes n}, M) \tag{7}$$

defined by sending a map f to the $\mathbb{F}_2$-linear map which sends $a_1 \otimes \cdots \otimes a_n$ to $f(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$. The maps $d_n^* \colon \underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes n}, M) \to \underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes(n+1)}, M)$, defined for every n by

$$\begin{aligned} d_n^*(f)(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 & f(a_2 \otimes \cdots \otimes a_{n+1}) \\ & + \sum_{i=1}^{n} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ & + f(a_1 \otimes \cdots \otimes a_n) a_{n+1}, \end{aligned}$$

turn $(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes *}, M), d^*)$ into a cochain complex. We can then check that isomorphism (7) induces an isomorphism of cochain complexes

$$(\underline{\mathrm{Hom}}_{A^e}(B_*(A), M), d_B^*) \xrightarrow{\cong} (\underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes *}, M), d^*).$$

Hence, for every n and s, we have an isomorphism

$$\mathrm{HH}^{n,s}(A, M) \cong H^n(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes *}, M[s]), d^*). \tag{8}$$

In fact, isomorphism (8) is used to define Hochschild cohomology in [36, Section 4, page 85] (see also [3, Section 4]).

We will now recall a criterion on the Hochschild cohomology of a graded algebra which implies formality. In fact, it implies the following stronger property.

Definition 4.4. A graded $\mathbb{F}_2$-algebra A is called *intrinsically formal* if every differential graded algebra C with $H^*(C) \cong A$ is formal.

Remark 4.5. We note that intrinsic formality is a property formulated for graded algebras while formality is a property formulated for dg-algebras. However, one may call a dg-algebra C intrinsically formal if its cohomology algebra is. Intrinsic formality is a much stronger property than formality.

An equivalent characterisation of intrinsic formality is given as follows:

Lemma 4.6. *Let A be a graded $\mathbb{F}_2$-algebra. Then A is intrinsically formal if and only if any two differential graded algebras with cohomology isomorphic to A are quasi-isomorphic.*

Proof. Let C be a dg-algebra with $H^*(C) \cong A$. If any two dg-algebras with cohomology isomorphic to A are quasi-isomorphic, then C and $H^*(C)$ are quasi-isomorphic. Hence C is formal. This shows that A is intrinsically formal.

To prove the other implication, let C_1 and C_2 be dg-algebras with cohomology algebras $H^*(C_1) \cong A \cong H^*(C_2)$. If A is intrinsically formal, then C_1 and C_2 are formal. This means that there are quasi-isomorphisms of dg-algebras $C_1 \leftarrow T_1 \to A$ and $A \leftarrow T_2 \to C_2$ where we consider A as a dg-algebra with trivial differential. By Remark 3.3, we can then find a dg-algebra T and quasi-isomorphisms of dg-algebras $C_1 \leftarrow T \to C_2$. Thus, C_1 and C_2 are quasi-isomorphic. □

The following theorem is a special case of a result due to Kadeishvili (see [16] and [17]). For a proof we refer to the work of Seidel and Thomas [36, Theorem 4.7, page 85].

Theorem 4.7 (Kadeishvili)**.** *Let $A = \bigoplus_{i\geq 0} A_i$ be a non-negatively graded $\mathbb{F}_2$-algebra with $A_0 = \mathbb{F}_2$. Assume that*

$$\mathrm{HH}^{n,2-n}(A) = 0 \textit{ for all } n \geq 3.$$

Then A is intrinsically formal. □

Theorem 4.7 and Lemma 4.6 then imply:

Corollary 4.8. *Let C be a dg-algebra over $\mathbb{F}_2$ with $C^0 = \mathbb{F}_2$. Assume that*

$$\mathrm{HH}^{n,2-n}(H^*(C)) = 0 \textit{ for all } n \geq 3.$$

Then C is formal. □

We will refer to the above criteria using the following terminology:

Definition 4.9. For a non-negatively graded $\mathbb{F}_2$-algebra $A = \bigoplus_{i\geq 0} A_i$ with $A_0 = \mathbb{F}_2$, we say that A satisfies *Kadeishvili's vanishing* if

$$\mathrm{HH}^{n,2-n}(A) = 0 \text{ for every } n \geq 3.$$

For a dg-algebra C over $\mathbb{F}_2$ with $C^0 = \mathbb{F}_2$, we say that C satisfies *Kadeishvili's vanishing* if the graded $\mathbb{F}_2$-algebra $H^*(C)$ satisfies Kadeishvili's vanishing.

Remark 4.10. For the reader familiar with A_∞-algebras we note that the condition of Theorem 4.7 can be interpreted and the proof can be summarised as follows: Let A be a graded algebra and let C be a dg-algebra with an isomorphism of graded algebras $A \xrightarrow{\cong} H^*(C)$. Recall that we can view C as as A_∞-algebra with trivial higher multiplication maps m_n for $n \geq 3$. Kadeishvili showed that it is always possible to equip A with the structure of an A_∞-algebra such that there is a quasi-isomorphism of A_∞-algebras $q\colon A \to C$ which lifts the isomorphism $A \xrightarrow{\cong} H^*(C)$ of graded algebras (see [16] and [17]). Since the higher level A_∞-maps m_n on A may be nontrivial for $n \geq 3$, the A_∞-algebra structure on A may be different from the one it inherits as a graded algebra. Moreover, q is not a morphism of dg-algebras in general. However, each map m_n is a cocycle in the Hochschild complex and represents a cohomology class in $\mathrm{HH}^{n,2-n}(A)$. Thus, if $\mathrm{HH}^{n,2-n}(A) = 0$ for all $n \geq 3$, then all higher multiplication maps on A must be trivial. One can then lift the A_∞-morphism $q\colon A \to C$ via a certain tensor algebra T to a zigzag of quasi-isomorphisms $A \leftarrow T \to C$ of dg-algebras. Then it suffices to compose with the algebra isomorphism $H^*(C) \cong A$ to conclude that C is formal. We refer to [36, Proof of Theorem 4.7] for the details.

5. Graded Hochschild cohomology of Koszul algebras

We continue to work over the field $\mathbb{F}_2$. Unless stated otherwise $\otimes$ will denote $\otimes_{\mathbb{F}_2}$.

Definition 5.1. A *quadratic algebra* is a non-negatively graded $\mathbb{F}_2$-algebra $A = \bigoplus_{i\geq 0} A_i$ such that $A_0 = \mathbb{F}_2$ and A is generated over $\mathbb{F}_2$ by A_1 with relations of degree two. Explicitly, let

$$T(A_1) = \mathbb{F}_2 \oplus A_1 \oplus (A_1 \otimes A_1) \oplus \cdots = \bigoplus_{i\geq 0} A_1^{\otimes i}$$

be the free tensor algebra of the $\mathbb{F}_2$-vector space A_1. Then A is quadratic if the canonical map $\tau\colon T(A_1) \to A$ is surjective and $\ker(\tau)$ is generated by its component $R = \ker(\tau) \cap (A_1 \otimes A_1)$ of degree two, so that $A = T(A_1)/(R)$, where (R) denotes the ideal generated by R. A quadratic algebra A is called *locally finite* if each A_i is a finite-dimensional vector space over $\mathbb{F}$.

For a quadratic algebra $A = \bigoplus_{i\geq 0} A_i$ we denote by A_+ the ideal $A_+ = \bigoplus_{i>0} A_i$. We will later use the following natural way to create new quadratic algebras out of old (see for e.g. [28, Chapter 3.1]):

Definition 5.2. Let $A = \bigoplus_{i\geq 0} A_i$ and $B = \bigoplus_{i\geq 0} B_i$ be quadratic algebras. The *connected sum* of A and B, denoted by $A \sqcap B$, is the graded ring with $(A \sqcap B)_0 = \mathbb{F}_2$, $(A \sqcap B)_i = A_i \oplus B_i$ for $i > 0$ and multiplication A_+B_+ and B_+A_+ is set to be zero.

The connected sum $C := A \sqcap B$ is a quadratic algebra. Let $R_A \subseteq A_1 \otimes A_1$ and $R_B \subseteq B_1 \otimes B_1$ denote the relations in the tensor algebras defining A and B, respectively. Then we can write C as a quotient of a tensor algebra

$$C = T(A_1 \oplus B_1)/(R_C)$$

where $R_C \subseteq C_1^{\otimes 2}$ is given by $R_C = R_A \oplus R_B \oplus (A_1 \otimes B_1) \oplus (B_1 \otimes A_1)$.

To every quadratic algebra we can associate the following chain complex:

Definition 5.3. Let V be an $\mathbb{F}_2$-vector space and let $T(V) = \bigoplus_{i\geq 0} V^{\otimes i}$ be its tensor algebra. We consider $T(V)$ as a graded $\mathbb{F}_2$-algebra with $|v| = 1$ for all $v \in V$. Let $R \subset V \otimes V$ and let $A = T(V)/(R)$ be the associated quadratic algebra where (R) denotes the ideal generated by R in $T(V)$. We denote by $K_i^i(A)$ the $\mathbb{F}_2$-linear subspace defined by $K_0^0(A) = \mathbb{F}_2$, $K_1^1(A) = V$, $K_2^2(A) = R$ and

$$K_i^i(A) = \bigcap_{0\leq j\leq i-2} V^{\otimes j} \otimes R \otimes V^{\otimes i-j-2} \subset V^{\otimes i} \text{ for } i \geq 3.$$

We set $K_i(A) = A \otimes K_i^i(A) \otimes A$. We consider $K_i(A)$ as an A^e-module with A acting on the leftmost factor A and A^{op} acting on the rightmost factor A, respectively. For each $i \geq 0$, we define a homomorphism $d = d_i\colon K_i(A) \to K_{i-1}(A)$ by

$$d_i(a \otimes x_1 \otimes \cdots \otimes x_i \otimes a') = (ax_1) \otimes x_2 \otimes \cdots \otimes x_i \otimes a' + a \otimes x_1 \otimes x_2 \otimes \cdots \otimes (x_i a').$$

Since $R \subset V^{\otimes 2}$ generates the relations in A, it is clear that $d^2 = 0$. We refer to the chain complex $(K(A), d)$ as the *Koszul complex of A*.

Let $(B(A), d_B)$ denote the bar complex of A. Since R describes the relations in A, the natural inclusion $K_i(A) \hookrightarrow B_i(A)$ defines a morphism of complexes $(K(A), d) \to (B(A), d_B)$. The multiplication map $\mu\colon A \otimes A \to A$ defines morphisms of complexes from the complexes $(K(A), d)$ and $(B(A), d_B)$ to the complex A concentrated in degree 0 with trivial differential. However, while $\mu\colon B(A) \to A$ is always a quasi-isomorphism, $\mu\colon K(A) \to A$ may not be a quasi-isomorphism for an arbitrary quadratic algebra A. This leads to the following definition.

Definition 5.4. A quadratic algebra A is called *Koszul* if the morphism of complexes induced by multiplication $\mu\colon K(A) \to A$ is a quasi-isomorphism.

Remark 5.5. We note that there are many equivalent definitions of the notion of a Koszul algebra. By [39, Proposition 3.1](where the notation $K'(A)$ is used for $K(A)$), Definition 5.4 is equivalent to the ones in [28], [30] and [31] (see also [4, Section 2.1]).

By [39, Proposition 3.1], the Koszul complex $(K(A), d)$ is a complex of free A^e-modules. Hence, if A is a Koszul algebra, we can use $K(A)$ to compute Hochschild cohomology. In fact, since $K(A) \hookrightarrow B(A)$ is a quasi-isomorphism when A is a Koszul algebra, we get the following result:

Proposition 5.6. *Let A be a Koszul algebra and M be a graded A-bimodule. The canonical inclusion of chain complexes $K(A) \hookrightarrow B(A)$ induces an isomorphism between the graded Hochschild cohomology of A with coefficients in M and the cohomology of the cochain complex $(\underline{\mathrm{Hom}}_{A^e}(K(A), M), d^*)$ where d^* denotes the differential induced by the differential d of the chcain complex $K(A)$.* □

We now show that for a Koszul algebra an even simpler complex can be used to compute its Hochschild cohomology.

Definition 5.7. Let A be a Koszul algebra and M be a graded A-bimodule. For every $n \geq 0$ and $s \in \mathbb{Z}$, let ∂ denote the $\mathbb{F}_2$-linear map

$$\partial_n\colon \mathrm{Hom}_{\mathbb{F}_2}(K_n^n(A), M_{n+s}) \to \mathrm{Hom}_{\mathbb{F}_2}(K_{n+1}^{n+1}(A), M_{n+1+s})$$

defined by setting

$$\partial_n(f)(x_1 \otimes \ldots \otimes x_{n+1}) = x_1 f(x_2 \otimes \ldots \otimes x_{n+1}) + f(x_1 \otimes \ldots \otimes x_n) x_{n+1}.$$

It is easily verified that $\partial \circ \partial = 0$. We refer to the cochain complex

$$(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(K_*^*(A), M[s]), \partial) \tag{9}$$

as the *simplified Koszul cochain complex.*

Let A be a Koszul algebra and M be a graded A-bimodule. For every n and every integer s, isomorphism (7) descends to an isomorphism

$$\underline{\mathrm{Hom}}_{A^e}(K_n(A), M[s]) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathbb{F}_2}(K_n^n(A), M_{n+s}). \tag{10}$$

It is straight-forward to check that isomorphism (10) yields an isomorphism of cochain complexes

$$(\underline{\mathrm{Hom}}_{A^e}(K_*(A), M[s]), d^*) \xrightarrow{\cong} (\underline{\mathrm{Hom}}_{\mathbb{F}_2}(K^*_*(A), M[s]), \partial). \tag{11}$$

By Proposition 5.6, the inclusion $K(A) \hookrightarrow B(A)$ induces an isomorphism between the Hochschild cohomology group $\mathrm{HH}^{n,s}(A, M)$ and the nth-cohomology of the left-hand side of (11). Since (11) is an isomorphism of cochain complexes, this proves the following result:

Proposition 5.8. *Let A be a Koszul algebra and M be a graded A-bimodule. For every pair (n, s), the composition of the restriction along the inclusion $K(A) \hookrightarrow B(A)$ and isomorphism (11) defines an isomorphism*

$$\mathrm{HH}^{n,s}(A, M) \xrightarrow{\cong} H^n(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(K^*_*(A), M[s]), \partial) \tag{12}$$

from the graded Hochschild cohomology group $\mathrm{HH}^{n,s}(A, M)$ to the nth cohomology group of the cochain complex $(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(K^*_*(A), M[s]), \partial)$. □

Theorem 4.7 and Proposition 5.8 then imply the following criterion for intrinsic formality:

Corollary 5.9. *Let A be a Koszul algebra over $\mathbb{F}_2$. Assume that*

$$H^n(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(K^*_*(A), A[2-n]), \partial) = 0 \text{ for all } n \geq 3.$$

Then A is intrinsically formal. □

Finally, to prove our main technical result in the next sections we will apply the following spectral sequence. We will provide a proof of the following proposition in appendix B.

Proposition 5.10. *Let A be a positively graded $\mathbb{F}_2$-algebra with $A_0 = \mathbb{F}_2$. Let $\{A_\alpha\}_\alpha$ be a directed system of graded subalgebras such that A is equal to the inductive limit $\varinjlim_\alpha A_\alpha$. Let M be a graded A^e-module. For each integer s, there is a first quadrant spectral sequence*

$$E_2^{p,q} = \varprojlim_\alpha{}^p \mathrm{HH}^{q,s}(A_\alpha, M) \Rightarrow \mathrm{HH}^{p+q,s}(A, M)$$

where $\varprojlim^p_\alpha$ denotes the pth right derived functor of the inverse limit functor.

6. Boolean and dual graded algebras

In this section we introduce the two types of graded algebras for which we will compute the Hochschild cohomology of their connected sums in the next sections. For more details and background on Boolean rings we refer to appendix A.

Definition 6.1. We say that a ring B is *Boolean* if $x^2 = x$ for every $x \in B$. For every Boolean ring B, let $B_* = \bigoplus_{n\geq 0} B_n$ denote the graded ring such that $B_0 = \mathbb{F}_2$ and for every $n \geq 1$ we set $B_n = B$. For every pair of positive integers m, n the multiplication $B_m \times B_n \to B_{m+n}$ is the multiplication in the ring $B = B_m = B_n = B_{m+n}$. We call B_* the *associated Boolean graded algebra*.

For the next lemma, we recall that the polynomial ring $A = \mathbb{F}_2[x]$ is a quadratic algebra with $A_1 = \mathbb{F}_2\langle x\rangle$, the one-dimensional $\mathbb{F}_2$-vector space generated by x, and trivial relations. We also recall from Corollary A.16 that every finite Boolean ring has an orthogonal basis, i.e., there are elements $x_1, x_2, \ldots, x_n$ in B which form a basis of B as a vector space over $\mathbb{F}_2$ and $x_i x_j = 0$ for $i \neq j$.

Lemma 6.2. *Let B be a finite Boolean ring, and let $x_1, x_2, \dots, x_n$ be an orthogonal basis of B. Then the associated Boolean graded algebra B_* is a quadratic algebra and is isomorphic to the connected sum of the $\mathbb{F}_2[x_j]$, i.e.,*

$$B_* \cong \mathbb{F}_2[x_1] \sqcap \cdots \sqcap \mathbb{F}_2[x_n].$$

Moreover, B_* is a Koszul algebra.

Proof. That B_* is isomorphic to the connected sum of the quadratic algebras $\mathbb{F}_2[x_j]$ follows immediately from Definition 6.1. By [28, Chapter 3, Corollary 1.2 on page 58], the connected sum of Koszul algebras is Koszul. Hence, in order to show that B_* is Koszul, it suffices to show that $\mathbb{F}_2[x]$ is a Koszul algebra. The latter follows from [28, Example on page 20] and Remark 5.5 since $\mathbb{F}_2[x]$ is the symmetric algebra in one generator. Thus, $\mathbb{F}_2[x]$ is Koszul and hence so is B_*. □

Proposition 6.3. *Let B be a Boolean ring and let B_* be its associated Boolean graded algebra. Then B_* is a Koszul algebra.*

Proof. This follows from the fact that every Boolean ring is the inductive limit of finite Boolean rings. Indeed, every ring is the inductive limit of its finitely generated subrings. Every subring of a Boolean ring is Boolean, and from Stone's representation theorem it is also clear that every finitely generated Boolean ring is finite. Also note that the functor $B \mapsto B_*$ of Definition 6.1 maps injective morphisms to injective morphisms, so for every Boolean ring B the graded $\mathbb{F}_2$-algebra B_* is the inductive limit of graded $\mathbb{F}_2$-algebras of the form A_* where $A \subseteq B$ is a finite Boolean ring. By Lemma 6.2, it now suffices to observe that the property of being a Koszul algebra is preserved under passing to inductive limits in the category of graded algebras over a field. This was shown by Bruns and Gubeladze in [6, Lemma 1.1, page 48]. □

As a generalisation of the algebra of dual numbers $k[\varepsilon]/(\varepsilon^2)$, we refer to the following type of graded algebras as *dual algebras* (even though we do not consider them as the dual of another algebra).

Definition 6.4. Let V be a vector space over $\mathbb{F}_2$. We call the non-negatively graded $\mathbb{F}_2$-algebra $V_* = \bigoplus_{j\geq 0} V_j$ with $V_0 = \mathbb{F}_2$, $V_1 = V$ and $V_j = 0$ for $j \geq 2$ the *associated dual algebra*. A dual algebra V_* is quadratic with relations $R = V_1 \otimes_{\mathbb{F}_2} V_1$.

Lemma 6.5. *For every $\mathbb{F}_2$-vector space V, the dual algebra V_* is Koszul. For $k + m \geq 2$, we have $\mathrm{HH}^{k,m}(V_*) = 0$.*

Proof. That V_* is Koszul follows from the fact that the inclusion map of $K(V_*)$ into the bar complex is the identity. The vanishing of $\mathrm{HH}^{k,m}(V_*)$ for $k + m \geq 2$ follows from the fact that there are no nontrivial cochains in the Koszul cochain complex for $k + m \geq 2$, since $V_j = 0$ for $j \geq 2$. □

Proposition 6.6. *Let V be an $\mathbb{F}_2$-vector space, let V_* be its associated dual algebra, B be a Boolean ring and let B_* be its associated Boolean graded algebra. Then $V_* \sqcap B_*$ is a Koszul algebra.*

Proof. By Lemma 6.5 we know that V_* is Koszul, and by Proposition 6.3 we know that B_* is Koszul. By [28, Chapter 3, Corollary 1.2 on page 58] the connected sum of Koszul algebras is Koszul. This proves the assertion. □

7. Kadeishvili's vanishing for connected sums of dual and Boolean graded algebras - the finite case

In this section we are going to prove that connected sums of dual algebras and the Boolean graded algebras associated to a Boolean ring have vanishing Hochschild cohomology when we restrict to a finite subring. In Section 9 we extend the computation to the case of arbitrary Boolean subrings.

We fix an $\mathbb{F}_2$-vector space V, and let V_* be its associated dual algebra. We also fix a Boolean ring B and a *finite* Boolean subring $A \subset B$. Let A_* and B_* be the associated Boolean graded algebras. Our next goal is to compute the groups $\mathrm{HH}^{k,m}(V_* \sqcap A_*, V_* \sqcap B_*)$ for $m < 0$.

By Propositions 5.8 and 6.6, we can do this by using the simplified Koszul cochain complex (9) of $V_* \sqcap A_*$ of Definition 5.7, i.e., we consider the cochain complex

$$(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(K^*_*(V_* \sqcap A_*), (V_* \sqcap B_*)[m]), \partial).$$

First we determine the $\mathbb{F}_2$-vector spaces $K^k_k(V_* \sqcap A_*)$. This will require some preparation.

Definition 7.1. Let I denote a basis of V. Let $x_1, x_2, \ldots, x_n$ be the orthonormal basis of A, see Corollary A.16. Let J denote the set $\{x_1, x_2, \ldots, x_n\}$. A sequence $t_1, t_2, \ldots, t_k \in I \cup J$ is *admissible* if for every $j = 1, 2, \ldots, k-1$ such that $t_j, t_{j+1} \in J$ we have $t_j \neq t_{j+1}$. We say that an admissible sequence $t_1, t_2, \ldots, t_k \in I \cup J$ is *stable* if $t_1, t_k \in J$ and $t_1 = t_k$, and it is *unstable* otherwise.

Notation 7.2. For every sequence $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup J$, let the same symbol $\mathbf{t}$ denote the element $t_1 \otimes t_2 \otimes \cdots \otimes t_k \in K^k_k(V_* \sqcap A_*)$ by slight abuse of notation.

Lemma 7.3. *Write $Q_* = V_* \sqcap A_*$. The subspace $K^k_k(Q) \subset Q_1^{\otimes k}$ is spanned by the basis consisting of the vectors $\mathbf{t}$, where $\mathbf{t}$ is any admissible sequence.*

Proof. Let $R_Q \subseteq Q_1 \otimes Q_1$ denote the relations in the tensor algebra defining the quadratic algebra Q. Note that the vectors $\mathbf{t}$, where $\mathbf{t}$ is any sequence in $I \cup J$, are linearly independent, so any subset of them will form a basis of their span. Therefore it will be sufficient to prove that for every index $j = 1, 2, \ldots, k-1$ the subspace:

$$Q_1^{\otimes j-1} \otimes R_Q \otimes Q_1^{\otimes k-j-1} \subset Q_1^{\otimes k}$$

is spanned by the vectors $\mathbf{t}$, where $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup J$ satisfies the condition $t_j \neq t_{j+1}$ if $t_j, t_{j+1} \in J$. Recall that the relations in A_* are generated by the vectors $x_i \otimes x_j$ for $i \neq j$. Let $T_Q \subset Q_1^{\otimes 2}$ be the subspace spanned by the vectors $x_j \otimes x_j$, where $j = 1, 2, \ldots, n$. Then $Q_1^{\otimes 2} = R_Q \oplus T_Q$, and hence

$$Q_1^{\otimes k} = \left(Q_1^{\otimes j-1} \otimes R_Q \otimes Q_1^{\otimes k-j-1}\right) \oplus \left(Q_1^{\otimes j-1} \otimes T_Q \otimes Q_1^{\otimes k-j-1}\right). \tag{13}$$

Now if

$$\underline{t} = \sum_{\mathbf{t}} a_{\mathbf{t}} \mathbf{t} = \left(\sum_{\{t_j, t_{j+1}\} \not\subseteq J} a_{\mathbf{t}} \mathbf{t} + \sum_{\substack{t_j, t_{j+1} \in J \\ t_j \neq t_{j+1}}} a_{\mathbf{t}} \mathbf{t} \right) + \sum_{\substack{t_j, t_{j+1} \in J \\ t_j = t_{j+1}}} a_{\mathbf{t}} \mathbf{t} \quad (a_{\mathbf{t}} \in \mathbb{F}_2)$$

lies in $Q_1^{\otimes j-1} \otimes R_Q \otimes Q_1^{\otimes k-j-1}$, then the sum outside of the parentheses must be zero, since it lies in the second summand of the decomposition in (13), while the sum within the parentheses lies in the first summand of the decomposition in (13). The claim is now clear. □

We are going to use the following fact about Boolean rings.

Lemma 7.4. *Let $x, y \in B$. Let (x), (y) and (x, y) denote the ideals in B generated by x, y, x and y, respectively. Then the following assertions hold:*

(i) for every $z \in B$, we have $z \in (x, y)$ if and only if $(1 + x + y + xy)z = 0$;
(ii) if $xy = 0$ then (x, y) is the direct sum of (x) and (y).

In particular, for every $x, z \in B$ we have $z \in (x)$ if and only if $(1 + x)z = 0$.

Proof. If $(1 + x + y + xy)z = 0$ then $z = z(x + y + xy) \in (x, y)$. If $z \in (x, y)$ then there are $a, b \in B$ such that $z = ax + by$, so

$$(1 + x + y + xy)z = (ax + by) + (ax + bxy) + (axy + by) + (axy + bxy) = 0,$$

so claim (i) is true. Now assume that $xy = 0$. Note that $(x, y) = (x) + (y)$, so we only need to show that $(x) \cap (y) = 0$. If $z \in (x) \cap (y)$ then there are $a, b \in B$ such that $z = ax = by$. Therefore

$$z = z^2 = axby = abxy = 0,$$

so claim (ii) holds, too. □

Now we assume that $j \geq 1$ and that $f \colon K^k_k(V_* \sqcap A_*) \to B_j$ is a cocycle in the simplified Koszul cochain complex $(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(K^*_*(V_* \sqcap A_*), B_*[j-k]), \partial)$ of Definition 5.7. For the sake of simple notation, we will identify B_j with B_1 in all that follows. Let $p \colon V_1 \oplus B_1 \to B_1$ be the unique linear map which is zero on V_1 and the identity on B_1.

Lemma 7.5. *For every admissible sequence $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup J$ we have:*

$$f(\mathbf{t}) \in (p(t_1), p(t_k)).$$

Proof. We define the element $x := 1 + p(t_1) + p(t_k) + p(t_1)p(t_k) \in A_1 \subset B_1 = B$. We claim that x satisfies

$$xt_1 = 0 = xt_k.$$

First, elements in I are orthogonal to all elements and the claim is true for $t_1, t_k \in I$. Second, if $t_1 \in J$, then $xt_1 = t_1 + t_1^2 + t_1p(t_k) + t_1^2p(t_k) = t_1 + t_1 + t_1p(t_k) + t_1p(t_k) = 0$. And we have a similar calculation if $t_k \in J$. This shows the claim.

Hence $x \otimes t_1 \otimes \cdots \otimes t_{k-1} \otimes t_k \in K^{k+1}_{k+1}(V_* \sqcap A_*)$. Since f is a cocycle:

$$xf(t_1 \otimes t_2 \otimes \cdots \otimes t_k) + f(x \otimes t_1 \otimes \cdots \otimes t_{k-1})t_k = 0.$$

Multiplying both sides of this equation by x we get that

$$0 = x^2 f(t_1 \otimes t_2 \otimes \cdots \otimes t_k) + f(x \otimes t_1 \otimes \cdots \otimes t_{k-1})xt_k = xf(t_1 \otimes t_2 \otimes \cdots \otimes t_k).$$

The claim now follows from part (i) of Lemma 7.4. □

Definition 7.6. For every stable admissible sequence $\mathbf{t}$, by Lemma 7.5, we have

$$f(\mathbf{t}) = \alpha(\mathbf{t})$$

for a unique $\alpha(\mathbf{t}) \in (p(t_1))$. Set $\beta(\mathbf{t}) = \alpha(\mathbf{t})$ in this case. For every unstable admissible sequence $\mathbf{t}$, Lemma 7.5 implies

$$f(\mathbf{t}) = \alpha(\mathbf{t}) + \beta(\mathbf{t}), \quad \alpha(\mathbf{t}) \in (p(t_1)), \beta(\mathbf{t}) \in (p(t_k)),$$

where $\alpha(\mathbf{t}), \beta(\mathbf{t})$ are uniquely determined by part *(ii)* of Lemma 7.4. For every admissible sequence $\mathbf{t}$ we call $\alpha(\mathbf{t}), \beta(\mathbf{t})$ the *head, tail value* of $f(\mathbf{t})$, respectively.

Let $A_k(I,n)$ denote the set of admissible sequences $t_1, t_2, \ldots, t_k \in I \cup J$, where we recall that n denotes the cardinality of J, i.e., the $\mathbb{F}_2$-dimension of the finite subring $A \subset B$, which determines the associated Boolean graded algebra A_* by Lemma 6.2.

Definition 7.7. We define two maps $L\colon A_k(I,n) \to A_k(I,n)$ and $R\colon A_k(I,n) \to A_k(I,n)$, the *left* and *right translations*, as follows. Let

$$L(t_1, t_2, \ldots, t_k) = \begin{cases} t_1, t_2, \ldots, t_k & \text{, if the sequence is stable,} \\ t_2, t_3, \ldots, t_k, t_1 & \text{, if the sequence is unstable,} \end{cases}$$

and similarly

$$R(t_1, t_2, \ldots, t_k) = \begin{cases} t_1, t_2, \ldots, t_k & \text{, if the sequence is stable,} \\ t_k, t_1, t_2, \ldots, t_{k-1} & \text{, if the sequence is unstable.} \end{cases}$$

These two maps translate the sequence to the left or to the right, respectively, and put the term which drops out to the other end. It is also clear that they are well-defined, i.e., they map admissible sequences to admissible sequences. Note that L and R are inverses of each other, so we get a permutation, or a cyclic group action on $A_k(I,n)$. The fixed points of this action are exactly the stable admissible sequences.

Lemma 7.8. *For every admissible sequence* $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup J$ *we have:*

$$\alpha(R(\mathbf{t})) = \beta(\mathbf{t}),$$

and equivalently:

$$\beta(L(\mathbf{t})) = \alpha(\mathbf{t}).$$

Proof. The case of a stable admissible sequence is trivial. If $\mathbf{t}$ is unstable, the two identities above follow from the two relations:

$$t_k f(t_1 \otimes t_2 \otimes \cdots \otimes t_k) + f(t_k \otimes t_1 \otimes \cdots \otimes t_{k-1})t_k = 0,$$
$$t_1 f(t_2 \otimes \cdots \otimes t_k \otimes t_1) + f(t_1 \otimes t_2 \otimes \cdots \otimes t_k)t_1 = 0,$$

respectively. □

Remark 7.9. Note that the lemma above covers all relations between head and tail values for cocycles. This is true because all relations for cocycles are spanned by the relations:

$$t_1 f(t_2 \otimes t_3 \otimes \cdots \otimes t_{k+1}) + f(t_1 \otimes t_2 \otimes \cdots \otimes t_k)t_{k+1} = 0, \tag{14}$$

where $\mathbf{t} = t_1, t_2, \ldots, t_{k+1} \in I \cup J$ is any admissible sequence. In particular there is no relation between head and tail values for admissible sequences lying in different orbits of R. Within every orbit of the right translation operator the head value of the right translate is equal to the tail value of the translated sequence, but otherwise we are free to choose the head and tail values within an orbit.

Definition 7.10. We define two maps

$$l\colon A_k(I,n) \to A_{k-1}(I,n) \quad \text{and} \quad r\colon A_k(I,n) \to A_{k-1}(I,n),$$

the *left and right truncations*, as follows. Let

$$l(t_1, t_2, \dots, t_k) = t_1, t_2, \dots, t_{k-1},$$

and similarly

$$r(t_1, t_2, \dots, t_k) = t_2, t_3, \dots, t_k.$$

These maps are clearly well-defined, i.e., they map admissible sequences to admissible sequences. We say that an orbit $O \subset A_k(I, n)$ of R is *stable* if it consists of one element, and it is *unstable* if it consists of at least two elements.

Lemma 7.11. *Let $O \subset A_k(I, n)$ be an unstable orbit and let $\mathbf{i}, \mathbf{j}$ be two elements of O. The following holds:*

- (i) *if $l(\mathbf{i}) = l(\mathbf{j})$ then $\mathbf{i} = \mathbf{j}$,*
- (ii) *if $r(\mathbf{i}) = r(\mathbf{j})$ then $\mathbf{i} = \mathbf{j}$,*
- (iii) *we have $r(\mathbf{i}) = l(\mathbf{j})$ if and only if $\mathbf{i} = R(\mathbf{j})$ and $\mathbf{j} = L(\mathbf{i})$.*

Proof. We first prove part (i). Since $\mathbf{j}$ is a permutation of $\mathbf{i}$, the number of occurrences of i_k in $\mathbf{j}$ is the same as in $\mathbf{i}$. Since $l(\mathbf{i}) = l(\mathbf{j})$, the number of occurrences of i_k in $l(\mathbf{j})$ is the same as in $l(\mathbf{i})$. This is only possible if $j_k = i_k$, and so part (i) is true. The proof of part (ii) is essentially the same as the proof of part (i) above, just left and right have to be exchanged. By definition $r(R(\mathbf{j})) = l(\mathbf{j})$ and $l(L(\mathbf{i})) = r(\mathbf{i})$, so the reverse implication of part (iii) is clearly true. On the other hand if $r(\mathbf{i}) = l(\mathbf{j})$ then $r(\mathbf{i}) = r(R(\mathbf{j}))$ by the above, so $\mathbf{i} = R(\mathbf{j})$ by part (ii), and hence $\mathbf{j} = L(\mathbf{i})$, too. So part (iii) holds, too. □

Definition 7.12. The *truncation set* $\mathbb{T}(O)$ of an unstable orbit O as above is its image with respect to the map l. By part (iii) of Lemma 7.11 this is the same as the image of O with respect to the map r. By parts (i) and (ii) of Lemma 7.11, respectively, we get that both $l : O \to \mathbb{T}(O)$ and $r : O \to \mathbb{T}(O)$ are bijective. Let $l^{-1} : \mathbb{T}(O) \to O$ and $r^{-1} : \mathbb{T}(O) \to O$ be the inverse of these two maps, respectively.

We are finally ready to prove the following theorem:

Theorem 7.13. *Let V be an $\mathbb{F}_2$-vector space and V_* its associated dual algebra. Let B be a Boolean ring, $A \subset B$ a finite Boolean subring, and let A_* and B_*, respectively, be the associated Boolean graded algebras. We have:*

$$\mathrm{HH}^{k,2-k}(V_* \sqcap A_*, V_* \sqcap B_*) = 0 \textit{ for all } k \geq 3.$$

Assume $|A| \geq 8$ and let $m < 0$ be a negative integer. Then we have:

$$\mathrm{HH}^{k,m}(V_* \sqcap A_*, V_* \sqcap B_*) = 0 \textit{ for all } k \geq 0 \textit{ with } k \neq 1 - m.$$

Proof. For $k \geq 2$ and $j \geq 2$, we are going to show that every cocycle

$$f : K_k^k(V_* \sqcap A_*) \to B_j$$

is a coboundary. This will imply that $\mathrm{HH}^{k,s}(V_* \sqcap A_*, V_* \sqcap B_*) = 0$ for all pairs (k, m) with $k \geq 2$ and $m \geq 2 - k$. The remaining cases will be discussed afterwards. By the above it will be enough to show the claim orbit by orbit, that is, we may assume without loss of generality that f is supported on an orbit $O \subset A_k(I, n)$ of the right translation R. We will let α and β denote the head and tail values of f, as above.

First assume that O is a stable orbit $\{\mathbf{t}\}$ with $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup J$. In particular, $t_1 = t_k$ and these two elements are in J. Let $g\colon K^{k-1}_{k-1}(V_* \sqcap A_*) \to B_{j-1}$ be the cochain given by the rule:

$$g(\mathbf{s}) = \begin{cases} \alpha(\mathbf{t})t_1^{j-1} & \text{, if } \mathbf{s} = r(\mathbf{t}), \\ 0 & \text{, otherwise.} \end{cases}$$

Lemma 7.14. *We have $\partial g = f$.*

Proof. Note that $t_{k-1} \neq t_k$ since $t_k \in J$ and $\mathbf{t}$ is admissible. Therefore $l(\mathbf{t}) = t_1, \ldots, t_{k-1}$ is different from $r(\mathbf{t})$, and hence

$$\partial g(\mathbf{t}) = t_1 g(r(\mathbf{t})) + g(l(\mathbf{t}))t_k = \alpha(\mathbf{t})t_1^j + 0 = f(\mathbf{t}).$$

Now let $\mathbf{u} = u_1, u_2, \ldots, u_k \in I \cup J$ be an admissible sequence different from $\mathbf{t}$. If $g(r(\mathbf{u})) \neq 0$, then $r(\mathbf{u}) = r(\mathbf{t})$, and hence $u_1 \neq t_1$, as $\mathbf{u}$ and $\mathbf{t}$ are different. Therefore $u_1 g(r(\mathbf{u})) = u_1\alpha(\mathbf{t})t_1^{j-1} = 0$ in this case since $j-1 \geq 1$ by assumption, $\alpha(\mathbf{t}) \in (p(t_1))$ and the product of u_1 and t_1 is zero when $u_1 \neq t_1$. Otherwise, $g(r(\mathbf{u})) = 0$, and hence $u_1 g(r(\mathbf{u})) = 0$ trivially in this case.

If $g(l(\mathbf{u})) \neq 0$, then $l(\mathbf{u}) = r(\mathbf{t})$, and so $u_{k-1} = t_k = t_1$. So u_k is different from t_1, since $\mathbf{u}$ is admissible. Hence we have $g(l(\mathbf{u}))u_k = \alpha(\mathbf{t})t_1^{j-1}u_k = 0$ in this case since $j-1 \geq 1$ and the product of t_1 and u_k is zero when $u_k \neq t_1$. Otherwise, $g(l(\mathbf{u})) = 0$, and hence $g(l(\mathbf{u}))u_k = 0$ trivially in this case. We get that

$$\partial g(\mathbf{u}) = u_1 g(r(\mathbf{u})) + g(l(\mathbf{u}))u_k = 0 + 0 = 0. \qquad \square$$

Now assume that O is an unstable orbit. For every sequence $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup J$, let $a(\mathbf{t}), b(\mathbf{t})$ denote the first term t_1 and the last term t_k of $\mathbf{t}$, respectively. Let $g\colon K^{k-1}_{k-1}(V_* \sqcap A_*) \to B_{j-1}$ be the cochain given by the rule:

$$g(\mathbf{s}) = \begin{cases} \alpha(r^{-1}(\mathbf{s}))p(a(r^{-1}(\mathbf{s})))^{j-1} & \text{, if } \mathbf{s} \in \mathbb{T}(O), \\ 0 & \text{, otherwise.} \end{cases}$$

In particular, g is supported on the truncation set of O.

Lemma 7.15. *We have:*

$$g(\mathbf{s}) = \beta(l^{-1}(\mathbf{s}))p(b(l^{-1}(\mathbf{s})))^{j-1}$$

for every $\mathbf{s} \in \mathbb{T}(O)$.

Proof. Since $l(l^{-1}(\mathbf{s})) = \mathbf{s} = r(r^{-1}(\mathbf{s}))$, we get that $l^{-1}(\mathbf{s}) = L(r^{-1}(\mathbf{s}))$ by part (*iii*) of Lemma 7.11. Therefore, $b(l^{-1}(\mathbf{s})) = a(r^{-1}(\mathbf{s}))$ and $\beta(l^{-1}(\mathbf{s})) = \alpha(r^{-1}(\mathbf{s}))$ by Lemma 7.8. Therefore,

$$g(\mathbf{s}) = \alpha(r^{-1}(\mathbf{s}))p(a(r^{-1}(\mathbf{s})))^{j-1} = \beta(l^{-1}(\mathbf{s}))p(b(l^{-1}(\mathbf{s})))^{j-1}. \qquad \square$$

Lemma 7.16. *We have $\partial g = f$.*

Proof. Let $\mathbf{u} = u_1, u_2, \ldots, u_k \in I \cup J$ be an admissible sequence. First assume that $\mathbf{u} \in O$. Then

$$\partial g(\mathbf{u}) = u_1 g(r(\mathbf{u})) + g(l(\mathbf{u}))u_k = \alpha(\mathbf{u})p(u_1)^j + \beta(\mathbf{u})p(u_k)^j = f(\mathbf{u})$$

where we use the definition of g and Lemma 7.15 in rewriting the first and second summand, respectively. Now assume that $\mathbf{u}$ is not in O. If $g(r(\mathbf{u})) \neq 0$, then $r(\mathbf{u}) = r(\mathbf{t})$ for some $\mathbf{t} \in O$. Then $u_1 \neq t_1$, as $\mathbf{u}$ and $\mathbf{t}$ are different when $\mathbf{u} \notin O$. Therefore,

$u_1 g(r(\mathbf{u})) = u_1\alpha(\mathbf{t})p(t_1)^{j-1} = 0$ in this case since $j-1 \geq 1$ by assumption and $u_1 \neq t_1$. Otherwise, $g(r(\mathbf{u})) = 0$, and hence $u_1 g(r(\mathbf{u})) = 0$ trivially in this case.

If $g(l(\mathbf{u})) \neq 0$, then $l(\mathbf{u}) = l(\mathbf{t})$ for some $\mathbf{t} \in O$. Then $u_k \neq t_k$, as $\mathbf{u}$ and $\mathbf{t}$ are different. Therefore, since $j-1 \geq 1$ and $t_k \neq u_k$, we have $g(l(\mathbf{u}))u_k = \beta(\mathbf{t})p(t_k)^{j-1}u_k = 0$ by Lemma 7.15 in this case. Otherwise, $g(l(\mathbf{u})) = 0$, and hence $g(l(\mathbf{u}))u_k = 0$ trivially in this case. We get that

$$\partial g(\mathbf{u}) = u_1 g(r(\mathbf{u})) + g(l(\mathbf{u}))u_k = 0 + 0 = 0. \qquad \square$$

This finishes the proof for $k \geq 2$ and $m \geq 2-k$. Since both summands and hence the direct sum $V_j \oplus B_j$ are trivial for negative j, there are no nontrivial cochains and hence no nontrivial cocycles for $k+m < 0$. Hence it remains to check the values $k \geq 1$ and $m = -k$. For $m = -k$, the first nontrivial differentials in the cochain complex we use to compute Hochschild cohomology are

$$\mathrm{Hom}_{\mathbb{F}_2}(K_k^k(V_* \sqcap A_*), \mathbb{F}_2) \xrightarrow{\partial_k} \mathrm{Hom}_{\mathbb{F}_2}(K_{k+1}^{k+1}(V_* \sqcap A_*), V_1 \oplus B_1) \xrightarrow{\partial_{k+1}} \dots$$

Let $f\colon K_k^k(V_* \sqcap A_*) \to \mathbb{F}_2$ be a cocycle. Let $\mathbf{t} = t_{i_1} \otimes \ldots \otimes t_{i_k}$ be an admissible sequence. Assuming that the set $I \cup J$ consists of at least three elements, there is a $t_{i_{k+1}}$ different from both t_{i_1} and t_{i_k}. Then

$$t_{i_1} f(t_{i_2} \otimes t_{i_3} \otimes \cdots \otimes t_{i_{k+1}}) + f(t_{i_1} \otimes t_{i_2} \otimes \cdots \otimes t_{i_k})t_{i_{k+1}} = 0$$

is only possible if the coefficients of t_{i_1} and $t_{i_{k+1}}$ in the above sum on the left-hand side, i.e., $f(t_{i_2} \otimes t_{i_3} \otimes \cdots \otimes t_{i_{k+1}})$ and $f(t_{i_1} \otimes t_{i_2} \otimes \cdots \otimes t_{i_k})t$, are both 0. In particular, we must have $f(\mathbf{t}) = 0$. Thus there are no nontrivial cocycles in this case. This finishes the proof of Theorem 7.13. $\square$

Remark 7.17. We note that the assumption $|A| \geq 8$ cannot be dropped. For example, if $A_* = \mathbb{F}_2[x_1] \sqcap \mathbb{F}_2[x_2]$, then $\mathrm{HH}^{k,-k}(A_*) \neq 0$ for even $k > 0$.

8. Mittag-Leffler functors and transfinite recursion on subrings

In this section we recall and prove some results that we will need to extend the claim of Theorem 7.13 to infinite subrings in the next section. First, we recall some facts about Mittag-Leffler functors.

Definition 8.1. Let $\mathcal{AB}$ denote the category of abelian groups. For every ordinal κ, by slight abuse of notation we let the same symbol denote the associated small category induced by an ordering. Let $F\colon \kappa^{op} \to \mathcal{AB}$ be a functor. We say that the functor F is *Mittag-Leffler* if

(ML1) for every $\alpha \in \beta \in \kappa$ the map $F_\beta \to F_\alpha$ is surjective,
(ML2) for every limit ordinal $\alpha \in \kappa$ the map $F_\alpha \to \lim_{\beta\in\alpha} F_\beta$ is surjective.

Lemma 8.2. *If $F\colon \kappa^{op} \to \mathcal{AB}$ is a functor such that*

(ML0) for every $\alpha + 1 \in \kappa$ the map $F_{\alpha+1} \to F_\alpha$ is surjective,
(ML2) for every limit ordinal $\alpha \in \kappa$ the map $F_\alpha \to \lim_{\beta\in\alpha} F_\beta$ is surjective.

Then F is Mittag-Leffler.

Proof. We only need to check condition (ML1) of Definition 8.1 for F. Note that we may assume without the loss of generality that $\alpha = \emptyset$ by replacing κ with the unique ordinal isomorphic to the segment $[\alpha, \kappa)$ and F with $F|_{[\alpha,\kappa)}$. For every $\alpha \leq \beta$ in κ, let $\phi_{\beta,\alpha} : F_\beta \to F_\alpha$ be the transition map, and for every limit ordinal $\beta \in \kappa$, let $\phi_\beta : F_\beta \to \lim_{\alpha\in\beta} F_\alpha$ be the limit of the transition maps $\phi_{\beta,\alpha}$. Now let $\beta \in \kappa$ and $x \in F_\emptyset$ be arbitrary. We are going to construct using transfinite recursion for every $\delta \leq \beta$ an element $x_\delta \in F_\delta$ with the following properties:

(i) we have $x_\emptyset = x$,
(ii) for every $\gamma \leq \delta \leq \beta$ we have $\phi_{\delta,\gamma}(x_\delta) = x_\gamma$,
(iii) for every limit ordinal $\delta \leq \beta$ we have $\phi_\delta(x_\delta) = \lim_{\gamma\in\delta} x_\gamma$.

This is sufficient to conclude that claim, as $\phi_{\beta,\emptyset}(x_\beta) = x_\emptyset = x$, so $\phi_{\beta,\emptyset}$ is surjective, as x was arbitrary. Also note that the limit $\lim_{\gamma\in\delta} x_\gamma$ in (iii) makes sense because of the compatibility in (ii). Let us start the transfinite recursion: for the initial $\delta = \emptyset$ just take $x_\emptyset = x$. This satisfies (i), the only non-trivial condition in this case. When δ is a limit ordinal there is an $x_\delta \in F_\delta$ such that $\phi_\delta(x_\delta) = \lim_{\gamma\in\delta} x_\gamma$ by condition (ML2). Clearly $\phi_{\delta,\gamma}(x_\delta) = x_\gamma$ for every $\gamma \in \delta$. When $\delta = \gamma + 1$ is a successor ordinal then there is an $x_\delta \in F_\delta$ such that $\phi_{\delta,\gamma}(x_\delta) = x_\gamma$ by condition (ML0). If $\alpha \in \delta$, then either $\alpha = \gamma$, and hence $\phi_{\delta,\alpha}(x_\delta) = x_\alpha$ by the above, or $\alpha \in \gamma$, and hence $\phi_{\delta,\alpha}(x_\delta) = \phi_{\gamma,\alpha}(\phi_{\delta,\gamma}(x_\delta)) = \phi_{\gamma,\alpha}(x_\gamma) = x_\alpha$ by the induction hypothesis. □

Proposition 8.3. *If $F\colon \kappa^{op} \to \mathcal{AB}$ is a Mittag-Leffler functor then $\varprojlim^n F = 0$ for every $n > 0$.*

Proof. This is Corollary A.3.14 of [26]. □

Lemma 8.4. *Let $F, G, H\colon \kappa^{op} \to \mathcal{AB}$ be functors which form a short exact sequence:*

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

If F and G are Mittag–Leffler then $\varprojlim^n H = 0$ for every $n > 0$.

Proof. Since the higher limit functors $\varprojlim^n$ are derived functors, we have a long cohomological exact sequence (see [40, §3.5]):

$$\cdots \longrightarrow \varprojlim{}^n G \longrightarrow \varprojlim{}^n H \longrightarrow \varprojlim{}^{n+1} F \longrightarrow \cdots .$$

associated to the short exact sequence above. Now we only need to apply Proposition 8.3. □

Next we prove a result on the structure of the subrings of a Boolean ring. Let B be a Boolean ring. For every subring $A \subset B$ and for every $x \in B$, let $A\langle x\rangle$ denote the subring of B generated by A and x. We assume now that B is infinite and let $A \subset B$ be an infinite subring. Let $\kappa = |A|$ be its cardinality and choose a bijection $x\colon \kappa \to A$. Fix a subring $A_0 \subset A$ such that $|A_0| = 8$. Using transfinite recursion we can construct a set of subrings

$$\{A_\alpha \subseteq A \mid \alpha \in \kappa\} \tag{15}$$

of A such that

(a) we have $A_\emptyset = A_0$,
(b) we have $A_{\alpha+1} = A_\alpha\langle x(\alpha)\rangle$ for every $\alpha \in \kappa$,
(c) we have $\bigcup_{\beta\in\alpha} A_\beta = A_\alpha$ for every limit ordinal $\alpha \in \kappa$.

For this collection of subrings we have some additional properties:

Lemma 8.5. *The following holds:*

(i) *we have $\bigcup_{\alpha\in\kappa} A_\alpha = A$,*
(ii) *we have $A_\beta \subseteq A_\alpha$ for every $\beta \in \alpha \in \kappa$,*

(*iii*) *we have* $|A_\alpha| < \omega$ *for every finite* $\alpha \in \kappa$,
(*iv*) *we have* $|A_\alpha| \leq |\alpha|$ *for every infinite* $\alpha \in \kappa$,
(*v*) *we have* $8 \leq |A_\alpha| < |A|$ *for every* $\alpha \in \kappa$.

Proof. For every $\beta \in \kappa$ we have $x(\beta) \in A_\beta\langle x(\beta)\rangle = A_{\beta+1} \subseteq \bigcup_{\alpha\in\kappa} A_\alpha$, so the latter is A, and hence (i) holds. We are going to show (ii) by transfinite induction. When α is a limit ordinal then $A_\beta \subseteq \bigcup_{\beta\in\alpha} A_\beta = A_\alpha$ by condition (c). When $\alpha = \gamma + 1$ is a successor ordinal then $A_\gamma \subseteq A_\gamma\langle x_\gamma\rangle = A_\alpha$ by (b). If $\beta \in \alpha$, then either $\beta = \gamma$, and hence $A_\beta \subseteq A_\alpha$ by the above, or $\beta \in \gamma$, and hence $A_\beta \subseteq A_\gamma \subseteq A_\alpha$ by the induction hypothesis. So (ii) is true. Since every finitely generated subring of A is finite, we get (iii) by induction, as every positive finite ordinal is a successor ordinal, so condition (b) applies.

Now we are going to show (iv) by transfinite induction. When $\alpha = \omega$ or a limit ordinal more generally, we have $|A_\beta| \leq |\alpha|$ for every $\beta \in \alpha$ either by part (iii) (when $\alpha = \omega$) or by the induction hypothesis. Therefore

$$|A_\alpha| = |\bigcup_{\beta\in\alpha} A_\beta| \leq |\alpha|^2 = |\alpha|$$

using condition (c). When $\alpha = \beta + 1$ is a successor ordinal then by condition (b) we have $A_\alpha = A_\beta\langle x_\beta\rangle$, so the latter is generated by $|\alpha| + 1 = |\alpha|$ elements, and hence its cardinality is at most $|\alpha|$. Therefore (iv) holds, too. For every $\alpha \in \kappa$ we have $A_\emptyset \subseteq A_\alpha$ by part (ii), so $8 = |A_\emptyset| \leq |A_\alpha|$. On the other hand, $|A_\alpha| < |A|$ for every $\alpha \in \kappa$ by part (iii), when α is finite, and by part (iv), when α is infinite since $|A| = \kappa > \alpha$. Therefore (v) is true. □

9. Kadeishvili's vanishing for connected sums of dual and Boolean graded algebras - the infinite case

Our next goal is to extend Theorem 7.13 to the case of infinite subrings $A \subseteq B$. In particular, we will extent the theorem to the case $A = B$. This will require a considerable amount of work.

Notation 9.1. We fix a vector space V over $\mathbb{F}_2$ and let V_* denote the associated dual algebra over $\mathbb{F}_2$. We fix a Boolean ring B and let B_* denote the associated graded Boolean algebra. For every subring $A \subseteq B$ with associated Boolean graded algebra A_* consider the cochain complex:

$$\mathrm{Hom}_{\mathbb{F}_2}(K^k_k(V_* \sqcap A_*), \mathbb{F}_2) \xrightarrow{\partial_k} \mathrm{Hom}_{\mathbb{F}_2}(K^{k+1}_{k+1}(V_* \sqcap A_*), V_1 \oplus B_1) \xrightarrow{\partial_{k+1}} \cdots .$$

Let $F^{k,d-k}(A)$ denote the group $\mathrm{Hom}_{\mathbb{F}_2}(K^{k+d}_{k+d}(V_* \sqcap A_*), (V_* \sqcap B_*)_d)$ of degree d cochains in this complex, and let $G^{k,d-k}(A) \subseteq F^{k,d-k}(A)$ be the subgroup of cocycles.

Lemma 9.2. *Let* $A \subset B$ *be a subring such that* $|A| \geq 8$. *Then* $G^{k,-k}(A) = 0$ *for all* $k \geq 1$.

Proof. Since $|A| \geq 8$ there is a finite subring $A_0 \subseteq A$ such that $|A_0| = 8$. Since every finitely generated subring of A is finite, the ring A is the union of its finite subrings containing A_0. We already know the claim for the latter by the proof of Theorem 7.13. So by taking the limit we get the claim for A, too. □

As a consequence we get the following

Corollary 9.3. *Let $A \subset B$ be a subring such that $|A| \geq 8$. Then*

$$\mathrm{HH}^{k,-k+m}(V_* \sqcap A_*, V_* \sqcap B_*) = 0$$

for all $k \geq 1, m \leq 0$.

Proof. For $m = 0$, the assertion follows from Lemma 9.2. For $m < 0$, there are no nontrivial maps in $\mathrm{Hom}_{\mathbb{F}_2}(K^{k+1}_{k+1}(V_* \sqcap A_*), V_m \oplus B_m)$, since both V_m and B_m are trivial for negative m. □

Recall that, for every subring $A \subset B$ and for every $x \in B$, $A\langle x\rangle$ denotes the subring of B generated by A and x.

Proposition 9.4. *Let $A \subset B$ be a subring. Then for every $x \in B$ the induced map $G^{k,1-k}(A\langle x\rangle) \to G^{k,1-k}(A)$ is surjective for all $k \geq 2$.*

Proof. Let $f \in G^{k,1-k}(A)$ be an arbitrary cocycle. We need to show that f has a lift to $G^{k,1-k}(A\langle x\rangle)$. Note that for every subring $C \subset B$ every function in

$$\mathrm{Hom}_{\mathbb{F}_2}(K^k_k(V_* \sqcap C_*), V_1) \subseteq \mathrm{Hom}_{\mathbb{F}_2}(K^k_k(V_* \sqcap C_*), V_1 \oplus B_1)$$

is a cocycle, since the multiplication with elements of degree ≥ 1 in V_* vanishes. Moreover, every $\mathbb{F}_2$-vector space homomorphism $K^k_k(V_* \sqcap A_*) \to V_1$ can be factored through $K^k_k(V_* \sqcap A_*) \to K^k_k(V_* \sqcap A\langle x\rangle_*)$ by sending x to zero. This implies that the map

$$\mathrm{Hom}_{\mathbb{F}_2}(K^k_k(V_* \sqcap A\langle x\rangle_*), V_1) \to \mathrm{Hom}_{\mathbb{F}_2}(K^k_k(V_* \sqcap A_*), V_1)$$

is surjective. Since every $\mathbb{F}_2$-vector space homomorphism $K^k_k(V_* \sqcap A_*) \to V_1 \oplus B_1$ can be written as a sum of homomorphisms into V_1 and B_1, respectively, it remains to show that cocycles with values in B_1 can be lifted.

We therefore assume from now on that the cocycle $f \in G^{k,1-k}(A)$ takes values in B_1. Consider B_1 as a B-module and write $f = xf + (1-x)f$. Note that xf is again a cocycle, since multiplication of x with elements in $V_* \sqcap A_*$ vanishes. The definition of the differential then yields that xf is a cocycle. That implies that $(1-x)f$ is a cocycle as well. Therefore it will be enough to show that xf and $(1-x)f$ have lifts to $G^{k,1-k}(A\langle x\rangle)$. Since the roles of x and $1-x$ are symmetrical, we may assume without the loss of generality that $f = xf$. We are going to construct the lift when A is finite first. We will need some preparations to do so. For more background on Boolean rings and proofs of the assertions on Boolean rings we refer to appendix A.

Notation 9.5. For every Boolean ring C, let $\mathbf{S}(C) = \mathrm{Spec}(C)$ denote the spectrum of C. For every set X, let $\mathbf{B}(X)$ denote the ring of $\mathbb{F}_2$-valued functions on X. Note that every prime ideal in a Boolean ring C is the kernel of a ring homomorphism $C \to \mathbb{F}_2$, see Proposition A.3 and Theorem A.10. So every element $x \in C$ gives rise to a function in $\mathbf{B}(\mathbf{S}(C))$ which furnishes a ring homomorphism $\mathfrak{f}\colon C \to \mathbf{B}(\mathbf{S}(C))$. This map is an isomorphism by Theorem A.6.

Remark 9.6. We say that an element $x \in C$ is an *atom* if the support of $\mathfrak{f}(x)$ is a one element set, i.e., we have $\mathfrak{f}(x)(p) \neq 0$ for exactly one $p \in \mathbf{S}(C)$. From this perspective the unique orthonormal basis of a finite Boolean ring C is just the collection of all atoms of C, see also Corollary A.16. Therefore, we have a natural bijection between the elements of the unique orthonormal basis of C and the elements of $\mathbf{S}(C)$. We will identify these two sets in all that follows.

Definition 9.7. For every homomorphism $h\colon C \to D$ between Boolean rings let $h^*\colon \mathbf{S}(D) \to \mathbf{S}(C)$ be the induced map. Let $a\colon A \to A\langle x\rangle$ be the inclusion map. Since a is injective, the induced map $a^*\colon \mathbf{S}(A\langle x\rangle) \to \mathbf{S}(A)$ is surjective. Since every ring homomorphism $A\langle x\rangle \to \mathbb{F}_2$ is uniquely determined by its restriction to A and x, the fibres of a^* have at most two elements. Let $s\colon \mathbf{S}(A) \to \mathbf{S}(A\langle x\rangle)$ be the section of a^* such that $s(p)$ is the unique element of $(a^*)^{-1}(p)$, if $(a^*)^{-1}(p)$ is a one element set, and $s(p)$ is the unique element of $(a^*)^{-1}(p)$ such that $\mathfrak{f}(x)(s(p)) = 1$, otherwise, for every $p \in \mathbf{S}(A)$.

Notation 9.8. Fix a basis I of V. For every homomorphism $h\colon C \to D$ between Boolean rings let by slight abuse of notation h^* also denote the map $I \cup \mathbf{S}(D) \to I \cup \mathbf{S}(C)$ which is the identity on I and the map induced by h restricted to $\mathbf{S}(D)$. Similarly let s also denote the map $I \cup \mathbf{S}(A) \to I \cup \mathbf{S}(A\langle x\rangle)$ which is the identity on I and the section introduced above, when restricted to $\mathbf{S}(A)$. Moreover for every sequence $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup \mathbf{S}(D)$ let $h^*(\mathbf{t})$ denote the sequence $h^*(t_1), h^*(t_2), \ldots, h^*(t_k) \in I \cup \mathbf{S}(C)$, and for every sequence $t_1, t_2, \ldots, t_k \in I \cup \mathbf{S}(A)$ let $s(\mathbf{t})$ denote the sequence $s(t_1), s(t_2), \ldots, s(t_k) \in I \cup \mathbf{S}(A\langle x\rangle)$.

Definition 9.9. Let C be a subring of B. Under the identification in Remark 9.6 above, admissible sequences for C are taking values in $I \cup \mathbf{S}(C)$. A sequence $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup \mathbf{S}(A\langle x\rangle)$ is *x-admissible* if it is admissible and $\mathbf{t} = s(a^*(\mathbf{t}))$.

Lemma 9.10. *The following holds:*

(*i*) *if $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup \mathbf{S}(A\langle x\rangle)$ is x-admissible then $a^*(\mathbf{t})$ is admissible,*
(*ii*) *the maps a^* and s induce a bijection between the set of x-admissible sequences in $I \cup \mathbf{S}(A\langle x\rangle)$ and admissible sequences in $I \cup \mathbf{S}(A)$.*

Proof. If $a^*(\mathbf{t})$ were not admissible, there would be an index i such that $a^*(t_i) = a^*(t_{i+1})$. Since for every $u \in I \cup \mathbf{S}(A)$ the element $s(u)$ is unique, this means that $t_i = t_{i+1}$ which contradicts that $\mathbf{t}$ is admissible. So claim (i) is true. By construction s maps admissible sequences in $I \cup \mathbf{S}(A)$ to x-admissible sequences in $I \cup \mathbf{S}(A\langle x\rangle)$. Therefore a^* and s are maps between these sets, and since they are inverses of each other, we get that claim (ii) holds, too. □

Now we are going to define a function $f_x \in \mathrm{Hom}_{\mathbb{F}_2}(K^k_k(V_* \sqcap A\langle x\rangle_*), B_1)$ as the unique function such that for every admissible sequence $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup \mathbf{S}(A\langle x\rangle)$ we have:

$$f_x(x_{\mathbf{t}}) = \begin{cases} f(x_{a^*(\mathbf{t})}) & \text{, if } \mathbf{t} \text{ is } x\text{-admissible,} \\ 0 & \text{, otherwise.} \end{cases}$$

For every admissible sequence $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup \mathbf{S}(A)$ we compute:

$$f_x(x_{\mathbf{t}}) = \sum_{\mathbf{u} \in (a^*)^{-1}(\mathbf{t})} f_x(x_{\mathbf{u}}) = f_x(x_{s(\mathbf{t})}) = f(x_{\mathbf{t}}),$$

where we used the additivity of f_x and part (ii) of Lemma 9.10 in the first equation, and its definition in the second and third equations. Therefore the restriction of f_x onto $K^k_k(V_* \sqcap A_*)$ is f. Since the map a^* commutes with translations, and the translates of x-admissible sequences are x-admissible, in order to show that f_x is a cocycle it will be sufficient to show for every x-admissible sequence $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup \mathbf{S}(A\langle x\rangle)$ that

$$f_x(x_{\mathbf{t}}) \in (t_1, t_k).$$

Note that $xt \in (s(t))$ for every $t \in I \cup \mathbf{S}(A)$. Therefore

$$f_x(x_{\mathbf{t}}) = f(x_{a^*(\mathbf{t})}) = xf(x_{a^*(\mathbf{t})}) \in x(a^*(t_1), a^*(t_k)) \subseteq (xa^*(t_1), xa^*(t_k)) \subseteq (t_1, t_k)$$

using that $s(a^*(\mathbf{t})) = \mathbf{t}$. This concludes the proof when A is finite.

Since every subring of a Boolean ring is Boolean, and every Boolean ring is the union of the directed set of its finite subrings, we need to show that the lifts we have constructed for finite subrings are compatible in order to prove the claim in general. Compatibility means the following: let $A \subseteq \widetilde{A}$ be a pair of finite subrings of B. Let $\widetilde{f} \in G^{k,1-k}(\widetilde{A})$ be a cocycle such that $\widetilde{f} = x\widetilde{f}$. (The latter condition implies that $\widetilde{f}$ takes values in B_1.) Let $f \in G^{k,1-k}(A)$ be the restriction of $\widetilde{f}$ onto $K^k_k(V_* \sqcap A_*)$. Then f_x is the restriction of the similarly constructed lift $\widetilde{f}_x$ of $\widetilde{f}$ onto $K^k_k(V_* \sqcap A\langle x\rangle_*)$.

Notation 9.11. Let $\widetilde{a}\colon \widetilde{A} \to \widetilde{A}\langle x\rangle$ be the inclusion map. Let $\widetilde{s}\colon I \cup \mathbf{S}(\widetilde{A}) \to I \cup \mathbf{S}(\widetilde{A}\langle x\rangle)$ be the section of $\widetilde{a}^*$ defined the same way as s and let the same symbol denote its extension to a map from series in $I \cup \mathbf{S}(\widetilde{A})$ to series in $I \cup \mathbf{S}(\widetilde{A}\langle x\rangle)$ by our usual abuse of notation. Let $i : A \to \widetilde{A}$ and $j : A\langle x\rangle \to \widetilde{A}\langle x\rangle$ be the inclusion maps.

Lemma 9.12. *The following holds:*

- (i) *if* $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup \mathbf{S}(\widetilde{A}\langle x\rangle)$ *is x-admissible then* $j^*(\mathbf{t})$ *is x-admissible, too,*
- (ii) *for every admissible* $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup \mathbf{S}(A)$ *the maps* $\widetilde{a}^*$ *and* $\widetilde{s}$ *induce a bijection between the set of x-admissible sequences in* $(j^*)^{-1}(s(\mathbf{t}))$ *and admissible sequences in* $(i^*)^{-1}(\mathbf{t})$.

Proof. By part (i) of Lemma 9.10 we already know that $j^*(\mathbf{t})$ is admissible. So if $a^*(\mathbf{t})$ were not x-admissible, there would be an index i such that $j^*(t_i) \neq s(a^*(j^*(t_i)))$. Then $t_i \neq \widetilde{s}(\widetilde{a}^*(t_i))$ either, since $j^* \circ \widetilde{s} \circ \widetilde{a}^* = s \circ a^* \circ j^*$, which is a contradiction. So claim (i) holds. Note that $\widetilde{a}^* \circ i^* = j^* \circ a^*$ and $\widetilde{s} \circ j^* = i^* \circ s$. Therefore $\widetilde{a}^*$ and $\widetilde{s}$ are maps between these sets, and since they are inverses of each other, we get that claim (ii) holds, too. □

Now we are ready to show that f_x is the restriction of $\widetilde{f}_x$. It will be sufficient to check this on $x_{\mathbf{t}}$ where $\mathbf{t} = t_1, t_2, \ldots, t_k \in I \cup \mathbf{S}(A\langle x\rangle)$ is admissible. If $\mathbf{t}$ is not x-admissible, then there is no x-admissible sequence in $(j^*)^{-1}(\mathbf{t})$ by Lemma 9.12, so

$$\widetilde{f}_x(x_{\mathbf{t}}) = \sum_{\mathbf{u} \in (j^*)^{-1}(\mathbf{t})} \widetilde{f}_x(x_{\mathbf{u}}) = 0 = f_x(x_{\mathbf{t}}),$$

where we used the additivity of $\widetilde{f}_x$ in the first equation, and the definition of $\widetilde{f}_x$ and f_x in the second and third equations, respectively. If $\mathbf{t}$ is x-admissible, then

$$\begin{aligned}\widetilde{f}_x(x_{\mathbf{t}}) = \sum_{\mathbf{u} \in (j^*)^{-1}(\mathbf{t})} \widetilde{f}_x(x_{\mathbf{u}}) &= \sum_{\substack{\mathbf{u} \in (j^*)^{-1}(\mathbf{t}) \\ \mathbf{u} = \widetilde{s}(\widetilde{a}^*(\mathbf{u}))}} \widetilde{f}(x_{\widetilde{a}^*(\mathbf{u})}) \\ &= \sum_{\mathbf{u} \in (i^*)^{-1}(a^*(\mathbf{t}))} \widetilde{f}(x_{\mathbf{u}}) = \widetilde{f}(x_{a^*(\mathbf{t})}) = f(x_{a^*(\mathbf{t})}) = f_x(x_{\mathbf{t}}),\end{aligned}$$

where we used the additivity of $\widetilde{f}_x$ in the first equation, the definition of $\widetilde{f}_x$ in the second equation, part (ii) of Lemma 9.12 in the third equation, the additivity of $\widetilde{f}$

in the fourth equation, the fact that f is the restriction of $\widetilde{f}$ in the fifth equation, and the definition of f_x in the last equation. This finishes the proof of Proposition 9.4. □

We can now use Proposition 9.4 to show the vanishing of Hochschild cohomology groups. Let $A \subset B$ be an infinite subring. Let $\kappa = |A|$ be its cardinality and choose a bijection $x\colon \kappa \to A$. Let $\mathcal{SA}$ denote the category of subrings of A with respect to the inclusions as morphisms. By part (ii) of Lemma 8.5 we have a functor $\iota\colon \kappa \to \mathcal{SA}$ with $\iota(\alpha) = A_\alpha$ for every $\alpha \in \kappa$. Let $F^k, G^k, H^k\colon \kappa^{op} \to \mathcal{AB}$ be the composition of ι with the contravariant functors:

$$C \mapsto F^{k-1,1-k}(C), \quad C \mapsto G^{k,1-k}(C), \quad C \mapsto H^k(C) = \mathrm{HH}^{k,1-k}(V_* \sqcap C_*, V_* \sqcap B_*)$$

from $\mathcal{SA}$ to $\mathcal{AB}$, respectively.

Proposition 9.13. *We have* $\varprojlim^n H^k = 0$ *for every* $n > 0$.

Proof. By Lemma 9.2 there is a short exact sequence:

$$0 \longrightarrow F^k \xrightarrow{\partial_k} G^k \longrightarrow H^k \longrightarrow 0$$

of functors. Therefore it will be enough show that both F^k and G^k are Mittag-Leffler by Lemma 8.4. Since F^k is just the dual of an inductive system of vector spaces, it is Mittag-Leffler. As we can check that an element of $F^{k,1-k}(C)$ is a cocycle on $(k{+}2)$-uples of elements of C, we have $G^{k,1-k}(A_\alpha) = \lim_{\beta \in \alpha} G^{k,1-k}(A_\beta)$ for every limit ordinal $\alpha \in \kappa$, and hence G^k satisfies property (ML2) of Definition 8.1 . By Proposition 9.4 the functor G^k satisfies property (ML0) of Lemma 8.2. Hence G^k is Mittag-Leffler by Lemma 8.2. □

Now we are ready to prove our main

Theorem 9.14. *Let B be an infinite Boolean ring. Let $A \subseteq B$ be a subring such that $|A| \geq 8$. Let $m < 0$ be a negative integer. Then*

$$\mathrm{HH}^{k,m}(V_* \sqcap A_*, V_* \sqcap B_*) = 0$$

for every $k \geq 0$ such that $k \neq 1 - m$. In particular, we have

$$\mathrm{HH}^{n,2-n}(V_* \sqcap B_*) = 0 \text{ for all } n \geq 3.$$

Hence the graded algebra $V_* \sqcap B_*$ is intrinsically formal.

Proof. We already know that this is true when A is finite by Theorem 7.13, so we may assume without the loss of generality that A is infinite. Let $\kappa = |A|$ be its cardinality. We prove Theorem 9.14 by transfinite induction on the cardinality κ of A. This means we may assume that the claim holds for every subring of A of cardinality less than κ. In particular, the claim of the theorem holds for A_α for every $\alpha \in \kappa$ by part (v) of Lemma 8.5, where $\{A_\alpha \subseteq A \mid \alpha \in \kappa\}$ denotes a system of subrings as constructed in (15) in Section 8. Fix an integer $m < 0$. Since the higher limits of identically zero functors vanish, all rows of the spectral sequence of Proposition 5.10:

$$E_2^{p,q} = \varprojlim_{\alpha}{}^p \mathrm{HH}^{q,m}(V_* \sqcap (A_\alpha)_*, V_* \sqcap B_*) \Rightarrow \mathrm{HH}^{p+q,m}(V_* \sqcap A_*, V_* \sqcap B_*)$$

are zero except when $q = 1 - m$. By Proposition 9.13 all terms of this row except $E_2^{0,1-m} = \varprojlim_\alpha \mathrm{HH}^{1-m,m}(V_* \sqcap (A_\alpha)_*, V_* \sqcap B_*)$ are zero, too. Hence the spectral sequence degenerates and consequently the theorem holds for A, too. The last assertion follows from Theorem 4.7. □

10. Proof of the main result

Let G be a profinite group which is real projective. Let $\mathcal{X}(G)$ denote the conjugacy classes of involutions of G equipped with its natural topology, and let $B = C(\mathcal{X}(\Gamma), \mathbb{F}_2)$ be the ring of continuous functions from $\mathcal{X}(G)$ to $\mathbb{F}_2$ where we equip the latter with the discrete topology. Note that B is a Boolean ring. Let B_* be the associated Boolean graded algebra.

By Scheiderer's Theorem 2.11, there is a surjective homomorphism of graded $\mathbb{F}_2$-algebras:

$$\pi\colon H^*(G, \mathbb{F}_2) \to B_*$$

such that the n-th degree component:

$$\pi_n\colon H^n(G, \mathbb{F}_2) \to B_n$$

is an isomorphism except perhaps when $n = 1$, and it is surjective when $n = 1$. Let $V \subseteq H^1(G, \mathbb{F}_2)$ be the kernel of π_1, and let V_* be the associated dual algebra over $\mathbb{F}_2$. By choosing a section

$$\phi\colon B_1 \to H^1(G, \mathbb{F}_2)$$

of π_1 we get an isomorphism of graded $\mathbb{F}_2$-algebras:

$$H^*(G, \mathbb{F}_2) \cong B_* \sqcap V_*.$$

By Theorem 9.14, the connected sum $B_* \sqcap V_*$ satisfies Kadeishvili's vanishing and is intrinsically formal. Thus $H^*(G, \mathbb{F}_2)$ satisfies Kadeishvili's vanishing and is intrinsically formal. By Corollary 4.8 this implies that the differential graded algebra $C^*(G, \mathbb{F}_2)$ is formal. This finishes the proof of Theorem 1.3.

Finally, as explained previously, the Galois groups of fields with virtual cohomological dimension one are real projective profinite groups. Hence, Proposition 6.6 implies Theorem 1.11.

Appendix A. Boolean rings

In this section we summarise the assertions on Boolean rings we use in Sections 6, 7 and 9.

Definition A.1. A ring R is called *Boolean* if $x^2 = x$ for every $x \in R$.

Examples A.2. The field with two elements $\mathbb{F}_2$ is a Boolean ring. In fact, since $x(1-x) = 0$ for all x in a Boolean ring, $\mathbb{F}_2$ is the only Boolean integral domain. The direct product of Boolean rings is Boolean, and so, for every set X, the direct product ring:

$$\mathbb{F}_2^X = \prod_{i \in X} \mathbb{F}_2$$

is a Boolean ring. Now let X be a topological space, and let $\mathbf{B}(X)$ denote the ring of functions $f\colon X \to \mathbb{F}_2$ which are continuous with respect to the discrete topology on $\mathbb{F}_2$. Since the subrings of Boolean rings are Boolean, and $\mathbf{B}(X)$ is a subring of $\mathbb{F}_2^X$, we get that $\mathbf{B}(X)$ is Boolean, too.

Proposition A.3. *In a Boolean ring R the following hold:*

(*i*) *we have $2x = 0$ for every $x \in R$,*

(*ii*) *every prime ideal* $\mathfrak{p}$ *is maximal, and* $R/\mathfrak{p}$ *is the field with two elements,*
(*iii*) *we have* $(x, y) = (x + y - xy)$ *for every* $x, y \in R$,
(*iv*) *every finitely generated ideal is principal.*

Proof. Since
$$2x = (2x)^2 = 4x^2 = 4x,$$
we get that $2x = 0$ by subtracting $2x$ from both sides. Now let $\mathfrak{p}$ be a prime ideal in R. Then the quotient $R/\mathfrak{p}$ is a Boolean ring and an integral domain. By Example A.2, this implies $R/\mathfrak{p} \cong \mathbb{F}_2$. Note that
$$x(x + y - xy) = x^2 + xy - x^2y = x + xy - xy = x.$$
Hence $x, y \in (x + y - xy)$. Since $x + y - xy \in (x, y)$, claim (*iii*) is clear. Let $I = (x_1, x_2, \ldots, x_n)$ be an finitely generated ideal of R. Since
$$I = ((x_1, x_2, \ldots, x_{n-1}), x_n),$$
we may assume by induction on n that $I = (x, y)$ for some $x, y \in R$. The claim now follows from part (*iii*). □

Proposition A.4. *The spectrum* Spec(R) *of a Boolean ring is compact and totally separated.*

Proof. Since the spectrum of a commutative ring with a unity is compact, the same holds for Spec(R), too. Recall that a topological space X is totally separated if for any two distinct points $x, y \in X$ there exist disjoint open sets $U \subset X$ containing x and $V \subset X$ containing y such that X is the union of U and V. Now let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ be two distinct points. Since they are maximal ideals, there is an $x \in R$ such that $x \in \mathfrak{p}$ and $x \notin \mathfrak{q}$. Since $R/\mathfrak{p} = \mathbb{F}_2$ by part (*ii*) of Proposition A.3, the former is equivalent to $1 - x \notin \mathfrak{p}$. As usual for every $f \in R$ let $D(f) \subseteq \mathrm{Spec}(R)$ denote the open subset

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\}. \tag{16}$$

Then $\mathfrak{p} \in D(1-x), \mathfrak{q} \in D(x)$, the intersection $D(x) \cap D(1-x)$ is empty by part (*ii*) of Proposition A.3, while the union $D(x) \cup D(1-x)$ is Spec(R), since if $x, 1-x \in \mathfrak{m}$ for some $\mathfrak{m} \in \mathrm{Spec}(R)$ then $1 \in \mathfrak{m}$ which is a contradiction. □

Notation A.5. Let R be a Boolean ring. Then for every $a \in R$ the corresponding section of the structure sheaf of Spec(R) is a continuous function
$$\sigma(a) \colon \mathrm{Spec}(R) \to \mathbb{F}_2$$
by part (*ii*) of Proposition A.3, and the furnished map
$$\sigma \colon R \to \mathbf{B}(\mathrm{Spec}(R))$$
is a ring homomorphism.

Theorem A.6. *For every Boolean ring* R*, the map* $\sigma \colon R \to \mathbf{B}(\mathrm{Spec}(R))$ *is an isomorphism.*

Proof. For every $x \in R$ we have $x^n = x$ by induction, so if x is nilpotent, then it is zero. Therefore the nilradical of R is zero, so by Krull's theorem σ is injective. So we only need to show that σ is surjective. Let $f \colon \mathrm{Spec}(R) \to \mathbb{F}_2$ be a continuous function. Then the set
$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f(x) = 1\}$$

is a closed subset, so it is compact. But it is also open, so it can be covered by open subsets of the form $D(a)$, where $a \in R$ by the definition of the topology of the spectrum of rings. Since $D(f)$ is compact, it can be covered by finitely many such, so

$$D(f) = D(a_1) \cup D(a_2) \cup \cdots \cup D(a_n)$$

for some $a_1, a_2, \ldots, a_n \in R$. By part (iv) of Proposition A.3, there is an $a \in R$ such that $(a) = (a_1, \ldots, a_n)$. Then

$$\begin{aligned} D(a) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid a \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid (a_1, \ldots, a_n) \not\subseteq \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid a_i \notin \mathfrak{p} \text{ for some } i\} \\ &= D(a_1) \cup D(a_2) \cup \cdots \cup D(a_n), \end{aligned}$$

so $D(f) = D(a)$. Since f and a take values in $\mathbb{F}_2$, we get that $f = a$. □

Notation A.7. Let X be a topological space. Then for every $p \in X$ the set

$$\beta(p) = \{x \in \mathbf{B}(X) \mid x(p) = 0\}$$

is the kernel of a surjective ring homomorphism $\mathbf{B}(X) \to \mathbb{F}_2$, so it is a maximal ideal in $\mathbf{B}(X)$. Consequently, we have an induced map

$$\beta \colon X \to \mathrm{Spec}(\mathbf{B}(X)).$$

For every $x \in \mathbf{B}(X)$ the pre-image

$$\beta^{-1}(D(x)) = \{p \in X \mid x(p) = 1\}$$

is open. Since the sets $\{D(x) \mid x \in \mathbf{B}(X)\}$ form a sub-basis of $\mathrm{Spec}(\mathbf{B}(X))$, we get that $\beta \colon X \to \mathrm{Spec}(\mathbf{B}(X))$ is continuous.

Theorem A.8. *When X is compact and totally separated, then β is a homeomorphism.*

Proof. Since X is compact and $\mathrm{Spec}(\mathbf{B}(X))$ is Hausdorff, it will be sufficient to show that β is a bijection. Let $x, y \in X$ be two distinct points. Since X is totally separated, so there exist disjoint open sets $U \subset X$ containing x and $V \subset X$ containing y such that X is the union of U and V. Let $f \colon X \to \mathbb{F}_2$ be the characteristic function of U. It is in $\mathcal{B}(X)$ since the complement of U is open, too. Clearly $f \in \beta(y)$, but $f \notin \beta(x)$, so β is injective.

For every ideal $I \lhd \mathbf{B}(X)$ let $Z(I) \subseteq X$ denote the closed subset

$$Z(I) = \{x \in X \mid f(x) = 0 \quad (\forall f \in I)\}.$$

We claim that for every proper ideal $I \lhd \mathbf{B}(X)$ the set $Z(I)$ is non-empty. First consider the case when $I = (f)$ for some $f \in \mathbf{B}(X)$. Then

$$Z(I) = \{x \in X \mid f(x) = 0\},$$

so if this set is empty, then f is the identically one function, and hence $I = (1) = \mathcal{B}(X)$ is not proper, a contradiction. Next consider the case when I is finitely generated; then it is principal by part (iv) of Proposition A.3, so $Z(I)$ is non-empty by the above. Finally, consider the general case. Then $Z(I)$ is the intersection of sets of the form $Z(J)$ where J is a finitely generated ideal of I. Since the latter collection of sets is closed under finite intersections, and each member is non-empty by the above, we get $Z(I)$ is also non-empty, since X is compact.

Now let $\mathfrak{m} \lhd \mathbf{B}(X)$ be a maximal ideal. By the above there is an $x \in Z(\mathfrak{m})$. Clearly $\beta(x) \supseteq \mathfrak{m}$, but $\mathfrak{m}$ is maximal, and hence $\beta(x) = \mathfrak{m}$. Therefore β is surjective, too. □

Notation A.9. Let **BO** denote the category of Boolean rings where morphisms are ring homomorphisms, and let **CTS** denote the category of compact, totally separated topological spaces where morphisms are continuous maps. There are two contravariant functors

$$\mathbf{B}\colon \mathbf{CTS} \to \mathbf{BO}, \quad X \mapsto \mathbf{B}(X),$$

which is well-defined as we saw in Examples A.2, and

$$\mathbf{S}\colon \mathbf{BO} \to \mathbf{CTS}, \quad R \mapsto \mathrm{Spec}(R),$$

well-defined by Proposition A.4.

Theorem A.10 (Stone duality)**.** *The functors* **B** *and* **S** *are a pair of dualities of categories.*

Proof. By Theorem A.6, the map σ is a natural isomorphism between the identity of **BO** and $\mathbf{B} \circ \mathbf{S}$. By Theorem A.8, the map β is a natural isomorphism between the identity of **CTS** and $\mathbf{S} \circ \mathbf{B}$. □

Corollary A.11. *Let R be a finite Boolean ring. Then the following are equivalent:*

(*i*) *R is finite.*
(*ii*) *R is finitely generated as an $\mathbb{F}_2$-algebra.*
(*iii*) *R is Noetherian.*
(*iv*) *R is Artinian.*
(*v*) $\mathrm{Spec}(R)$ *is finite.*
(*vi*) *$R \cong \mathbb{F}_2^X$ for a finite set X.*

In this case $R \cong \mathbb{F}_2^{\mathrm{Spec}(R)}$.

Proof. Every Boolean ring is an $\mathbb{F}_2$-algebra, so if R is finite, then it is finitely generated as an $\mathbb{F}_2$-algebra, and hence (*i*) implies (*ii*). Every finitely generated $\mathbb{F}_2$-algebra is Noetherian by Hilbert's basis theorem, so (*ii*) implies (*iii*). Every Boolean ring is zero dimensional by part (*ii*) of Proposition A.3, so if it is Noetherian, it is Artinian by a standard theorem in commutative algebra (see Theorem 8.5 of [1] on page 90). Therefore (*iii*) implies (*iv*). Every Artinian ring is a finite direct product of Artinian local rings (see Theorem 8.7 of [1] on page 90), so its spectrum is finite. Therefore (*iv*) implies (*v*). Now let R be a Boolean ring whose spectrum is finite. Since $\mathrm{Spec}(R)$ is totally separated by Proposition A.4, it is discrete, and hence $R \cong \mathbb{F}_2^{\mathrm{Spec}(R)}$ by Theorem A.6. In particular, (*v*) implies (*vi*). If $R \cong \mathbb{F}_2^X$ for a finite set X, then R is clearly finite, so (*vi*) implies (*i*). □

Notation A.12. Let **FBO** denote the category of finite Boolean rings where morphisms are ring homomorphisms, and let **FTS** denote the category of finite, totally separated topological spaces where morphisms are continuous maps. Note that the latter is the same as the category of finite, discrete topological spaces. There are two restrictions of functors

$$\mathbf{B}|_{\mathbf{FTS}}\colon \mathbf{FTS} \to \mathbf{FBO}, \quad X \mapsto \mathbf{B}(X)$$

and

$$\mathbf{S}|_{\mathbf{FBO}}\colon \mathbf{FBO} \to \mathbf{FTS}, \quad R \mapsto \mathrm{Spec}(R),$$

where the latter is well-defined by Corollary A.4.

Theorem A.13. *The functors* $\mathbf{B}|_{\mathbf{FTS}}$ *and* $\mathbf{S}|_{\mathbf{FTS}}$ *are a pair of dualities of categories.*

Proof. The restrictions of the natural isomorphisms β and σ onto **FTS** and **FBO** are the respective required natural isomorphisms. □

Definition A.14. Let R be a Boolean ring. We say that two elements $x, y \in R$ are orthogonal if $xy = 0$. We say that an element $a \in R$ is an *atom* if it is non-zero and cannot be written as the sum of two non-zero orthogonal elements of R. The *support* of an element $x \in R$ is the subset $D(x) \subseteq \mathrm{Spec}(X)$ introduced in (16).

Lemma A.15. *Let R be a Boolean ring. Then the following holds:*

- (i) *Two elements $x, y \in R$ are orthogonal if and only if the intersection of their supports is empty.*
- (ii) *A non-zero element $a \in R$ is an atom if and only if its support cannot be written as the disjoint union of two non-empty open and closed subsets.*
- (iii) *Every pair of different atoms of R are orthogonal to each other.*

Proof. Note that the support of the product xy is the intersection of the supports of x and y. Since the support of an element of R is empty if and only if it is zero by Theorem A.6, claim (i) follows. If $a = x + y$ such that x, y are orthogonal and both non-zero, then the support of a is the disjoint union of the supports of x and y by part (i). On the other hand if the support of a is the disjoint union of the non-empty open and closed subsets X and Y, there are elements $x, y \in R$ whose support is X, Y, respectively, by Theorem A.6. By part (i) these are orthogonal, non-zero and their sum has the same support as a. So $a = x + y$, and hence claim (ii) is true.

Let $a, b \in R$ be two atoms whose product is non-zero. Then $a = ab + a(1-b)$ and $aba(1-b) = a^2(b - b^2) = 0$, so $a(1-b) = 0$ by the definition of atoms. We get that $a = ab$. The same reasoning for b show that $b = ab$. Therefore $a = b$, and hence (iii) holds. □

Corollary A.16. *Let R be a finite Boolean ring. Then the atoms of R form a natural basis of R whose elements are orthogonal to each other. Moreover, every orthogonal basis consists of atoms.*

Proof. By Corollary A.11, the topological space $\mathrm{Spec}(R)$ is finite, discrete, and $R \cong \mathbb{F}_2^{\mathrm{Spec}(R)}$. Therefore, an element of R is an atom if and only if its support is a point by claim (ii) of Lemma A.15. These clearly form a basis of R and they are orthogonal by claim (iii) of Lemma A.15.

Now let $e_1, e_2, \ldots, e_n$ be an orthogonal basis. We need to show that each e_i is an atom. We may assume without the loss of generality that $i = 1$. Let x be characteristic function of an element of the support of e_1. Then x is an atom. The support of e_1 and e_i, $i \neq 1$, is disjoint since they are orthogonal. Hence x and e_i are orthogonal, so $xe_i = 0$. Now write x as a linear combination $\sum_{j \in J} e_j$ of the e_i. Since $0 = xe_i = \sum_{j \in J} e_j e_i$ only if i is not in J, the above argument implies $x = 0$ or $x = e_1$. The former is not possible, since x is not zero. Hence we get $x = e_1$, and e_1 is an atom. □

Appendix B. Proof of Proposition 5.10

For lack of a reference known to the authors, we provide a proof of the existence of the spectral sequence claimed in Proposition 5.10. The proof is an adjustment of the construction of a spectral sequence for the composition of a right derived functor with an inductive limit in [15, Chapter 4]. Even though Hochschild cohomology is a right derived functor we need to adjust for the fact that we take Hochschild cohomology of a system over varying base rings.

Proof of Proposition 5.10. We recall from Remark 4.3 and (8) that we can identify the graded Hochschild cohomology of A and M with the cohomology

$$\mathrm{HH}^{n,s}(A,M) = H^n(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes *}, M[s]), d^*).$$

To compute the higher inverse limits occurring in the spectral sequence we will use the resolutions of [15, Chapter 4]. Let $\{S_\alpha, f_{\alpha\beta}\}_\alpha$ be a projective system of $\mathbb{F}_2$-vector spaces. We denote by $(\Pi^* S_\alpha, \delta^*)$ the cochain complex defined in [15, Chapter 4, page 32] whose nth space $\Pi^n S_\alpha$ is the product

$$\prod_{\alpha_0 \le \ldots \le \alpha_n} S_{\alpha_0 \ldots \alpha_p} \text{ with } S_{\alpha_0 \ldots \alpha_n} = S_{\alpha_0}$$

consisting of families $s_{\alpha_0 \ldots \alpha_n}$ of elements in S_{α_0} indexed over tuples $\alpha_0 \le \ldots \le \alpha_n$ in I. These families inherit the structure of $\mathbb{F}_2$-vector spaces with componentwise operations. By [15, Théorème 4.1], the pth higher inverse limit of the system $\{S_\alpha, f_{\alpha\beta}\}$ equals the pth cohomology group of this cochain complex, i.e.,

$$\varprojlim_\alpha{}^p S_\alpha = H^p(\Pi^* S_\alpha, \delta^*).$$

For an inductive system $\{T_\alpha, g_{\alpha\beta}\}_\alpha$, we denote by $(\Sigma^* T_\alpha, \partial^*)$ the chain complex defined in [15, page 33] with

$$\Sigma^n T_\alpha = \bigoplus_{\alpha_0 \le \ldots \le \alpha_n} T_{\alpha_0 \ldots \alpha_n} \text{ with } T_{\alpha_0 \ldots \alpha_n} = T_{\alpha_0}.$$

The chain complex $\Sigma^* T_\alpha$ is dual to Π^* in the sense that

$$\underline{\mathrm{Hom}}_{\mathbb{F}_2}(\Sigma^* T_\alpha, M) \cong \Pi^* \underline{\mathrm{Hom}}_{\mathbb{F}_2}(T_\alpha, M). \tag{17}$$

Now we construct the spectral sequence following [15, page 34–35]. We denote the transition maps of the directed system of subrings A_α of A by $\iota_{\alpha\beta} \colon A_\beta \hookrightarrow A_\alpha$. We define a double complex $C^{*,*} = \underline{\mathrm{Hom}}_{\mathbb{F}_2}(\Sigma A_\alpha, M)$ by setting $C^{p,q}$ to be

$$C^{p,q} = \underline{\mathrm{Hom}}_{\mathbb{F}_2}\left(\bigoplus_{\alpha_0 \le \ldots \le \alpha_p} (A_{\alpha_0 \ldots \alpha_p})^{\otimes q}, M \right).$$

The differential in the p-direction is the one induced by ∂^*. For fixed p and $A_{\alpha_0 \ldots \alpha_p} = A_{\alpha_0}$, the differential in the q-direction is the differential d^* of the complex $\underline{\mathrm{Hom}}_{\mathbb{F}_2}((A_{\alpha_0 \ldots \alpha_p})^{\otimes q}, M)$.

We consider the two spectral sequences that arise from this double complex. First let p be the filtration-index. Then, using isomorphism (17), we get

$$\begin{aligned} E_1'^{p,q} &= \prod_{\alpha_0 \leq \ldots \leq \alpha_p} H^q \underline{\mathrm{Hom}}_{\mathbb{F}_2}((A_{\alpha_0 \ldots \alpha_p})^{\otimes *}, M) \\ &= \prod_{\alpha_0 \leq \ldots \leq \alpha_p} \mathrm{HH}^{q,s}(A_{\alpha_0 \ldots \alpha_p}, M). \end{aligned}$$

By [15, Théorème 4.1] applied to the projective system $\{\mathrm{HH}^{q,s}(A_\alpha, M), \mathrm{HH}^{q,s}(f_{\alpha\beta}, M)\}_\alpha$, taking cohomology in the p-direction gives

$$E_2'^{p,q} = H^p \left(\prod_{\alpha_0 \leq \ldots \leq \alpha_p} \mathrm{HH}^{q,s}(A_{\alpha_0 \ldots \alpha_p}, M) \right) = \varprojlim_\alpha{}^p \mathrm{HH}^{q,s}(A_\alpha, M).$$

Now let q be the filtration-index. Taking cohomology in the p-direction gives

$$E_1''^{p,q} = H^p(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(\Sigma^* A_\alpha, M[s]).$$

Regarded as a graded $\mathbb{F}_2$-vector space, $M[s]$ is an injective object in the category of graded $\mathbb{F}_2$-vector spaces. Hence taking cohomology commutes with $\underline{\mathrm{Hom}}_{\mathbb{F}_2}(-, M[s])$. By [15, page 33], the complex $(\Sigma^* T_\alpha, \partial^*)$ is acyclic and $H_0(\Sigma^* T_\alpha) = \varinjlim_\alpha T_\alpha$. Since direct sums commute with tensor products, we get

$$E_1''^{p,q} = \begin{cases} \underline{\mathrm{Hom}}_{\mathbb{F}_2}(\varinjlim_\alpha (A_\alpha)^{\otimes q}, M[s]) & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

Since inductive limits commute with tensor products, we have

$$\underline{\mathrm{Hom}}_{\mathbb{F}_2}(\varinjlim_\alpha (A_\alpha)^{\otimes q}, M[s]) \cong \underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes q}, M[s])$$

for every q. Finally, taking cohomology in the direction of q gives

$$E_2''^{p,q} = \begin{cases} H^q(\underline{\mathrm{Hom}}_{\mathbb{F}_2}(A^{\otimes *}, M[s])) = \mathrm{HH}^{q,s}(A, M) & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

This spectral sequence degenerates. Hence we get that there is a spectral sequence of the form

$$E_2^{p,q} = \varprojlim_\alpha{}^p \mathrm{HH}^{q,s}(A_\alpha, M) \Rightarrow \mathrm{HH}^{p+q,s}(A, M). \qquad \square$$

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