FORMALITY AND STRONG MASSEY VANISHING FOR REAL PROJECTIVE GROUPS

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Extended abstract for
MFO-RIMS Tandem Workshop 2023
Arithmetic Homotopy and Galois Theory

1. Formality and Massey vanishing

A differential graded algebra $C^\bullet$ is called formal if there is a zig-zag of quasi-isomorphisms of differential graded algebras between $C^\bullet$ and its cohomology algebra with trivial differential. Formality is a rather strong property and implies, for example, the vanishing of all Massey products. Hence one may view Massey products as invariants which detect whether a differential graded algebra may contain more information than its cohomology. While there are many examples of non-vanishing Massey products in arithmetic, Hopkins and Wickelgren showed in [8] that all triple Massey products of degree one classes in the mod 2-Galois cohomology of global fields of characteristic different from 2 vanish, i.e., contain zero, whenever they are defined. Mináč and Tân then showed the vanishing of mod 2-triple Massey products of degree one classes for all fields. Moreover, they formulated the Massey vanishing conjecture stating that, for all fields k, all $n \geq 3$ and all primes p, the n-fold Massey product of degree one classes in mod p-Galois cohomology should vanish whenever it is defined (see [12]). The work of Hopkins–Wickelgren and Mináč–Tân has inspired a lot of activity in recent years. We now list the main cases we are aware of for which the Massey vanishing conjecture is now known to be true, in addition to our own work which we will describe below:

- By the work of Matzri [9], Efrat–Matzri [2] and Mináč–Tân [12] for all fields, all primes p and $n = 3$.
- By the work of Harpaz–Wittenberg [7] for all number fields, all primes p and all $n \geq 3$.
- By the work of Pál–Szabó [16] for all fields with virtual cohomological dimension at most one and all pseudo p-adically closed fields, all primes and all $n \geq 3$.
- By the work of Guillot–Mináč–Topaz–Wittenberg [4] for number fields and by Merkurjev–Scavia [11] for all fields, $p = 2$ and $n = 4$.
- Quadrelli shows that Efrat's Elementary Type Conjecture for pro-p-groups implies the Massey vanishing conjecture and proves the Massey vanishing conjecture in several further cases (see [20, Corollary 1.4] for a list).

We note that Ekedahl showed in [3] that there are non-vanishing triple Massey products of classes in $H^1_{\text{et}}(X, \mathbb{F}_p)$ for X an absolutely irreducible smooth projective

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Partially supported by the RCN Project No. 313472 *Equations in Motivic Homotopy*.

variety of dimension two over $\mathbb{C}$. Bleher–Chinburg–Gillibert show in [1] that triple Massey products of classes in $H^1_{\mathrm{et}}(X, \mathbb{F}_p)$ for X an absolutely irreducible smooth projective curve over a field of characteristic different from p may not vanish.

Based on their computations of Massey products in Galois cohomology, Hopkins–Wickelgren asked in [8] whether the mod 2-Galois cohomology algebra of fields actually is formal. Positselski showed in [19] that the answer to this question is negative in general, as there are local fields which are not formal. Harpaz–Wittenberg show in [4, Example A.15] that certain fourfold Massey products are not defined, even though the neighbouring cup products vanish, and thereby show that $\mathbb{Q}(\sqrt{2}, \sqrt{17})$ is not formal. Further examples of non-formal fields have recently been discussed by Merkurjev–Scavia in [10].

In [15], however, we show that there is a large class of fields for which formality holds. In particular, this implies the Massey vanishing conjecture for all primes p and all $n \geq 3$ and all nonzero cohomological degrees for these fields. We will now briefly describe the main results of [15].

2. Formality of real projective groups

Let G be a profinite group. An *embedding problem for G* is a solid diagram

$$\begin{tikzcd} & G \arrow[dl, dotted, "\widetilde{\phi}"'] \arrow[d, "\phi"] \\ B \arrow[r, "\alpha"] & A, \end{tikzcd}$$

where A, B are finite groups, the solid arrows are continuous homomorphisms and α is surjective. A solution of this embedding problem is a continuous homomorphism $\widetilde{\phi}\colon G \to B$ which makes the diagram commutative. The embedding problem above is called *real* if for every involution $t \in G$ with $\phi(t) \neq 1$ there is an involution $b \in B$ with $\alpha(b) = \phi(t)$. Following Haran and Jarden [6], a profinite group G is called *real projective* if G has an open subgroup without 2-torsion, and if every real embedding problem for G has a solution. In [15] we prove the following result:

Theorem 1 ([15])**.** *Let G be a real projective profinite group and p be a prime number, and let $C^\bullet(G, \mathbb{F}_p)$ denote the differential graded $\mathbb{F}_p$-algebra of continuous cochains of G with values in $\mathbb{F}_p$. Then $C^\bullet(G, \mathbb{F}_p)$ is formal.*

To prove the theorem we use the work of Scheiderer on the cohomology of real projective groups and calculate the graded Hochschild cohomology groups of a sum of certain quadratic algebras. Then we apply a criterion for formality due to Kadeishvili which states that the formality of a dg-algebra $C^\bullet$ is implied by the vanishing of the Hochschild cohomology groups $\mathrm{HH}^{n,2-n}(H^\bullet(C^\bullet), H^\bullet(C^\bullet))$ for all $n \geq 3$.

3. Implications for absolute Galois groups of fields

Now we describe the implications of Theorem 1 for fields. By the work of Haran and Jarden, the class of real projective groups is associated to the following class of fields. Recall that a field k has *virtual* cohomological dimension ≤ 1 if there is a finite separable extension K/k with $\mathrm{cd}(K) \leq 1$. Since the only torsion elements in the absolute Galois group of k are the involutions coming from the orderings of k, it is equivalent to require $\mathrm{cd}(K) \leq 1$ for any fixed finite separable extension K

of k without orderings, for example for $K = k(\mathbf{i})$ where $\mathbf{i} = \sqrt{-1}$. In particular, if k itself cannot be ordered (which is equivalent to -1 being a sum of squares in k), this condition is equivalent to $\mathrm{cd}(k) \leq 1$. Since by classical Artin–Schreier theory every involution in the absolute Galois group of a field is self-centralising, Haran's work [5, Theorem A on page 219] implies that the absolute Galois group of a field k is real projective if and only if k satisfies $\mathrm{cd}(k(\mathbf{i})) \leq 1$.

Examples 2. Examples of fields k which can be ordered with $\mathrm{cd}(k(\mathbf{i})) \leq 1$ include real closed fields, function fields in one variable over any real closed ground field, the field of Laurent series in one variable over any real closed ground field, and the field $\mathbb{Q}^{ab} \cap \mathbb{R}$ which is the subfield of $\mathbb{R}$ generated by the numbers $\cos(\frac{2\pi}{n})$ where $n \in \mathbb{N}$.

Furthermore, Haran and Jarden show in [6] that the following important class of fields has real projective absolute Galois groups. For a field k, let $\mathrm{Spr}(k)$ denote the real spectrum of k, i.e., the set of all orderings of k. For an ordering $<\in \mathrm{Spr}(k)$, let $k_<$ denote the real closure of the ordered field $(k, <)$. A field k is called *pseudo real closed* if every absolutely irreducible variety defined over k which has a $k_<$-rational simple point for every $<\in \mathrm{Spr}(k)$ has a k-rational point. In particular, a pseudo real closed field with no orderings is pseudo algebraically closed, i.e., every absolutely irreducible variety defined over the field has a rational point. Moreover, if k is pseudo real closed, then $k(\mathbf{i})$ is pseudo algebraically closed, as $k_<(\mathbf{i})$ is algebraically closed for every $<\in \mathrm{Spr}(k)$ by Artin–Schreier theory. Hence k has virtual cohomological dimension ≤ 1, and in particular the absolute Galois group $\Gamma(k)$ is real projective by Haran's work. In [6, Theorem on page 450] Haran–Jarden show that the absolute Galois group $\Gamma(k)$ of a pseudo real closed field k is real projective, and conversely, if G is a real projective group, then there is a pseudo real closed field k such that $\Gamma(k) \cong G$. Therefore, the Hopkins–Wickelgren formality conjecture for the class of fields of virtual cohomological dimension ≤ 1 is the same as for the class of pseudo real closed fields, and it is a purely group-theoretical problem for the class of real projective profinite groups. As a consequence of Theorem 1 we then obtain the following result, which provides the first example of a class of fields with infinite cohomological dimension for which formality holds.

Theorem 3 ([15]). *Let k be a field with virtual cohomological dimension ≤ 1 and let $\Gamma(k)$ denote its absolute Galois group. Then, for all primes p, $C^\bullet(\Gamma(k), \mathbb{F}_p)$ is formal and satisfies strong Massey vanishing, i.e., whenever the Massey product of any number of elements of any nonzero cohomological degrees is defined then it contains zero.*

4. Koszulity conjecture

Another consequence of our methods is a positive case of a conjecture by Positselski and Voevodsky on the Koszulity of Galois cohomology [18, §0.1, page 128]. One way to formulate the conjecture is that, for every field k containing a primitive pth root of unity and absolute Galois group $\Gamma(k)$, the algebra $H^\bullet(\Gamma(k), \mathbb{F}_p)$ is Koszul (see also [14]). One of the significances of the conjecture is that Positselski and Vishik show in [17] that Koszulity of the Galois cohomology would be a key ingredient in a potential alternative way to prove the Milnor–Bloch–Kato conjecture, i.e., the Norm Residue Theorem. In [15] we prove the following result:

Theorem 4 ([15]). *Let k be a field with virtual cohomological dimension ≤ 1 and p be a prime number. Then the cohomology algebra $H^\bullet(\Gamma(k), \mathbb{F}_p)$ is Koszul.*

Recently, other positive cases of the Koszulity conjecture of Positselski and Voevodsky were proven in [14, Theorem D].

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