GEOMETRIC REPRESENTATION OF COHOMOLOGY CLASSES FOR THE LIE GROUPS SPIN(7) AND SPIN(8)

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Abstract. By constructing concrete complex-oriented maps we show that the eight-fold of the generator of the third integral cohomology of the spin groups Spin(7) and Spin(8) is in the image of the Thom morphism from complex cobordism to singular cohomology, while the generator itself is not in the image. We thereby give a geometric construction for a nontrivial class in the kernel of the differential Thom morphism of Hopkins and Singer for the Lie groups Spin(7) and Spin(8). The construction exploits the special symmetries of the octonions.

1. Introduction

Let $\mathrm{Spin}(n)$ be the double cover of the real special orthogonal group $SO(n)$. The group $\mathrm{Spin}(n)$ is a compact smooth manifold of dimension $\frac{1}{2}n(n-1)$ and plays an important role in many areas of mathematics and mathematical physics. It is a natural question which elements of its cohomology can be represented by smooth manifolds. More precisely, based on Quillen's work in [12], we may ask which of the generators of $H^*(\mathrm{Spin}(n);\mathbb{Z})$ are in the image of the Thom morphism

$$\tau\colon MU^*(\mathrm{Spin}(n)) \longrightarrow H^*(\mathrm{Spin}(n);\mathbb{Z}),$$

where MU denotes complex cobordism. While it is well-known that the Thom morphism is, in general, not surjective in degrees ≥ 3 (see for example [13, Theorem 2.2]), τ is indeed surjective for the spin groups $\mathrm{Spin}(n)$ for $n \leq 6$ by [7, Proposition 3.4]. For $n \geq 7$, however, the Steenrod square Sq^3 does not vanish on the mod 2-reduction of the non-torsion generator $\gamma_3 \in H^3(\mathrm{Spin}(n);\mathbb{Z})$. By [1], this implies that γ_3 cannot be represented by a smooth manifold (see [7] for a further discussion and see for example [10] and [14, 15] for related computations for Lie groups). In this note, we show that $8 \cdot \gamma_3$ is, in fact, in the image of τ, and we construct a concrete geometric representative. More precisely, we prove the following result (see Theorem 4.1 below):

Theorem 1.1. *Let $\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)$ denote the Grassmannian of oriented k-planes in $\mathbb{R}^n$ na dlet S^1 denote the unit circle. Let γ_3 be a non-torsion generator of $H^3(\mathrm{Spin}(7);\mathbb{Z})$. There is a complex-oriented map*

$$\varphi = f_7 \times f_5 \colon \widetilde{\mathrm{Gr}}_2(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}}_2(\mathbb{R}^5) \times S^1 \longrightarrow \mathrm{Spin}(7)$$

such that the Thom morphism $MU^3(\mathrm{Spin}(7)) \to H^3(\mathrm{Spin}(7);\mathbb{Z})$ maps the element $[\varphi]$ represented by φ to $8 \cdot \gamma_3$.

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Remark 1.2. In [7, Theorem 4.3], we constructed a geometric representative of two times the generator of $H^3(SO(5);\mathbb{Z})$. This construction then generalizes to provide geometric representatives of higher multiples of the generators of $H^{4k+3}(SO(n);\mathbb{Z})$ for $k \geq 0$ and $n \geq k+1$ (see [7, Theorem 4.7] for the precise formula). For spin groups, however, we do not know how to extend the construction of a corresponding map for $\mathrm{Spin}(n)$ with $n \geq 9$, despite the fact that such maps exist for all $n \geq 7$ by [7, Proposition 3.4]. The key features that make the construction of the map in Theorem 1.1 possible for Spin(7), and then also for Spin(8) in Corollary 4.2, are the special symmetries of the octonions which the groups Spin(7) and Spin(8) are related to.

There are at least two reasons why Theorem 1.1 is important. On the one hand, we obtain a more complete picture of the geometric representation of the generators of $H^*(\mathrm{Spin}(7);\mathbb{Z})$ and a geometric interpretation of the failure of the Thom morphism to be surjective in degree 3. On the other hand, Theorem 1.1 yields a geometric construction of a nontrivial element of degree 4 in the kernel of the differential refinement of the Thom morphism of Hopkins–Singer as we will explain next (for a more general and more detailed discussion we refer to [7, Section 2.3]). For the smooth manifold Spin(7), let $\check{MU}(4)^4(\mathrm{Spin}(7))$ and $\check{H}(4)^4(\mathrm{Spin}(7))$ denote the differential refinements of complex cobordism and singular cohomology of Hopkins–Singer in [5], respectively. By [5, diagram (4.57)], the Thom morphism $\tau\colon MU \to H\mathbb{Z}$ induces a commutative diagram

$$\begin{array}{ccc} MU^3(\mathrm{Spin}(7)) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} & \longrightarrow & \check{MU}(4)^4(\mathrm{Spin}(7)) \\ \downarrow {\scriptstyle \tau_{\mathbb{R}/\mathbb{Z}}} & & \downarrow {\scriptstyle \check{\tau}} \\ H^3(\mathrm{Spin}(7);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} & \longrightarrow & \check{H}(4)^4(\mathrm{Spin}(7)) \end{array} \tag{1}$$

in which the horizontal maps are injective. This implies that every element in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$ induces an element in the kernel of $\check{\tau}$. We say that an element in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$ or $\check{\tau}$ is *nontrivial* if it is not contained in the respective ideal generated by $MU^{*<0}$. We claim that the element $[\varphi] \otimes 1/8 \in MU^3(\mathrm{Spin}(7)) \otimes \mathbb{R}/\mathbb{Z}$ is a nontrivial element in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$: By Theorem 1.1, we have

$$\tau_{\mathbb{R}/\mathbb{Z}}([\varphi] \otimes 1/8) = \tau([\varphi]) \otimes 1 = 8 \cdot \gamma_3 = 0.$$

Hence $[\varphi] \otimes 1/8$ lies in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$. However, $[\varphi] \otimes 1/8$ is not 0 in $MU^3(\mathrm{Spin}(7)) \otimes \mathbb{R}/\mathbb{Z}$, since if $[\varphi]$ had been of the form $8 \cdot [\varphi']$ with $[\varphi'] \in MU^3(\mathrm{Spin}(7))$, then $[\varphi']$ would map to γ_3 or $\gamma_3 + y$ for some class $y \in H^3(\mathrm{Spin}(7);\mathbb{Z})$ with $8y = 0$. The latter is not the case by [7, Proposition 3.4]. Moreover, $[\varphi] \otimes 1/8$ does not lie in the ideal generated by $MU^{*<0}$, since otherwise $[\varphi]$ would be in the kernel of τ. Since $[\varphi]$ maps to $8 \cdot \gamma_3 \neq 0$ in $H^3(\mathrm{Spin}(7);\mathbb{Z})$, the latter is not the case. Hence Theorem 1.1 provides a geometric description of a nontrivial element in the kernel of the differential Thom morphism for Spin(7):

Theorem 1.3. *The class $\frac{1}{8}[\varphi]$ is a nontrivial element in the kernel of*

$$\check{\tau}\colon \check{MU}(4)^4(\mathrm{Spin}(7)) \longrightarrow \check{H}(4)^4(\mathrm{Spin}(7))$$

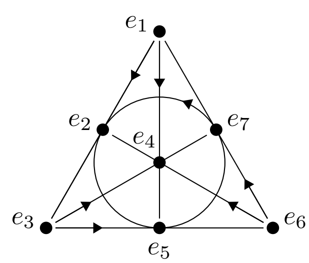
which is not contained in the ideal generated by $MU^{*<0}$.

Elements in the kernel of the differential refinement of the Thom morphism are particularly interesting as they show in which way differential cobordism is a finer invariant than ordinary differential cohomology. Finally, we note that in [5, §2.7] the group $\check{H}(4)^4(M)$, for a certain spin manifold M, explains the behavior of an important partition function in mathematical physics. We refer to [3, Example 48] for other interesting phenomena in mathematical physics related to the study of the morphisms between generalised differential cohomology theories. We do not know of a potential similar application of Theorem 1.3 yet. We expect, however, that the techniques used to prove Theorem 1.3 at least will help to shed new light on such phenomena and on the Abel–Jacobi invariant for complex cobordism of [4].

The paper is organised as follows. In section 2 we recall some basic facts about the geometry of the octonions and special orthogonal and spin groups. In section 3 we construct the maps that feature in Theorem 1.1 by using the special symmetries of the octonions. The nice properties of oriented Grassmannians and compact Lie groups then imply that the map is complex-oriented. In section 4 we prove our main technical results Theorem 4.1 and Corollary 4.2 by computing the effect of the relevant maps in homology via local degrees.

2. Octonions and Spin(7)

We let $\mathbb{O}$ denote the *octonions*, i.e., the normed division algebra with $\mathbb{R}$-basis $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ which satisfies the following relations: We have $e_0 = 1$ and $e_i^2 = -1$ for $i \neq 0$, and the multiplication of distinct basis elements is given by the following Fano plane as described in [8].



For example, we have $e_3e_2 = -e_1$, where the negative sign corresponds to the fact that the path from e_3 to e_2 goes against the direction of the arrow. We note that several versions of this Fano plane exist, producing different bases for $\mathbb{O}$. The subspace of $\mathbb{O}$ spanned by $\{e_1, \ldots, e_7\}$ is referred to as the *purely imaginary octonions*, denoted Im $\mathbb{O}$.

By associating the octonions with $\mathbb{R}^8$ and the purely imaginary octonions with $\mathbb{R}^7$, the group $SO(8)$ acts on the octonions $\mathbb{O}$, while $SO(7)$ acts on Im $\mathbb{O}$. We then have the following construction of the group Spin(7), from [6] and [16]. For every $g \in SO(7)$ there exists a unique, up to sign, $\tilde{g} \in SO(8)$ such that $g(x)\tilde{g}(y) = \tilde{g}(xy)$ for all $x, y \in \mathbb{O}$. Then we can define Spin(7) as a subgroup of $SO(8)$ by

$$\mathrm{Spin}(7) = \{\tilde{g} \in SO(8) \mid g \in SO(7)\}.$$

Thus, elements of Spin(7) can be considered as transformations of 8-dimensional Euclidean space. The canonical double covering $\mathrm{Spin}(7) \to SO(7)$ is the map $\tilde{g} \mapsto g$. For future reference we now state without proof some well-known properties of the octonions [2, Chapters 6.5 and 6.7].

Lemma 2.1. *Let $x, y, z \in \mathbb{O}$. Then we have*

(i) $x(xy) = (xx)y$,
$(yx)x = y(xx)$,
(ii) $(x(yz))x = x((yz)x) = (xy)(zx)$,
$(x(yx))z = x(y(xz))$,
$y(x(zx)) = ((yx)z)x$.

Now let $x, y, z \in \operatorname{Im} \mathbb{O}$. Then we have

(iii) $xy = z \Leftrightarrow x = yz$,
(iv) $xy = -yx$ *if* $x \perp y$.

We then state and prove one more property of the octonions.

Lemma 2.2. *Let $x, y, z \in \operatorname{Im} \mathbb{O}$ such that x, y, z, xy are mutually orthogonal. Then $x(yz) = -(xy)z$.*

Proof. Using Lemma 2.1, we compute

$$\begin{aligned} x(yz) &= -x(zy) = (zy)x = -(xx)((zy)x) = -x(x((zy)x)) \\ &= -x((xz)(yx)) = x((yx)(xz)) = (x((yx)x))z = -(xy)z. \end{aligned}$$ □

We will use the following notation.

Definition 2.3. We call a pair (L, σ) consisting of a 2-dimensional sub-vector space $L \subseteq \mathbb{R}^n$ with an orientation σ of L an *oriented plane*. When x and y are orthonormal vectors in $\mathbb{R}^n$, we write $[x, y]$ for the oriented plane spanned by x and y where the orientation is given by the ordering of the two vectors, i.e., $[x, y]$ and $[y, x]$ have opposite orientations.

Definition 2.4. Let $(L, \sigma) \subseteq \mathbb{R}^n$ be an oriented plane, and let $t \in \mathbb{R}$. We write $r_{L,\sigma,t}$ for the rotation of the plane L by the angle t along the orientation σ.

We note that if $(L, \sigma) = [x, y]$, then $r_{L,\sigma,t}$ is given by

$$\begin{aligned} x &\longmapsto \cos tx + \sin ty \\ y &\longmapsto -\sin tx + \cos ty. \end{aligned}$$

We now extend $r_{l,\sigma,t}$ to a transformation of $\mathbb{R}^8$.

Definition 2.5. Let $(L, \sigma) \subset \mathbb{R}^8$ be an oriented plane, $t \in \mathbb{R}$ and $z \in \mathbb{R}^8$. For $z \in \mathbb{R}^8$, let $z_1 \in L$ and $z_2 \in L^\perp$ be the unique vectors such that $z = z_1 + z_2$. Then we define $\psi_{(L,\sigma),t} \in SO(8)$ by

$$\psi_{(L,\sigma),t}(z) = r_{L,\sigma,t}(z_1) + z_2.$$

We will refer to an element of the form $\psi_{(L,\sigma),t}$ as a *rotation*.

3. Constructing the map

Let $\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)$ denote the Grassmannian of oriented k-planes in $\mathbb{R}^n$. We think of the circle S^1 as embedded in the complex plane, and we will write its points on the form e^{it} parameterized by $t \in \mathbb{R}$. We now construct a map which we subsequently prove is well-defined.

Definition 3.1. For $[x, y] \in \widetilde{\mathrm{Gr}}_2(\mathbb{R}^7)$, let w denote a unit vector in $\mathbb{R}^8$ such that $w \perp \operatorname{Span}\{e_0, x, y, xy\}$. We then define

$$\begin{aligned} f_7 \colon \widetilde{\mathrm{Gr}}_2(\mathbb{R}^7) \times S^1 &\longrightarrow SO(8) \\ ([x, y], e^{it}) &\longmapsto \psi_{[x,y],t} \cdot \psi_{[e_0,xy],t} \cdot \psi_{[w,w(xy)],t} \cdot \psi_{[wx,wy],t}. \end{aligned}$$

Lemma 3.2. *The map f_7 is well-defined.*

Proof. Firstly, we show that the map is invariant under the choice of basis $[x, y]$. A different basis for the same oriented plane is given by

$$[x', y'] = [\cos sx + \sin sy, -\sin sx + \cos sy], \quad s \in \mathbb{R}.$$

We then see that

$$\begin{aligned} x'y' &= (\cos sx + \sin sy)(-\sin sx + \cos sy) \\ &= \cos^2(s)xy - \sin^2(s)yx - \cos s \sin sx^2 + \cos s \sin sy^2 = xy. \end{aligned}$$

Thus, we have

$$\begin{aligned} [x', y'] = [x, y],\ [e_0, x'y'] = [e_0, xy],\ [w, w(x'y')] = [w, w(xy)], \\ [wx', wy'] = [\cos swx + \sin swy, -\sin swx + \cos swy] = [wx, wy], \end{aligned}$$

which implies that

$$\psi_{[x,y],t} \cdot \psi_{[e_0,xy],t} \cdot \psi_{[w,w(xy)],t} \cdot \psi_{[wx,wy],t} = \psi_{[x',y'],t} \cdot \psi_{[e_0,x'y'],t} \cdot \psi_{[w,w(x'y')],t} \cdot \psi_{[wx',wy'],t}.$$

Next, we show that f_7 is invariant under the choice of w. Since the rotations $\psi_{[x,y],t}$ and $\psi_{[e_0,xy],t}$ do not depend on the choice of w, it suffices to show that the map

$$\widetilde{f_7} \colon ([x, y], e^{it}) \mapsto \psi_{[w,w(xy)],t} \cdot \psi_{[wx,wy],t}$$

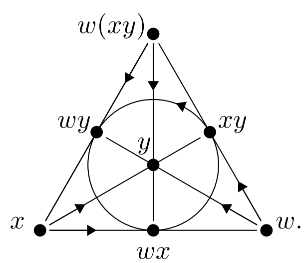
is well-defined. Given some w, we claim that

$$\mathcal{B} = \{e_0, x, y, xy, w, wx, wy, w(xy)\}$$

is an orthogonal basis for $\mathbb{O}$. The vectors e_0, x, y, xy, w are mutually orthogonal by construction. It follows from Lemmas 2.1 and 2.2 that the remaining ones are also mutually orthogonal. For example, we have

$$-wx \perp -e_0 \implies -(wx)y \perp -y \implies w(xy) \perp y.$$

Thus, $\mathcal{B}$ is an orthogonal basis. Furthermore, Lemmas 2.1 and 2.2 imply that the multiplication of basis elements is given by



(2)

Given this basis, a different choice of vector w must be given by $w' = aw + bwx + cwy + dw(xy)$, where $a, b, c, d \in \mathbb{R}$. Using Diagram (2), we find that

$$\begin{aligned} w'x &= -bw + awx - dwy + cw(xy) \\ w'y &= -cw + dwx + awy - bw(xy) \\ w'(xy) &= -dw - cwx + bwy + aw(xy). \end{aligned}$$

Let $\psi = \psi_{[w,w(xy)],t} \cdot \psi_{[wx,wy],t}$ and $\psi' = \psi_{[w',w'(xy)],t} \cdot \psi_{[w'x,w'y],t}$. It is then straight forward to check that ψ and ψ' act the same way on all elements of $\mathcal{B}$. For example, we have

$$\begin{aligned}\psi(w') &= a\psi(w) + b\psi(wx) + c\psi(wy) + d\psi(w(xy)) \\ &= a(\cos tw + \sin tw(xy)) + b(\cos twx + \sin twy) \\ &+ c(-\sin twx + \cos twy) + d(-\sin tw + \cos tw(xy)) \\ &= \cos t\,[aw + bwx + cwy + dw(xy)] + \sin t\,[-dw - cwx + bwy + aw(xy)] \\ &= \cos tw' + \sin tw'(xy) = \psi'(w').\end{aligned}$$

Thus, the map f_7 is well-defined. □

We now show how the map f_7 interacts with the groups $\mathrm{Spin}(7)$ and $SO(7)$. Let q denote the double covering $\mathrm{Spin}(7) \longrightarrow SO(7)$.

Proposition 3.3. *The image of the map f_7 is $\mathrm{Spin}(7) \subset SO(8)$. Moreover, for all $([x,y], e^{it}) \in \widetilde{\mathrm{Gr}}_2(\mathbb{R}^7) \times S^1$, we have $(q \circ f_7)([x,y], e^{it}) = \psi_{[x,y],2t} \in SO(7)$.*

Proof. Let $g = \psi_{[x,y],2t}$ and $\psi_t = \psi_{[e_0,xy],[x,y],[wx,wy],[w,w(xy)]}$. It suffices to show that $g(\alpha)\psi_t(\beta) = \psi_t(\alpha\beta)$ for all $\alpha, \beta \in \mathcal{B}$, where $\mathcal{B}$ is the basis constructed in the proof of Lemma 3.2. We first claim that $\alpha\psi_{\frac{\pi}{2}}(\beta) = \psi_{\frac{\pi}{2}}(\alpha\beta)$ for all $\alpha, \beta \in \mathcal{B}$ with $\alpha \notin \{x,y\}$. Observe that if $\beta \notin \{x,y\}$, then $\psi_{\frac{\pi}{2}}(\beta) = \beta(xy)$, and if $\beta \in \{x,y\}$, then $\psi_{\frac{\pi}{2}}(\beta) = -\beta(xy)$. Thus, we have

$$\alpha\psi_{\frac{\pi}{2}}(\beta) = \begin{cases} \alpha(\beta(xy)), & \beta \notin \{x,y\} \\ -\alpha(\beta(xy)), & \beta \in \{x,y\}. \end{cases}$$

Furthermore, Lemmas 2.1 and 2.2 imply that

$$\alpha(\beta(xy)) = \begin{cases} (\alpha\beta)(xy), & \alpha = \beta,\ \alpha = xy,\ \beta = xy, \text{ or } \alpha\beta = \pm xy \\ -(\alpha\beta)(xy), & \text{otherwise.} \end{cases}$$

Combined with the assumption that $\alpha \notin \{x,y\}$, we get that

$$\alpha\psi_{\frac{\pi}{2}}(\beta) = \left\{ \begin{matrix} (\alpha\beta)(xy), & \alpha\beta \notin \{\pm x, \pm y\} \\ -(\alpha\beta)(xy), & \alpha\beta \in \{\pm x, \pm y\} \end{matrix} \right\} = \psi_{\frac{\pi}{2}}(\alpha\beta).$$

This proves the claim. Then, for $\alpha \in \mathcal{B} \setminus \{x,y\}$, we have

$$\begin{aligned} g(\alpha)\psi_t(\beta) &= \alpha(\cos t\beta + \sin t\psi_{\frac{\pi}{2}}(\beta)) = \cos t\alpha\beta + \sin t\alpha\psi_{\frac{\pi}{2}}(\beta) \\ &= \cos t\alpha\beta + \sin t\psi_{\frac{\pi}{2}}(\alpha\beta) = \psi_t(\alpha\beta). \end{aligned}$$

It remains to show that $g(\alpha)\psi_t(\beta) = \psi_t(\alpha\beta)$ for $\alpha \in \{x,y\}$. These 16 cases are easily checked by direct computation, for example

$$g(x)\psi_t(y) = (\cos 2tx + \sin 2ty)(-\sin tx + \cos ty) = -\sin te_0 + \cos txy = \psi_t(xy).$$

This completes the proof. □

Definition 3.4. Let f_5 denote the restriction of f_7 to $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^5) \times S^1 \subseteq \widetilde{\mathrm{Gr}}_2(\mathbb{R}^7) \times S^1$. We then define the map

$$\begin{aligned} f_7 \times f_5 \colon \widetilde{\mathrm{Gr}}_2(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}}_2(\mathbb{R}^5) \times S^1 &\longrightarrow \mathrm{Spin}(7) \\ \left([x,y], e^{it}, [x'y'], e^{it'}\right) &\longmapsto f_7\left([x,y], e^{it}\right) \cdot f_5\left([x',y'], e^{it'}\right). \end{aligned}$$

Lemma 3.5. *The map $f_7 \times f_5$ admits a complex orientation. In particular, $f_7 \times f_5$ is a proper complex-oriented smooth map and represents an element in $MU^3(\mathrm{Spin}(7))$.*

Proof. The fact that g admits a complex orientation follows from the facts that S^1 is stably almost complex, $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^7)$ and $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)$ are almost complex, and Spin(7) is a compact Lie group. □

4. Computations in homology

We recall from [9, Corollary III.3.15 and Theorem IV.2.19] that, for $n = 7, 8$, there is an isomorphism $H^3(\mathrm{Spin}(n); \mathbb{Z}) \cong \mathbb{Z}$. Let $\gamma_3 \in H^3(\mathrm{Spin}(n); \mathbb{Z})$ be a generator. By [7, Proposition 3.4], γ_3 is not contained in the image of the Thom morphism. However, we will now show that the Thom morphism sends the cobordism class represented by $f_7 \times f_5$ to an integer multiple of γ_3 in $H^3(\mathrm{Spin}(7); \mathbb{Z})$.

Theorem 4.1. *The Thom morphism $MU^3(\mathrm{Spin}(7)) \to H^3(\mathrm{Spin}(7); \mathbb{Z})$ maps the element represented by $f_7 \times f_5$ to ± 8 times the generator $\gamma_3 \in H^3(\mathrm{Spin}(7); \mathbb{Z})$.*

Proof. By Poincaré duality it suffices to show that the fundamental class of $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1$ in homology is mapped to eight times the corresponding generator in $H_{18}(\mathrm{Spin}(7); \mathbb{Z})$. In [7] we constructed a class of maps into special orthogonal groups. In particular, there is the map

$$\begin{aligned} H_{7,0} \colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 &\longrightarrow SO(7) \\ \left([x,y], e^{it}, [x',y'], e^{it'}\right) &\longmapsto \psi_{[x,y],t} \cdot \psi_{[x',y'],t'}. \end{aligned}$$

By [7, Theorem 4.7], the induced map

$$(h_{7,0})_* \colon H_{18}(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1) \longrightarrow H_{18}(SO(7))$$

is given by a multiplication by ± 4 after identifying each homology group with $\mathbb{Z}$. We define the map

$$\begin{aligned} p \colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 &\longrightarrow \widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \\ \left([x,y], e^{it}, [x',y'], e^{it'}\right) &\longmapsto \left([x,y], e^{2it}, [x',y'], e^{2it'}\right). \end{aligned}$$

By Proposition 3.3, we get that $c \circ (f_7 \times f_5) = h_{7,0} \circ p$. Thus, we have the commutative diagram

$$\begin{array}{ccc} H_{18}(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1) & \xrightarrow{(f_7 \times f_5)_*} & H_{18}(\mathrm{Spin}(7)) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle c_*} \\ H_{18}(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1) & \xrightarrow{(h_{7,0})_*} & H_{18}(SO(7)), \end{array}$$

where all the homology groups are isomorphic to $\mathbb{Z}$. Since p consists of two maps of degree 2 on the circles, we see that p_* is a multiplication by 4. Furthermore, it follows from [11, 7.4] that c_* is a multiplication by 2. Thus, we conclude that $(f_7 \times f_5)_*$ is given by a multiplication by ± 8, which completes the proof. □

Since Spin(8) is homeomorphic to Spin(7) $\times S^7$ (see [6]), Theorem 4.1 implies the following result.

Corollary 4.2. *The Thom morphism* $MU^3(\mathrm{Spin}(8)) \to H^3(\mathrm{Spin}(8);\mathbb{Z})$ *maps the element represented by*

$$\widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \times S^7 \longrightarrow \mathrm{Spin}(7) \times S^7 \cong \mathrm{Spin}(8)$$
$$\Big([x,y], e^{it}, [x',y'], e^{it'}, s\Big) \longmapsto \Big(f_7\left([x,y], e^{it}\right) \cdot f_5\left([x',y'], e^{it'}\right), s\Big)$$

to ± 8 *times the generator* $\gamma_3 \in H^3(\mathrm{Spin}(8);\mathbb{Z})$.

Remark 4.3. By studying the differentials in the Atiyah–Hirzebruch spectral sequence for Spin(7), one can see that $2 \cdot \gamma_3 \in H^3(\mathrm{Spin}(7);\mathbb{Z})$ is in the image of the Thom morphism. Thus, the construction of $f_7 \times f_5$ is not the best possible one. Furthermore, using the same method as in the proof of Theorem 4.1, we see that the element of $MU^{10}(\mathrm{Spin}(7))$ represented by f_7 is mapped to ± 4 times the generator of $H^{10}(\mathrm{Spin}(7);\mathbb{Z})$. However, there must exist some element of $MU^{10}(\mathrm{Spin}(7))$ which maps to 2 times this generator.

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